Probability Models of Information Exchange on Networks

Lecture 6

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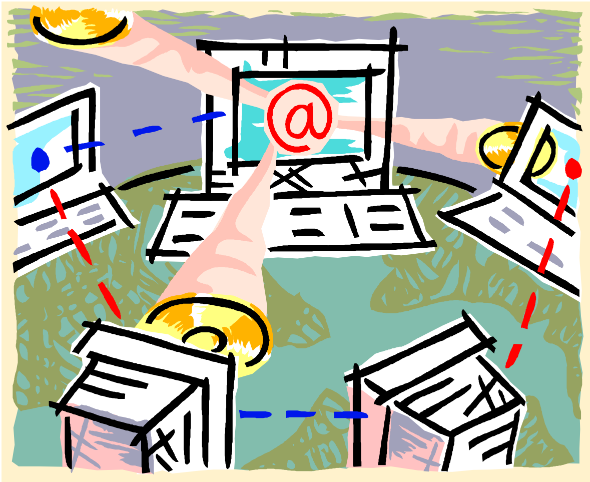


Many Other Models

• There are many models of information exchange on networks.
• Q: Which model to chose?

• My answer – good features of models include:

• “Canonical” models.
• Amenable to analysis.
• Studied intensively before.

• Ok to invent your own models.
• Models are always just models …

Ariel Rubinstein on theoretical economics

- When talking about economics:
- “Everything I say is personal, based upon the entire range of my life experience which also includes the fact that professionally I engage in economics theory. However, to the best of my understanding, economic theory has nothing to do say about the heart of the issue under discussion here. I am not sure I know what an opinion is. I am not attempting to predict the rate of inflation tomorrow …”

Some other natural models

- Growth models: percolation models, DLA etc.

- Competition models: Competing growth.

- Infection models: Contact process, SIR, SIS …

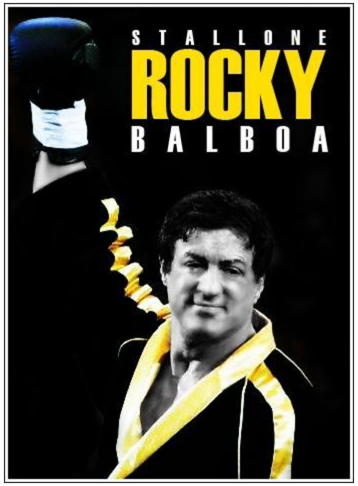
- Aspects of modeling:
- Dynamic networks
- Random networks
- …

- Today: two examples of percolation based processes.

Example1: models of collective behavior

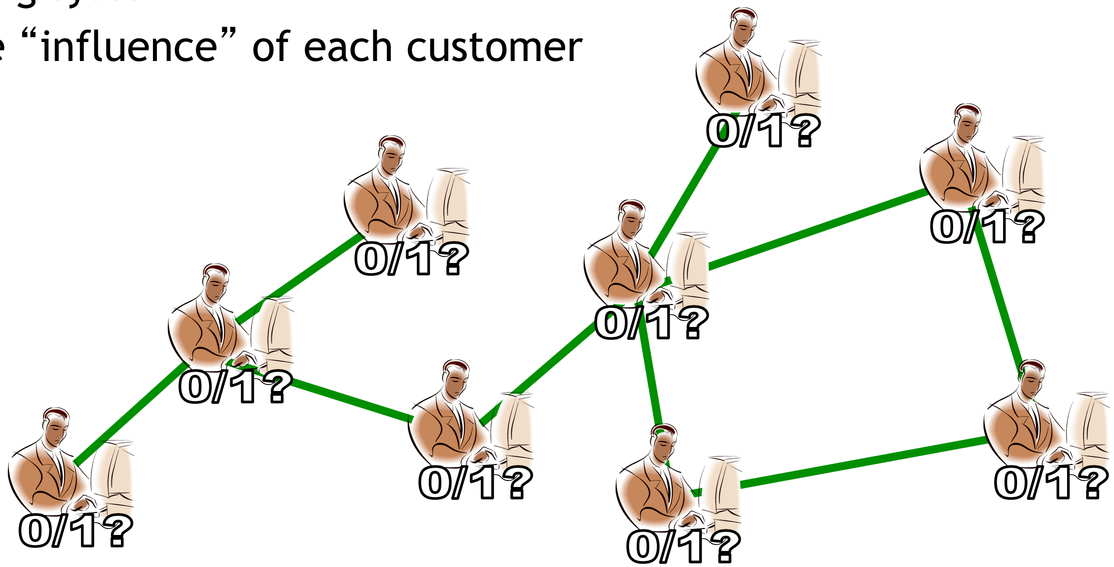
- examples:
  - joining a riot
  - adopting a product
  - going to a movie

- model features:
  - binary decision
  - cascade effect
  - network structure



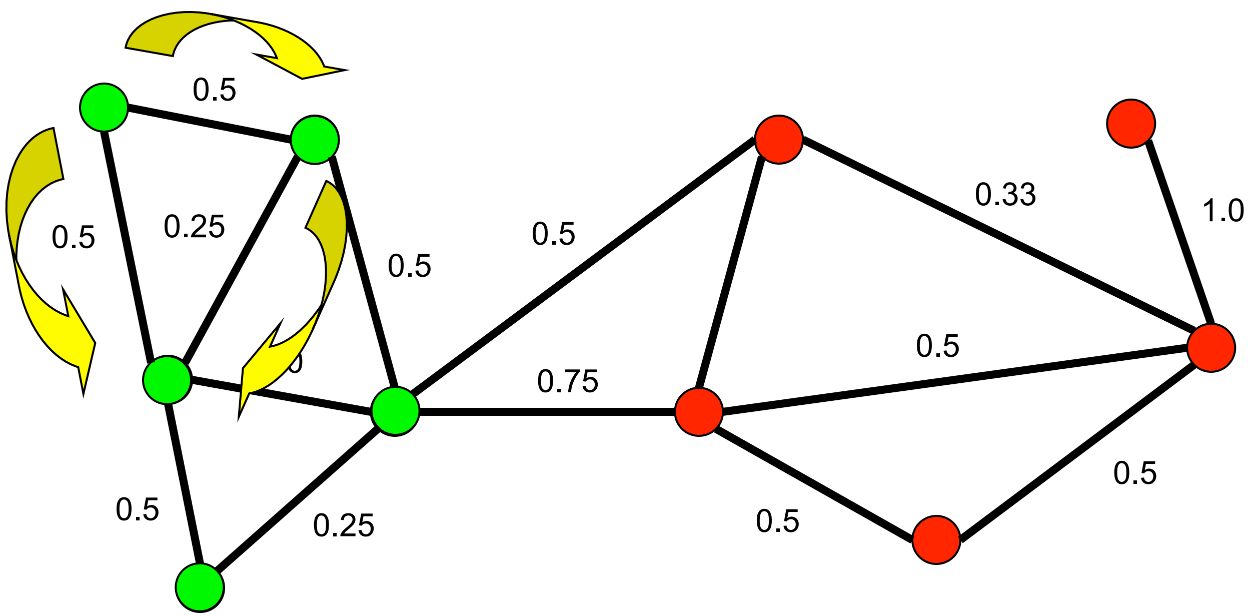
viral marketing

- referrals, word-of-mouth can be very effective
  - ex.: google+
- viral marketing
  - goal: **mining the network value** of potential customers
  - how: **target a small set of trendsetters**, seeds
- example [Domingos-Richardson' 02]
  - collaborative filtering system
  - use MRF to compute "influence" of each customer



independent cascade model

- when a node is activated
  - it gets one chance to activate each neighbour
  - probability of success from u to v is $p_{u,v}$



generalized models

- graph $G=(V,E)$; initial activated set S_0

- generalized threshold model [Kempe-Kleinberg-Tardos' 03,' 05]
  - activation functions: $f_u(S)$ where S is set of activated nodes
  - threshold value: θ_u uniform in $[0,1]$
  - dynamics: at time t,set S_t to S_{t-1} and add all nodes with $f_u(S_{t-1}) \geq \theta_u$
    (note the process stops after (at most) $n-1$ steps)
- generalized cascade model [KKT' 03,' 05]
  - when node u is activated:
    - gets one chance to activate each neighbours
    - probability of success from u to v: $p_u(v,S)$ where S is set of nodes who have already tried (and failed) to activate u
  - assumption: the $p_u(v,.)$' s are "order-independent"

- **theorem** [KKT' 03] - the two models are equivalent

influence maximization

- **definition** - the **influence** $\sigma(S)$ given the initial seed S is the expected size of the infected set at termination

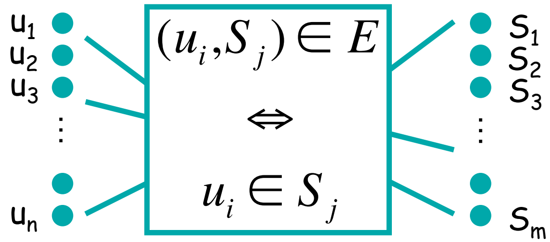
$$\sigma(S) = \mathrm{E}_S\left[\left|S_{n-1}\right|\right]$$

- **definition** - in the **influence maximization problem (IMP)**, we want to find the seed S of fixed size k that maximizes the influence

$$S^* = \arg\max\left\{\sigma(S) : S \subseteq V,\ |S| = k\right\}$$

- **theorem** [KKT' 03] - the IMP is **NP-hard**
  - reduction from *Set Cover*: ground set $U = \{u_1,\ldots,u_n\}$ and collection of cover subsets $S_1,\ldots,S_m$

independent cascade model



$$\exists S,\ |S| = k,\ \sigma(S) \ge n + k\,?$$

submodularity

- **definition** - a set function $f : V \rightarrow R$ is **submodular** if for all A, B in V

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

- example: $f(S) = g(|S|)$ where g is concave

- interpretation: “discrete concavity” or “diminishing returns”, indeed submodularity equivalent to

$$\forall S \subseteq T, \forall v \in V, \quad f(T \cup \{v\}) - f(T) \leq f(S \cup \{v\}) - f(S)$$

- threshold models:
  - it is natural to assume that the **activation functions have diminishing returns**
  - supported by observations of [Leskovec-Adamic-Huberman’06] in the context of viral marketing

main result

- **theorem** [M-Roch' 06; first conjectured in KKT' 03] - in the generalized threshold model, if all activation functions are monotone and submodular, then the **influence is also submodular**

- **corollary** [M-Roch' 06] - IMP admits a $(1 - e^{-1} - \varepsilon)$-approximation algorithm (for all $\varepsilon > 0$)
  - this follows from a general result on the approximation of submodular functions [Nemhauser-Wolsey-Fisher' 78]

- known special cases [KKT' 03, ' 05]:
  - linear threshold model, independent cascade model
  - decreasing cascade model, "normalized" submodular threshold model

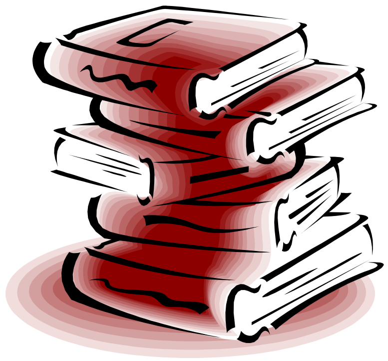
$$\forall S \subseteq T,\ p_u(v,S) \geq p_u(v,T) \text{ or equiv. } \frac{f_u(S \cup \{v\}) - f_u(S)}{1 - f_u(S)} \geq \frac{f_u(T \cup \{v\}) - f_u(T)}{1 - f_u(T)}$$

related work

- sociology
  - threshold models: [Granovetter'78], [Morris'00]
  - cascades: [Watts'02]

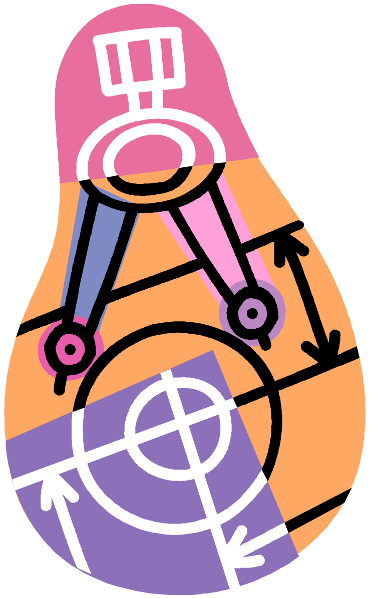
- data mining
  - viral marketing: [KKT'03,'05], [Domingos-Richardson'02]
  - recommendation networks: [Leskovec-Singh-Kleinberg'05], [Leskovec-Adamic-Huberman'06]

- economics
  - game-theoretic point of view: [Ellison'93], [Young'02]



- probability theory
  - Markov random fields, Glauber dynamics
  - percolation
  - interacting particle systems: voter model, contact process

proof sketch



coupling

- we use the generalized threshold model
- arbitrary sets A, B; consider 4 processes:
  - (A_t) started at A
  - (B_t) started at B
  - (C_t) started at $A \cap B$
  - (D_t) started at $A \cup B$
- it suffices to **couple the 4 processes** in such a way that for all t

$$C_t \subseteq A_t \cap B_t$$
$$D_t \subseteq A_t \cup B_t$$

- indeed, at termination

$$|A_{n-1}| + |B_{n-1}| \geq |A_{n-1} \cap B_{n-1}| + |A_{n-1} \cup B_{n-1}| \geq |C_{n-1}| + |D_{n-1}|$$

(note this works with $|.|$ replaced with any w monotone, submodular)

proof ideas

- our goal:

$$C_t \subseteq A_t \cap B_t \quad (1) \qquad D_t \subseteq A_t \cup B_t \quad (2)$$

- antisense coupling
  - obvious way to couple: use same θ_u's for all 4 processes
  - satisfies (1) but not (2)
  - "antisense": using θ_u for (A_t) and $(1-\theta_u)$ for (B_t) "maximizes union"
  - we combine both couplings
- piecemeal growth
  - seed sets can be introduced in stages
  - we add $A \cap B$ then $A \setminus B$ and finally $B \setminus A$
- need-to-know
  - not necessary to pick all θ_u's at beginning
  - can unveil only what we need to know:

$$\theta_v \in \left[f_v(S_{t-2}), f_v(S_{t-1}) \right]?$$

piecemeal growth

- process started at S: (S_t)
- **partition** of S: $S^{(1)},\ldots,S^{(K)}$
- consider the process (T_t):
  - pick θ_u's
  - run the process with seed $S^{(1)}$ until termination
  - add $S^{(2)}$ and continue until termination
  - add $S^{(3)}$ and so on

- **lemma** - the sets S_{n-1} and T_{Kn-1} have the same distribution

antisense coupling

- disjoint sets: S, T
- partition of S: $S^{(1)},\ldots,S^{(K)}$
- piecemeal process with seeds $S^{(1)},\ldots,S^{(K)},T$: (S_t)

- consider the process (T_t):
  - pick θ_u's
  - run piecemeal process with seeds $S^{(1)},\ldots,S^{(K)}$ until termination
  - add T and continue with **threshold values**

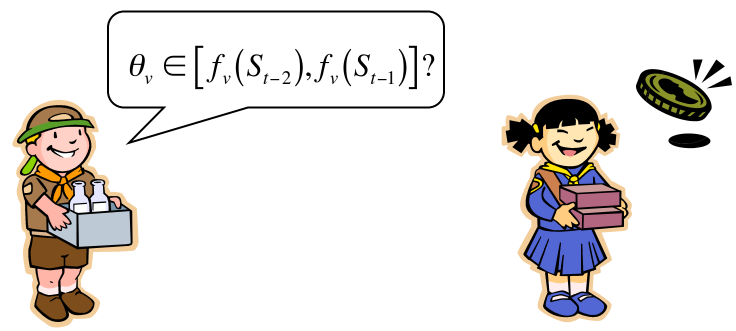
$$\theta_v' = 1 - \theta_v + f_v\left(T_{Kn-1}\right)$$

- **lemma** - the sets $S_{(K+1)n-1}$ and $T_{(K+1)n-1}$ have the same distribution

need-to-know

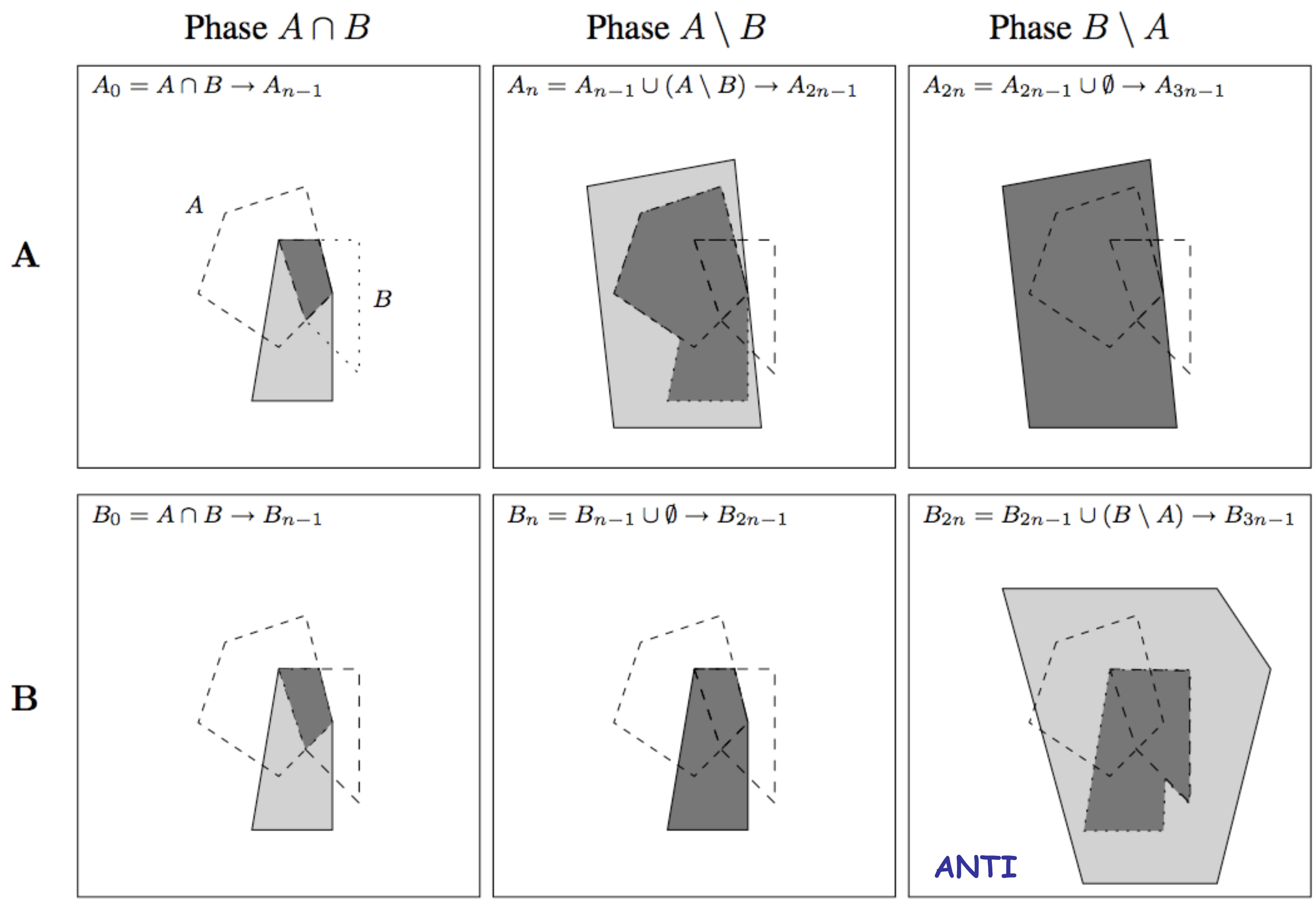
- proof of lemma
  - run the first K stages identically in both processes
  - note that for all v **not in** $S_{Kn-1} = T_{Kn-1}$, θ_v is uniformly distributed in $[f_v(T_{Kn-1}),1]$
  - but $\theta_v' = 1 - \theta_v + f_v(T_{Kn-1})$ has the same distribution

simulation 1



simulation 2

proof I



Phase $A \cap B$
Phase $A \setminus B$
Phase $B \setminus A$
A
B
$A_0 = A \cap B \rightarrow A_{n-1}$
$A_n = A_{n-1} \cup (A \setminus B) \rightarrow A_{2n-1}$
$A_{2n} = A_{2n-1} \cup \emptyset \rightarrow A_{3n-1}$
$B_0 = A \cap B \rightarrow B_{n-1}$
$B_n = B_{n-1} \cup \emptyset \rightarrow B_{2n-1}$
$B_{2n} = B_{2n-1} \cup (B \setminus A) \rightarrow B_{3n-1}$
A
B
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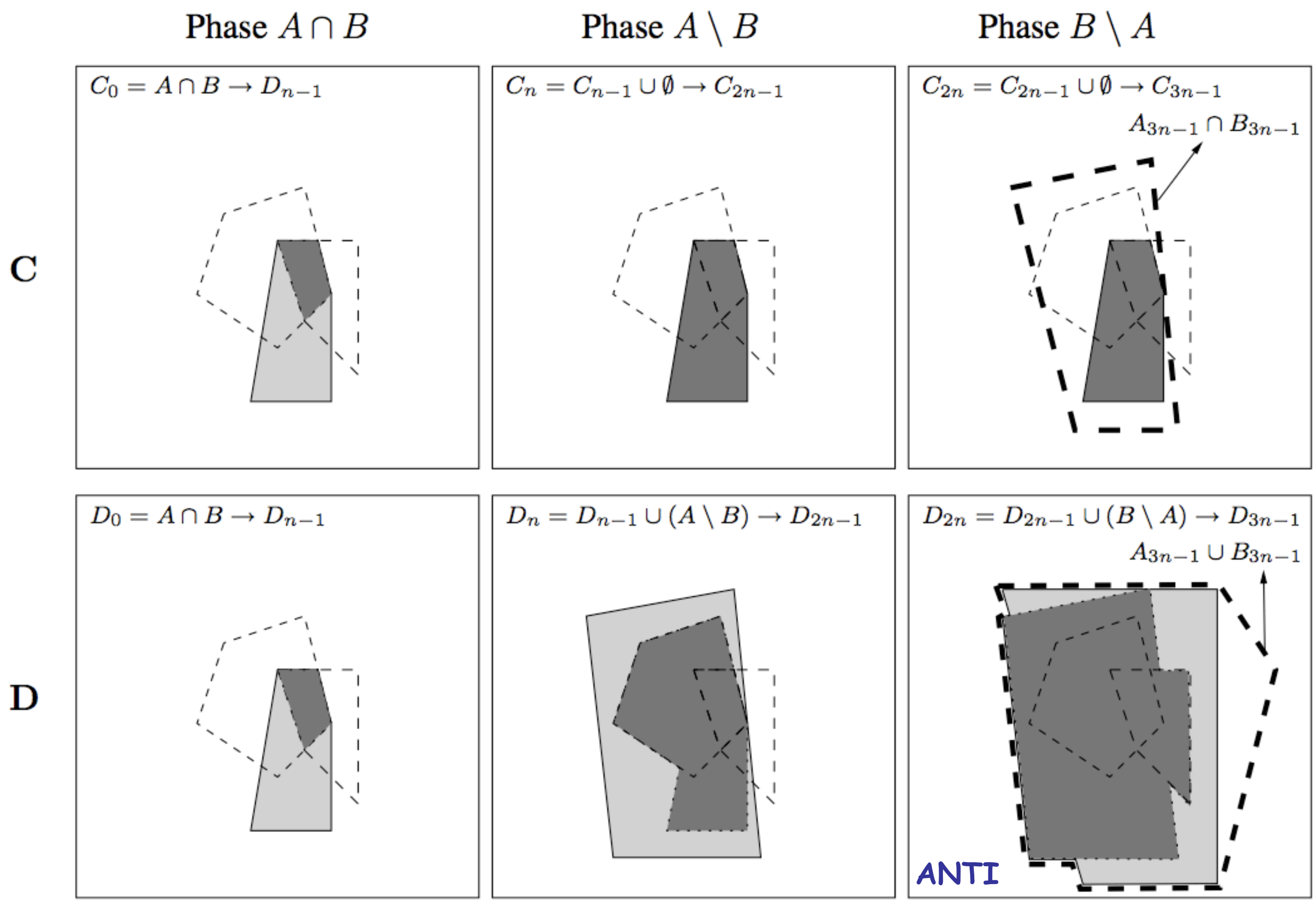
proof II


Phase $A \cap B$
Phase $A \setminus B$
Phase $B \setminus A$
C
D
$C_0 = A \cap B \to D_{n-1}$
$C_n = C_{n-1} \cup \emptyset \to C_{2n-1}$
$C_{2n} = C_{2n-1} \cup \emptyset \to C_{3n-1}$
$A_{3n-1} \cap B_{3n-1}$
$D_0 = A \cap B \to D_{n-1}$
$D_n = D_{n-1} \cup (A \setminus B) \to D_{2n-1}$
$D_{2n} = D_{2n-1} \cup (B \setminus A) \to D_{3n-1}$
$A_{3n-1} \cup B_{3n-1}$
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proof III

- new processes have correct final distribution
- up to time $2n-1$, $B_t = C_t$ and $A_t = D_t$ so that

$$C_t \subseteq A_t \cap B_t \qquad D_t \subseteq A_t \cup B_t$$

- for time $2n$, note that

$$B_{2n-1} \subseteq D_{2n-1}$$
$$B_{2n} = B_{2n-1} \cup (T \setminus S) \qquad D_{2n} = D_{2n-1} \cup (T \setminus S)$$

- so by **monotonicity** and **submodularity**

$$f_v(B_{2n}) - f_v(B_{2n-1}) \geq f_v(D_{2n}) - f_v(D_{2n-1})$$

- then proceed by induction

general result

- we have proved:

  theorem [Mossel-R' 06] - in the generalized threshold model, if all activation functions are submodular, then for any monotone, submodular function w, the generalized influence

  $$\sigma_w(S) = \mathrm{E}_S[w(S_{n-1})]$$

  is submodular
- Note: A closure property for sub-modular functions!

Competing first passage percolation on random regular graphs

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Based on a joint work with Tonći Antunović Yael Dekel, Elchanan Mossel and Yuval Peres

First passage percolation

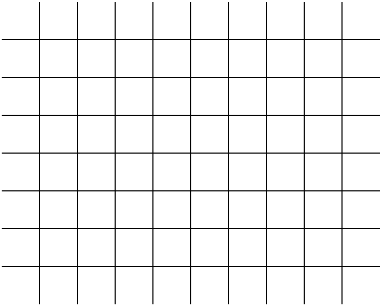
First passage percolation:
Fix a graph $G = (V, E)$, consider iid edge lengths $(\ell_e)_{e \in E}$.
Define the random metric on V

$$d(x, y) = \inf_{\Gamma} \ell(\Gamma),$$

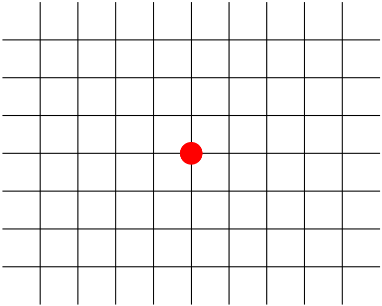
where the infimum is taken over all paths Γ connecting x and y and $\ell(\Gamma)$ is the sum of lengths of the edges on Γ.

An important case: $\ell_e \sim \exp(\lambda)$.
Process $r \mapsto B(0, r)$ evolves as a Markov process, new vertices are added at the rate $\lambda \times$ the number of neighbors in $B(0, r)$.

First passage percolation



First passage percolation



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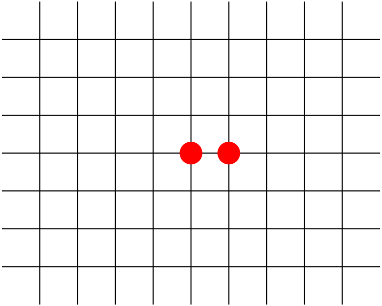

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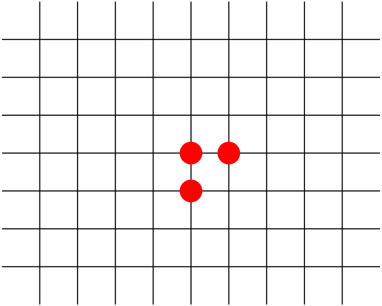

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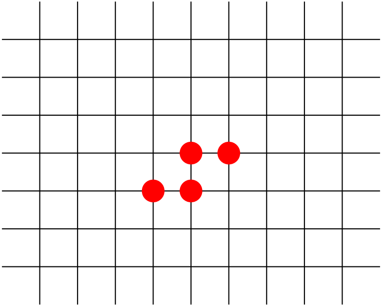

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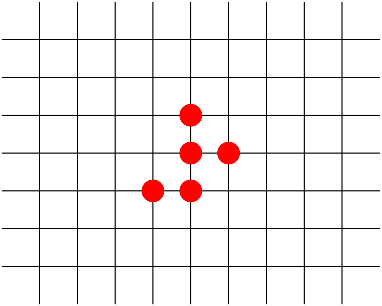

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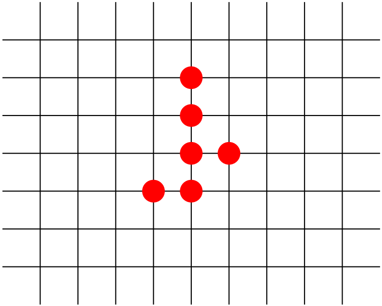

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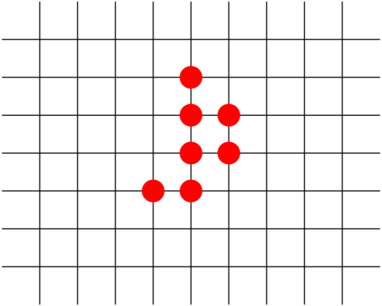

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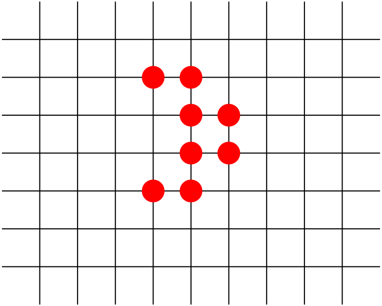

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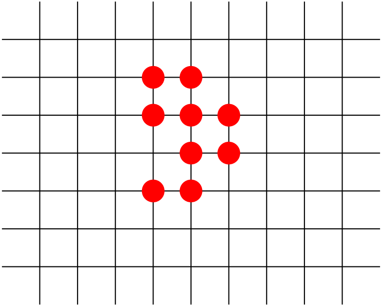

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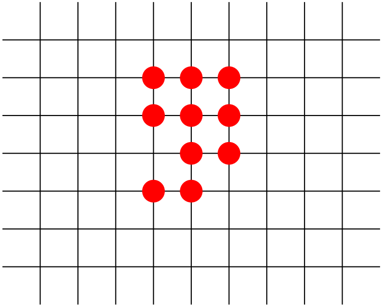

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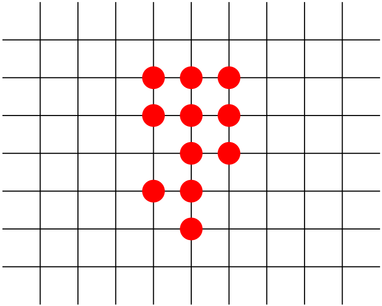

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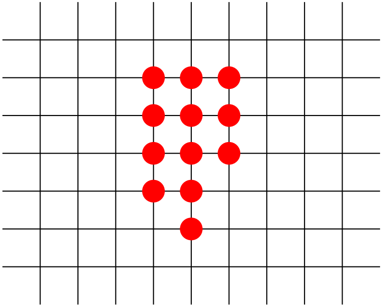

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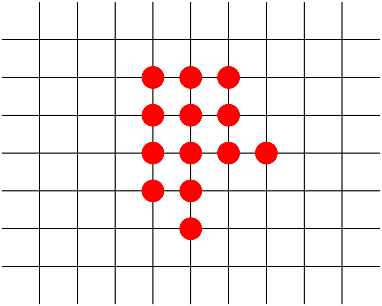

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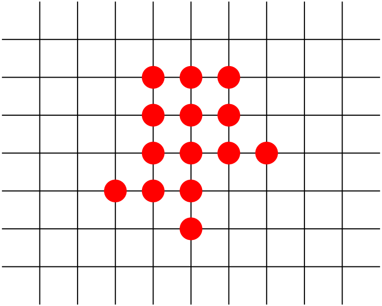

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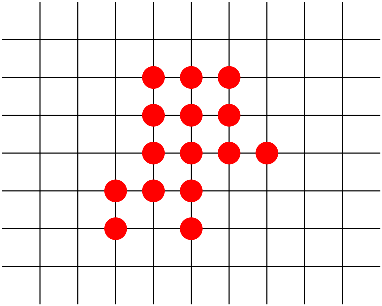

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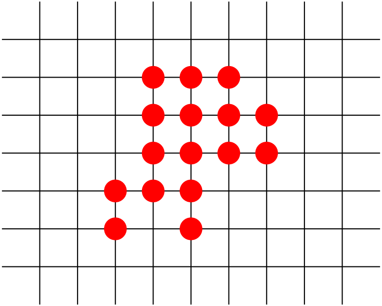

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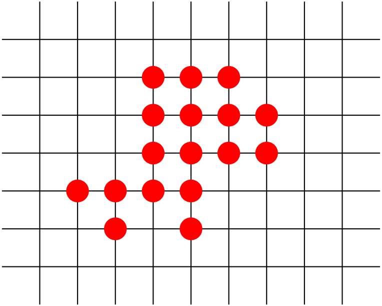

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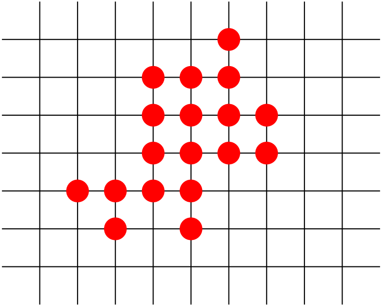

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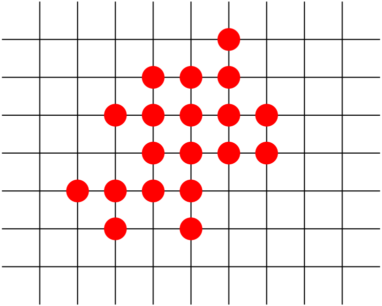

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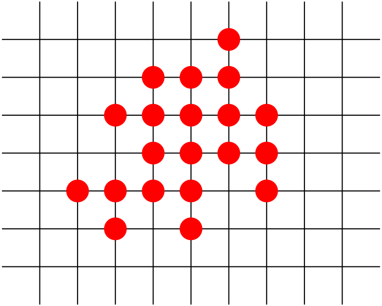

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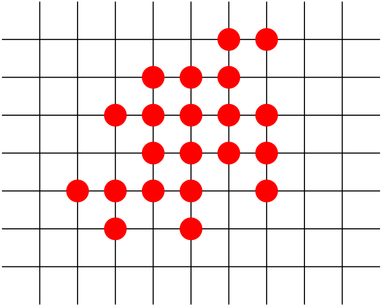

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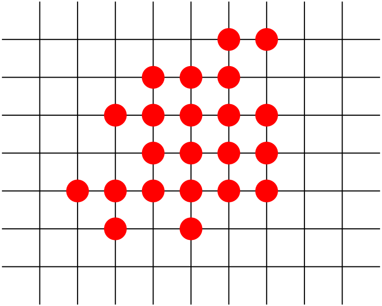

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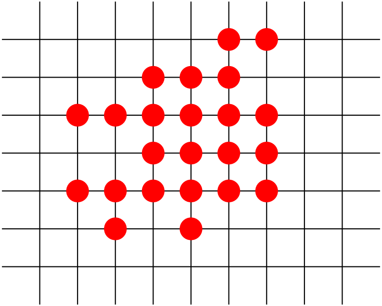

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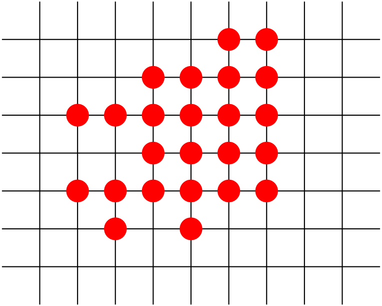

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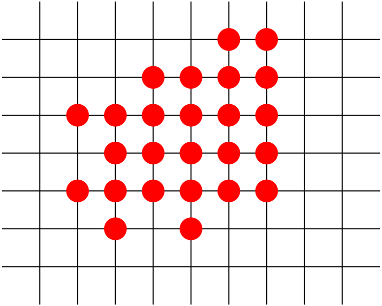

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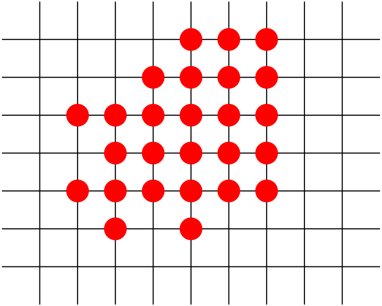

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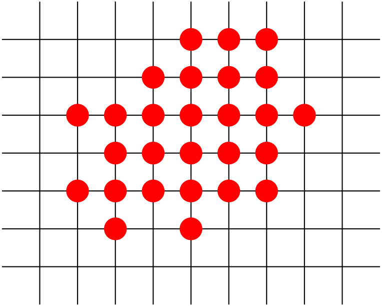
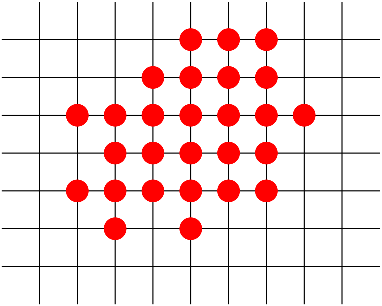


First passage percolation



Theorem (Cox-Durrett shape theorem)

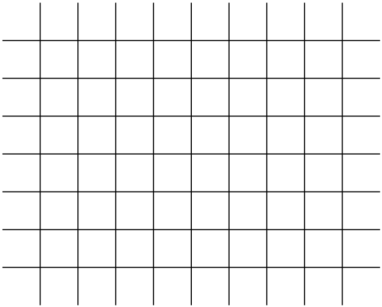
There exists a compact convex set A such that for any $\delta > 0$

$$\lim_{r \to \infty} \mathbb{P}\big((1-\delta) rA \subset B(0,r) \subset (1+\delta) rA\big) = 1.$$

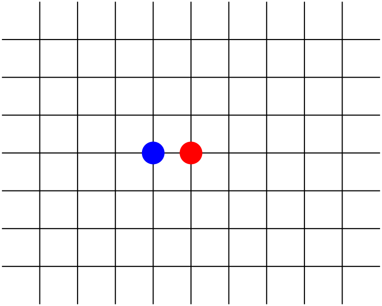
Competing first passage percolation (also called Two-type Richardson Model by Häggström, Pemantle):

- Start with one red vertex and one blue vertex, other uncolored.
- Uncolored vertices become red at the rate ($\lambda_R \times$ the number of red neighbors) and blue at the rate ($\lambda_B \times$ the number of blue neighbors).
- Once colored, vertices never change the color.

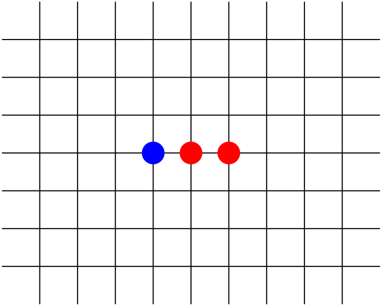
Competing first passage percolation



Competing first passage percolation



Competing first passage percolation



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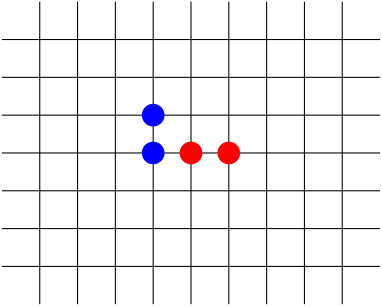

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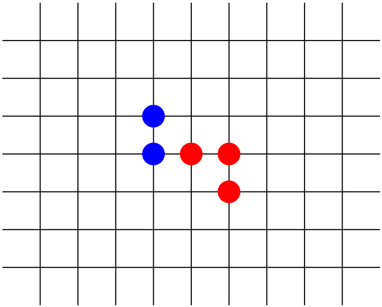

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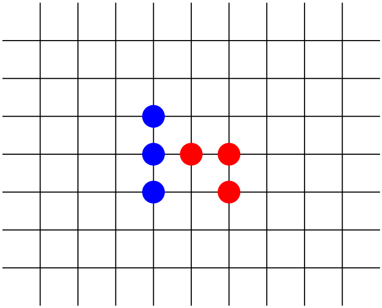

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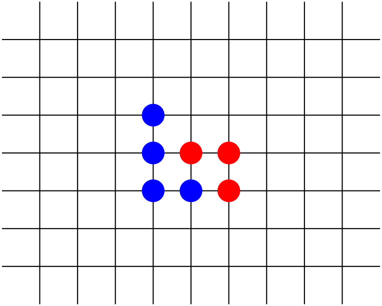

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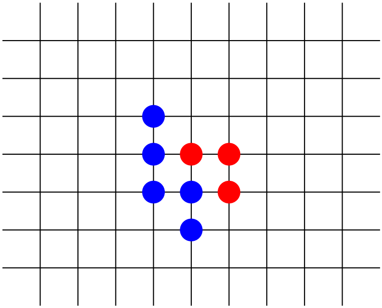

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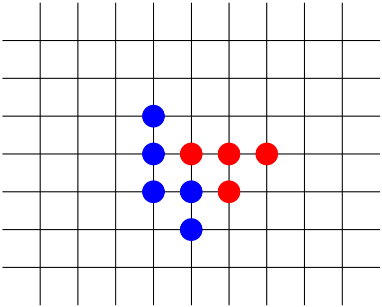

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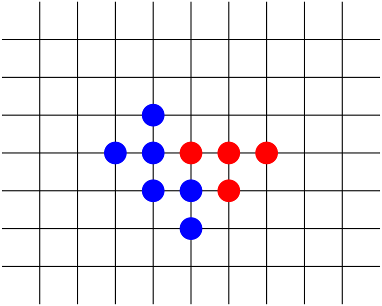

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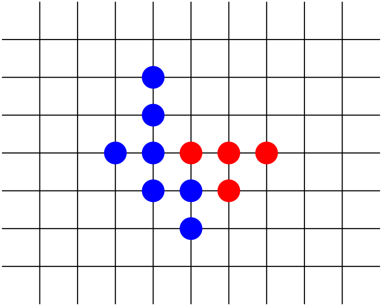

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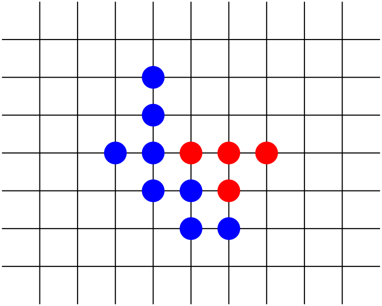

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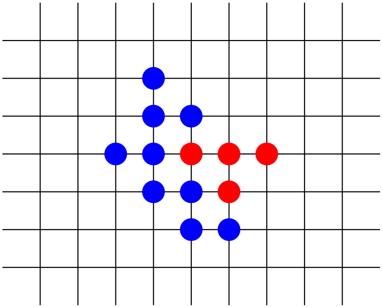

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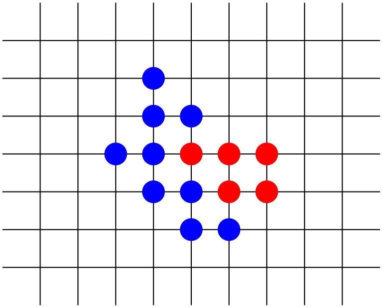

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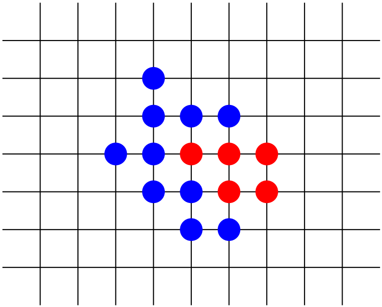

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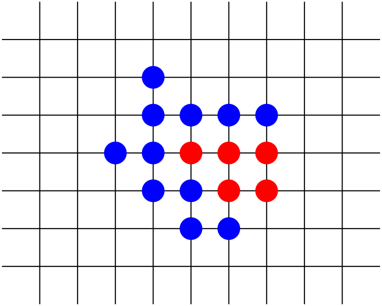

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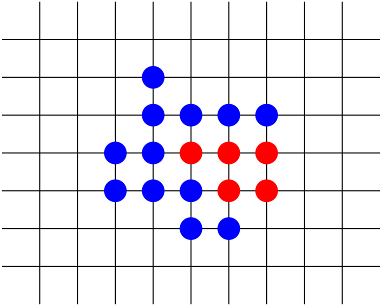

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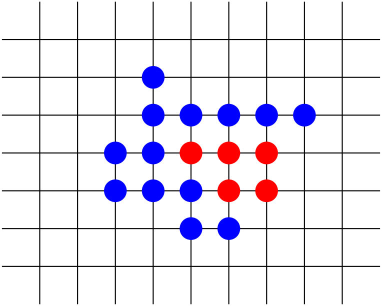

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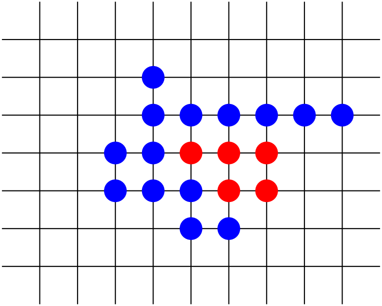

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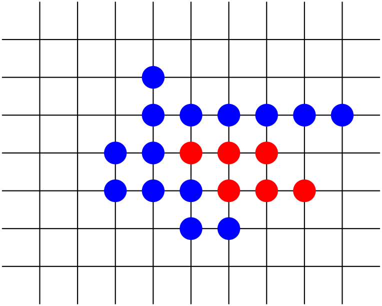

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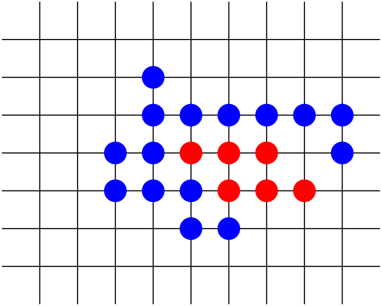

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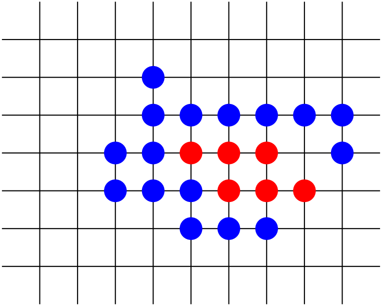

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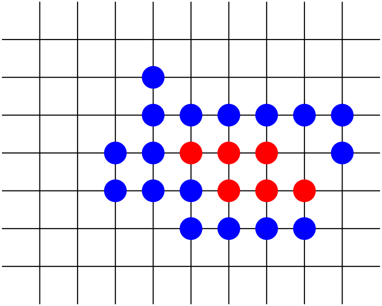

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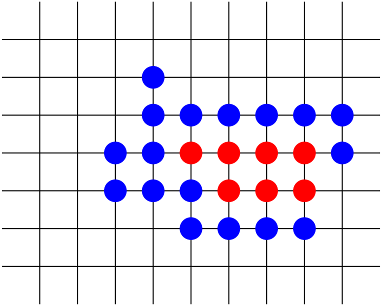

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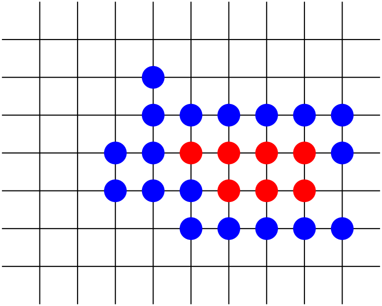

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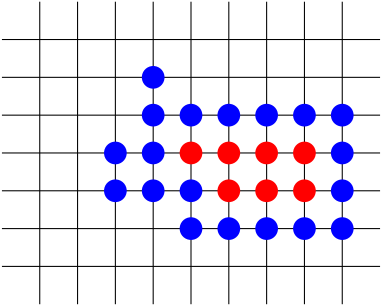


Theorem (Häggström, Pemantle)

On 2D lattice, for $\lambda_R = \lambda_B$

$$\mathbb{P}(\textit{both red and blue} \to \infty) > 0;$$

for at most countable set S

$$\frac{\lambda_R}{\lambda_B} \notin S \Rightarrow \mathbb{P}(\textit{both red and blue} \to \infty) = 0.$$

Random regular graphs

- Have only bounded number of short cycles.
- Neighborhoods or typical vertices are trees.
- Expander properties.
- **Configuration model**

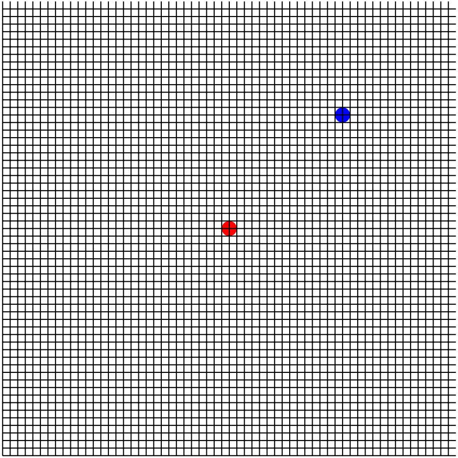
Competing process on random graphs

Let G_n be random d-regular graph on n vertices.

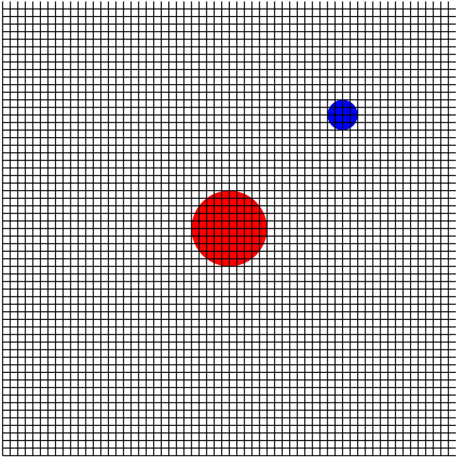
- Uniformly choose $\mathbf{r}(n)$ vertices of G_n and color it red and $\mathbf{b}(n)$ vertices and color it blue ($\mathbf{r}(n)$ and $\mathbf{b}(n)$ are given functions).
- Run the same dynamics as in the competing first passage percolation model with rates λ_R and λ_B.
- Consider the number of red and blue vertices R_n^{final} and B_n^{final} when the graphs is exhausted.

Question: Can we estimate R_n^{final} and B_n^{final}?

Compare with the Torus



Compare with the Torus



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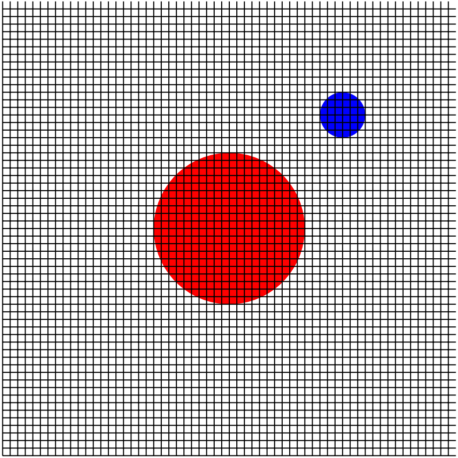

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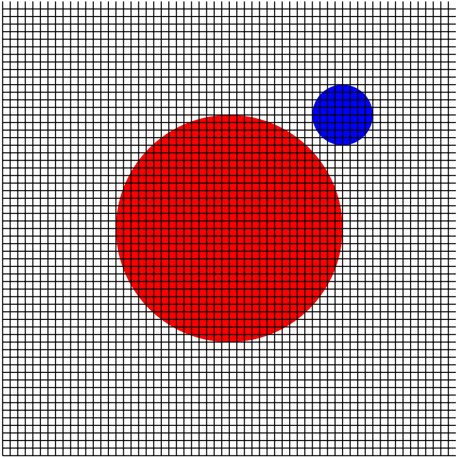

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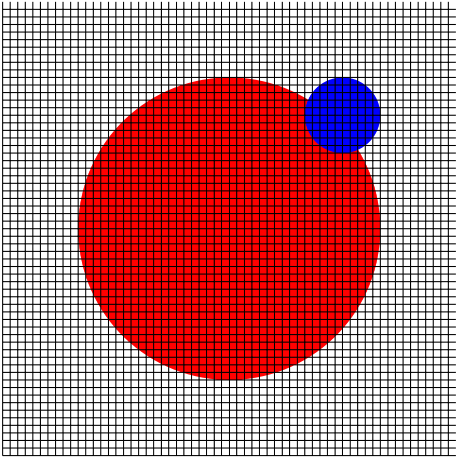

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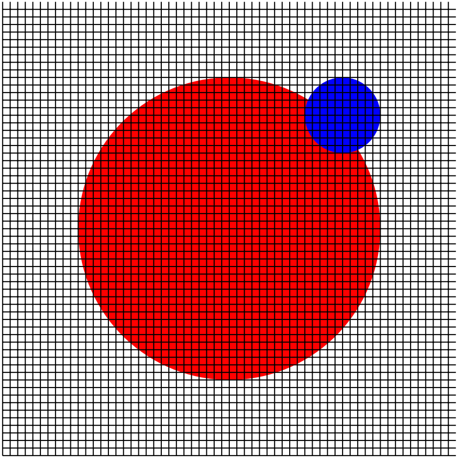


Both processes occupy $\Theta(n) = \Theta(k^2)$ vertices.

Results - asymptotics

Theorem (Antunović, Dekel, M, Peres)

Up to a constant factor with high probability

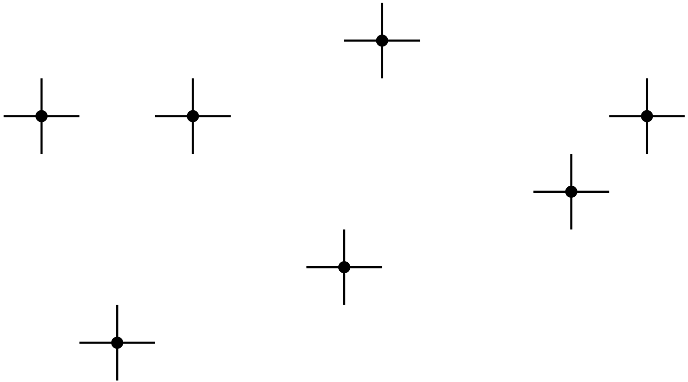
$$R_n^{total} \sim \mathbf{r}(n)\left(\frac{n}{\mathbf{b}(n)}\right)^{\lambda_R/\lambda_B} \wedge n.$$

In particular if $\mathbf{r}(n) = n^{\rho}$ *and* $\mathbf{b}(n) = n^{\beta}$ *then*

$$R_n^{total} \sim \begin{cases} n^{\rho+(1-\beta)\lambda_R/\lambda_B}, & \text{for } \rho < 1-(1-\beta)\lambda_R/\lambda_B, \\ n, & \text{for } \rho \geq 1-(1-\beta)\lambda_R/\lambda_B. \end{cases}$$

"Balance" occurs at $(1-\rho)\lambda_B = (1-\beta)\lambda_R$.

Configuration model



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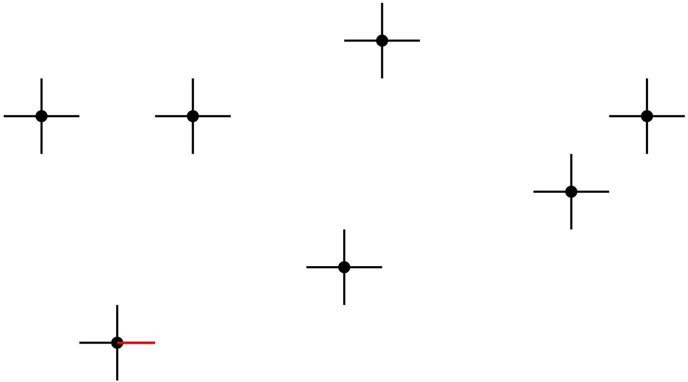

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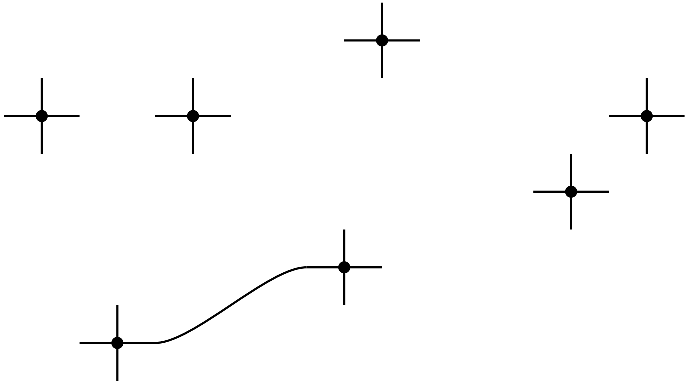

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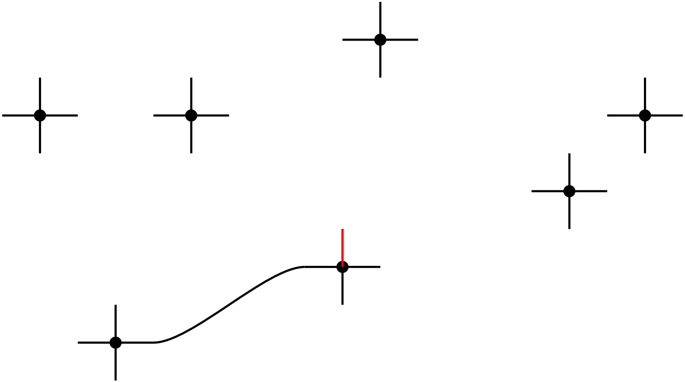

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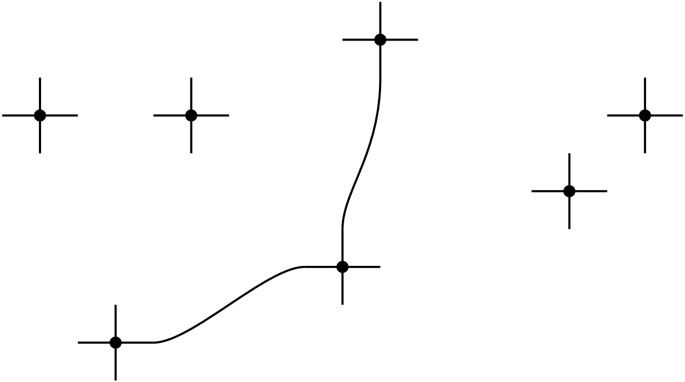

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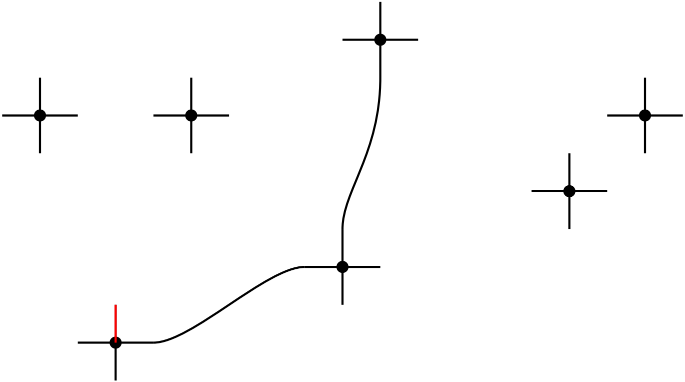

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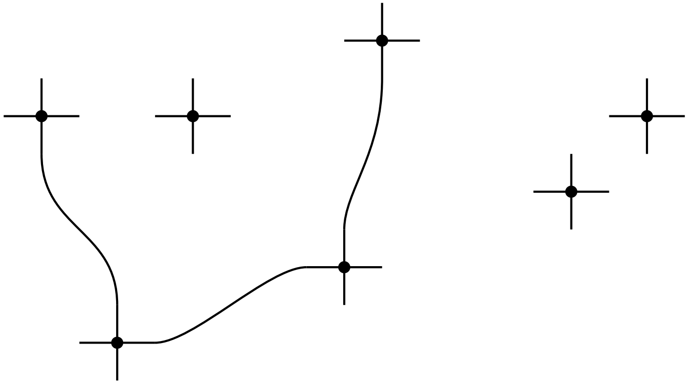

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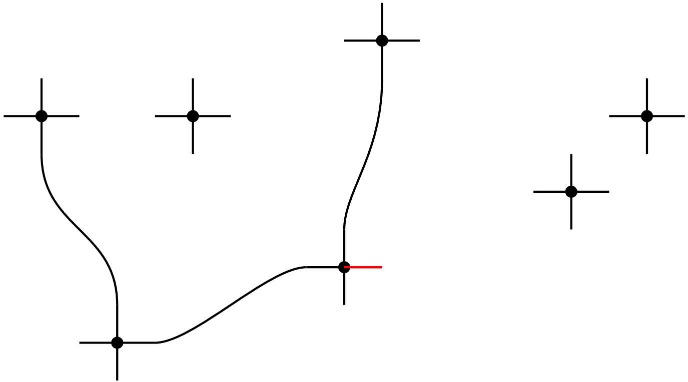

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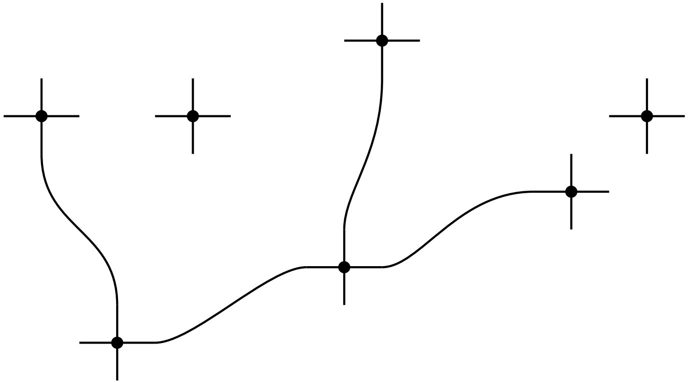

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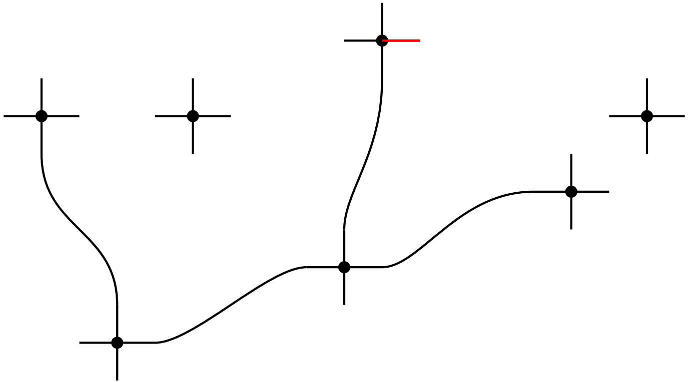

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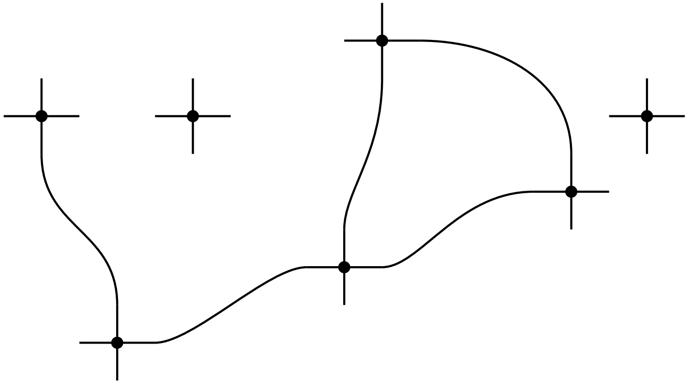

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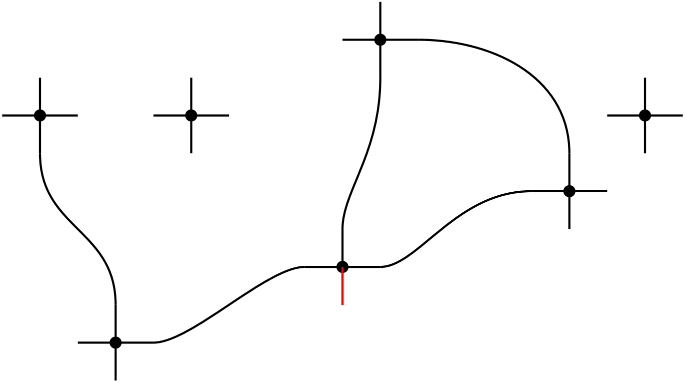

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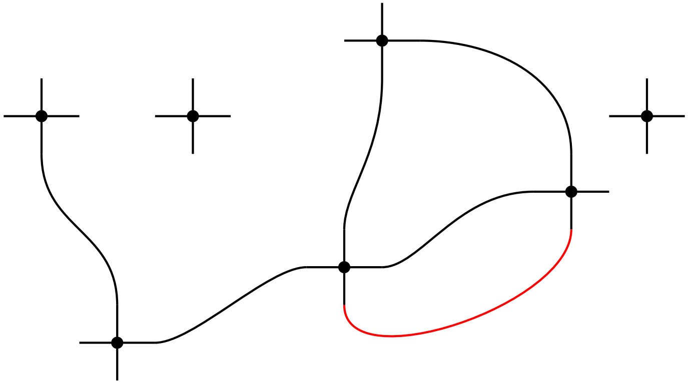

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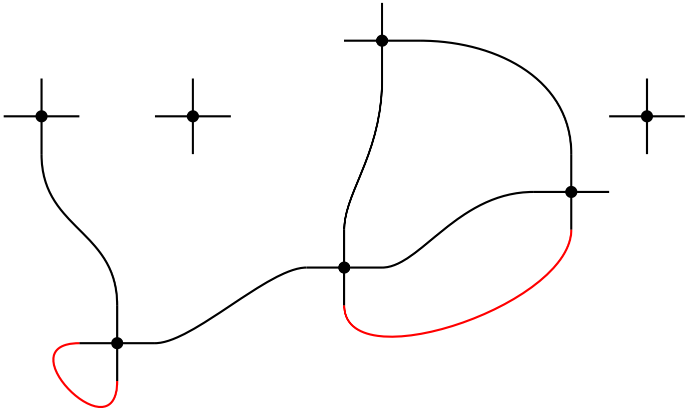

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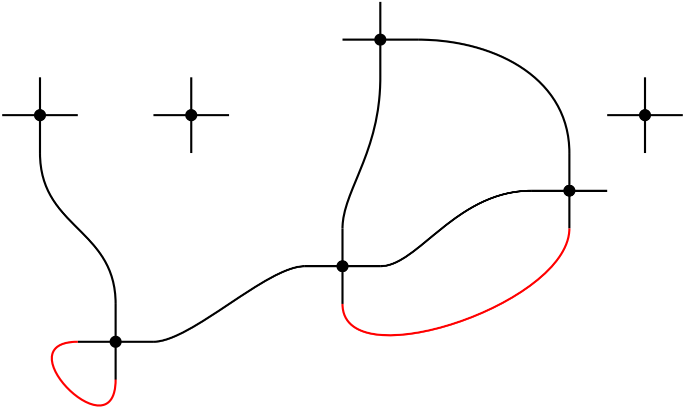
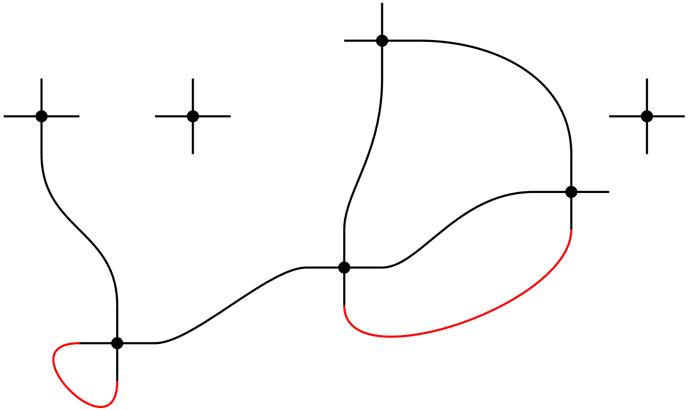


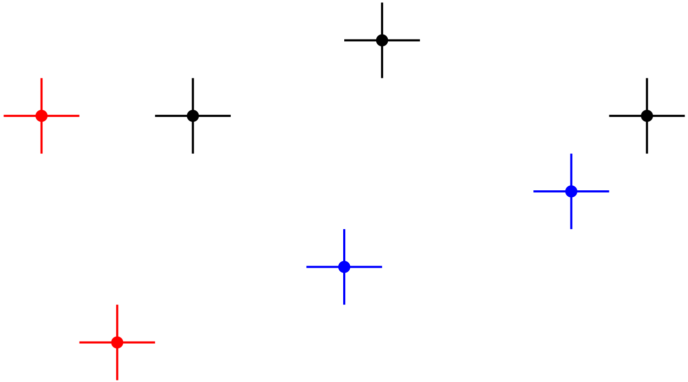
$\mathbb{P}$(Bad configuration) is bounded away from 1.

Configuration model



$\mathbb{P}$(Bad configuration) is bounded away from 1.
Conditioned that there are no Bad configuration the algorithm generates a random regular graph.

Couple CFPP and the configuration model



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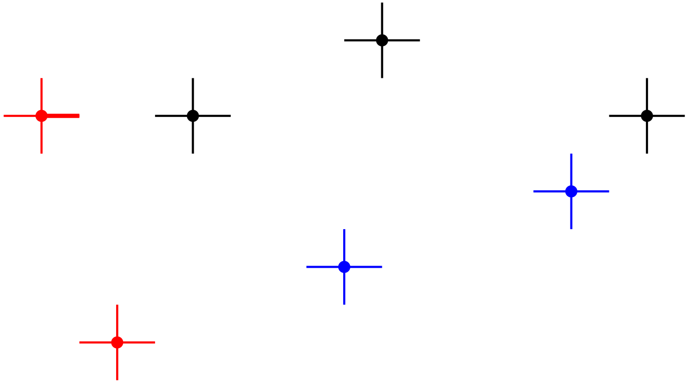

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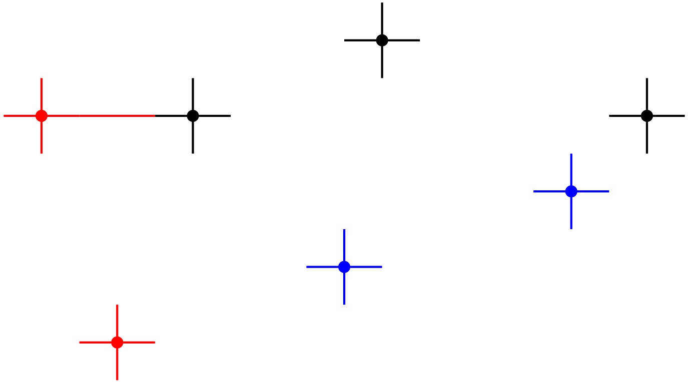

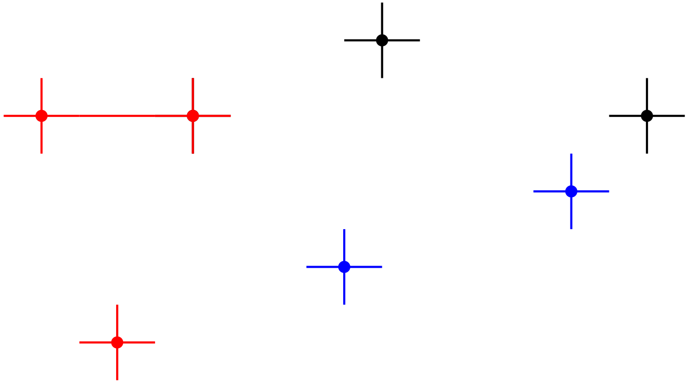

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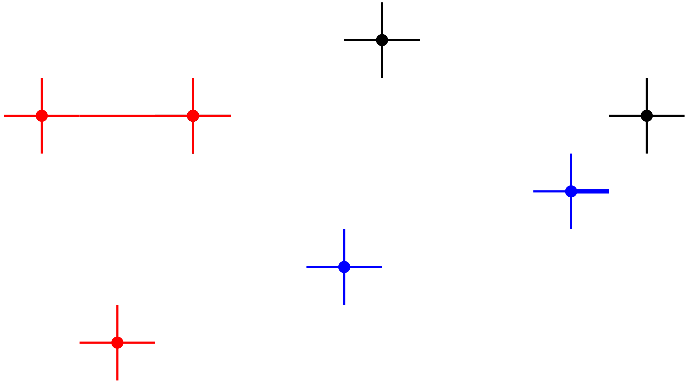

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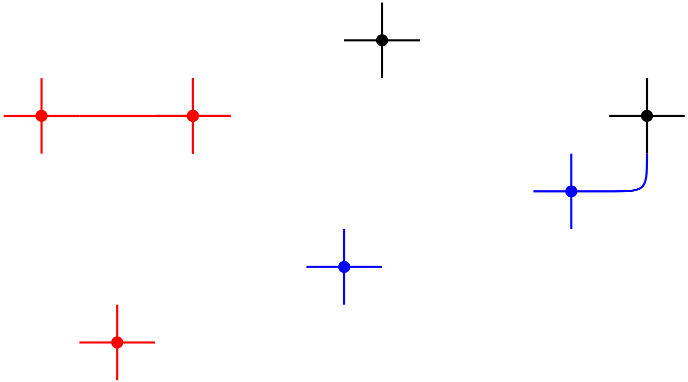

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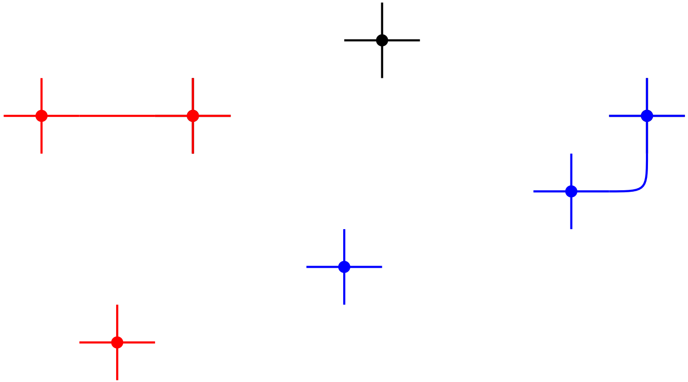

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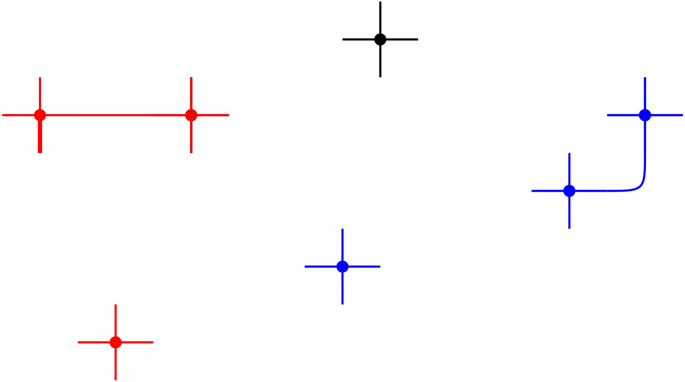

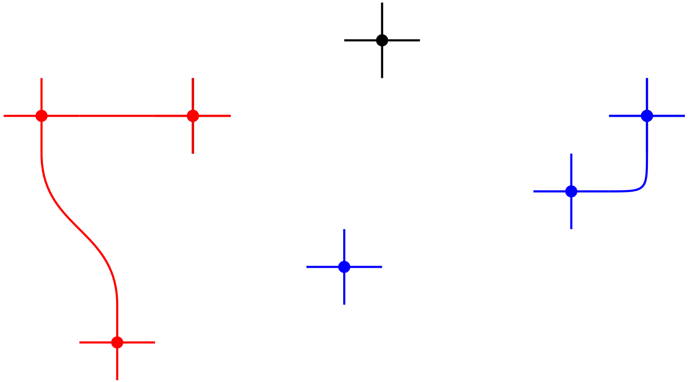

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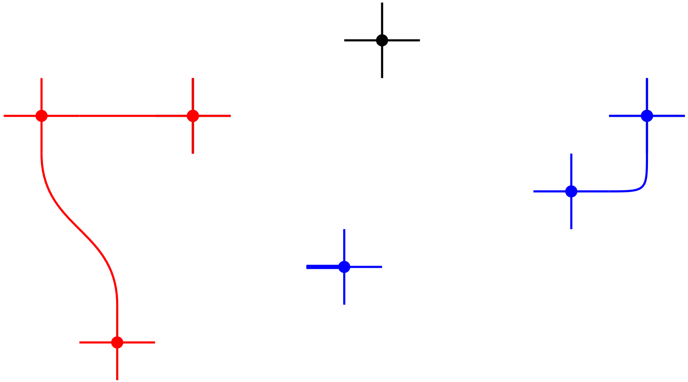

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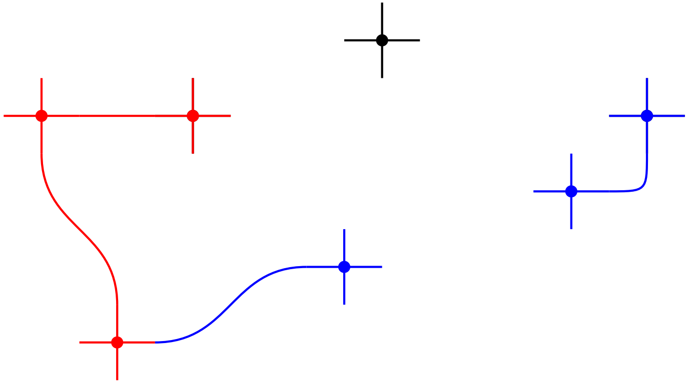

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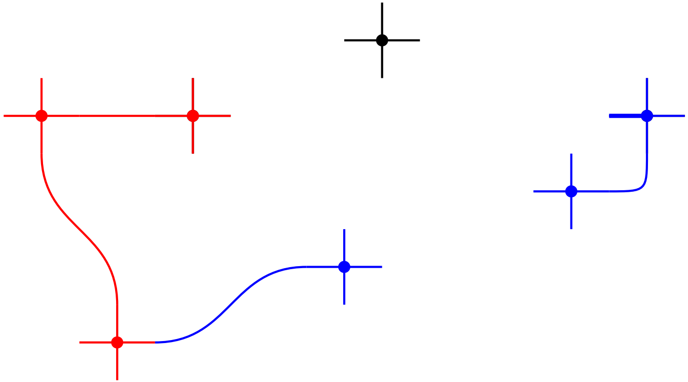

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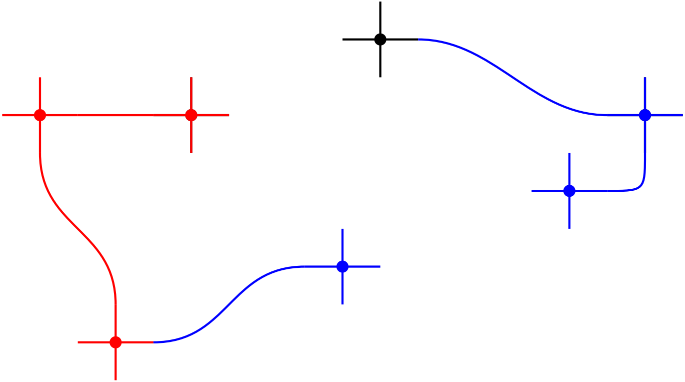

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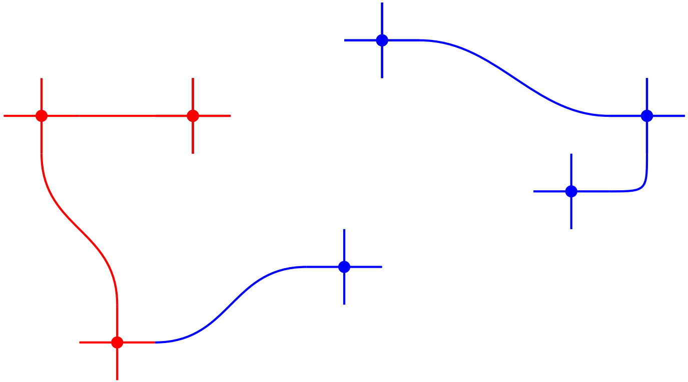

Couple CFPP and the configuration model

We will keep track of:
$\mathcal{R}_t$ and $\mathcal{B}_t$ ... number of red and blue "half-edges" at step t.

Couple CFPP and the configuration model

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$\mathcal{R}_t$ and $\mathcal{B}_t$ ... number of red and blue "half-edges" at step t.
Transition probabilities:

$$(\mathcal{R}_{t+1},\mathcal{B}_{t+1}) = \begin{cases} (\mathcal{R}_t + d - 2, \mathcal{B}_t), & w.p.\ \frac{\lambda_R \mathcal{R}_t}{\lambda_R\mathcal{R}_t+\lambda_B\mathcal{B}_t}\frac{dn-2t-\mathcal{R}_t-\mathcal{B}_t}{dn-2t-1} \\ (\mathcal{R}_t, \mathcal{B}_t + d - 2), & w.p.\ \frac{\lambda_B \mathcal{B}_t}{\lambda_R\mathcal{R}_t+\lambda_B\mathcal{B}_t}\frac{dn-2t-\mathcal{R}_t-\mathcal{B}_t}{dn-2t-1} \\ (\mathcal{R}_t - 2, \mathcal{B}_t), & w.p.\ \frac{\lambda_R \mathcal{R}_t}{\lambda_R\mathcal{R}_t+\lambda_B\mathcal{B}_t}\frac{\mathcal{R}_t-1}{dn-2t-1} \\ (\mathcal{R}_t, \mathcal{B}_t - 2), & w.p.\ \frac{\lambda_B \mathcal{B}_t}{\lambda_R\mathcal{R}_t+\lambda_B\mathcal{B}_t}\frac{\mathcal{B}_t-1}{dn-2t-1} \\ (\mathcal{R}_t - 1, \mathcal{B}_t - 1), & w.p.\ \frac{(\lambda_R+\lambda_B)\mathcal{B}_t\mathcal{R}_t}{(\lambda_R\mathcal{R}_t+\lambda_B\mathcal{B}_t)(dn-2t-1)} \end{cases}$$

Couple CFPP and the configuration model

To control $\mathcal{R}_t$ and $\mathcal{B}_t$ we use martingale techniques to control

$$X_t = \mathcal{R}_t + \mathcal{B}_t$$

and

$$Y_t = \frac{\mathcal{R}_t}{\mathcal{B}_t^{\lambda_R/\lambda_B}}$$

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X_t ... number of active half-edges in the configuration model
Y_t ... is the continuous time martingale when both processes evolve without any interactions (and self-interactions)

Couple CFPP and the configuration model

$$X_t = \mathcal{R}_t + \mathcal{B}_t$$

Process $(X_t, dn - 2t - X_t)$ evolves as an urn model

$$\begin{pmatrix} -2 & 0 \\ d-2 & -d \end{pmatrix}$$

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for all $0 \leq t < dn/2$.

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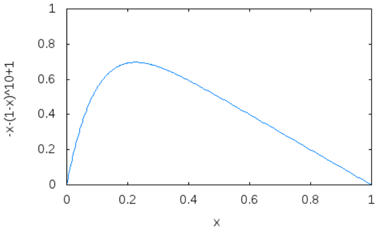

$$Y_t = \frac{\mathcal{R}_t}{\mathcal{B}_t^{\lambda_R/\lambda_B}}$$

As long as X_t is large

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Observe the "extra advantage" of the faster process.

THANKS!!!

Thank you!

- Speakers: …
- Organizers:
- Laurent, Lionel, Tasia
- Heather Peterson
- You!