Definition 0.1 *Let X be a smooth projective variety of dimension n and let $D \in N_1(X)$.*

We say that the class D is represented by a free curve if there is a non-constant morphism $f : \mathbb{P}^1 \to X$ such that $f_*[\mathbb{P}^1] = D$ and $f^*\mathcal{T}_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $a_1, \ldots, a_n \geq 0$.

We say that the class D is represented by a very free curve if there is a non-constant morphism $f : \mathbb{P}^1 \to X$ such that $f_*[\mathbb{P}^1] = D$ and $f^*\mathcal{T}_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $a_1, \ldots, a_n \geq 1$.

Let X be a del Pezzo surface. We want to determine which divisor classes $D \in \text{Pic}(X)$ are represented by (very) free curves. Note that if D is represented by a free curve, then D is a nef divisor. We start with the three cases $X \simeq \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $Bl_p(\mathbb{P}^2)$, or equivalently $\deg X \geq 8$.

- If $X \simeq \mathbb{P}^2$, denote by ℓ the divisor class of a line. Clearly $a\ell$ is represented by a (very) free curve if and only if $a \geq 1$.
- If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, denote by F_1 the divisor class of $\mathbb{P}^1 \times \{0\}$, and denote by F_2 the divisor class of $\{0\} \times \mathbb{P}^1$. A divisor class $a_1F_1 + a_2F_2$ is represented by a free curve if and only if $a_1, a_2 \geq 0$ and $(a_1, a_2) \neq (0, 0)$.
- If $X \simeq Bl_p(\mathbb{P}^2)$, denote by ℓ the divisor class of a the pull-back of a line from $\mathbb{P}^2$ and let e denote the divisor class of the exceptional divisor. It is immediate to check that a divisor $D = a\ell + be$ is nef if and only if $a \geq b \geq 0$. Let D be a nef divisor; we can therefore write $D = \alpha\ell + \beta(\ell - e)$, $\alpha, \beta \geq 0$. Obviously, $\alpha\ell$ is represented by a (very) free curve if $\alpha > 0$. Also, $\beta(\ell - e)$ is represented by a free divisor if $\beta > 0$ (the morphism $f : \mathbb{P}^1 \to X$ must in this case have degree β to its image). It now follows easily from Theorem II.7.6 of [Ko] that $D = \alpha\ell + \beta(\ell - e)$ is represented by a free curve if and only if $\alpha, \beta \geq 0$ and $(\alpha, \beta) \neq (0, 0)$ and that D is represented by a very free curve if and only if $\alpha > 0$ and $\beta \geq 0$.

Proposition 0.2 *Let X be a del Pezzo surface of degree $d \leq 7$. A divisor class $D \in \text{Pic}(X)$ is nef if and only if $D \cdot L \geq 0$ for all $(-1)-$curves $L \subset X$.*

Proof. The necessity of the conditions is obvious. To establish the sufficiency, we proceed by induction on $\delta := 9 - d$.

If $\delta = 2$ write $D = a\ell - b_1e_1 - b_2e_2$, in some standard basis $\{\ell, e_1, e_2\}$. By assumption we know that $b_i \geq 0$ and $a \geq b_1 + b_2$. Thus we can write

$$D = \big(a - b_1 - b_2\big)\ell + b_1\big(\ell - e_1\big) + b_2\big(\ell - e_2\big)$$

which shows that D is a non-negative combination of nef classes.

Suppose $\delta > 2$. Let $n := \min\{D \cdot L \ ; \ L \subset X$ is a $(-1)-$curve$\}$; by assumption we know that $n \geq 0$. Let $\tilde{D} := D + nK_X$; for any $(-1)-$curve $L \subset X$ we have $\tilde{D} \cdot L = D \cdot L - n \geq 0$, and there is a $(-1)-$curve L' such that $\tilde{D} \cdot L' = 0$, by the definition of n.

Let $b : X \to X'$ be the contraction of the curve L' and note that X' is a del Pezzo surface of degree $9 - (\delta - 1)$. We have $\tilde{D} = b^* b_* \tilde{D} - rL'$ and

$$0 = \tilde{D} \cdot L' = b^* b_* \tilde{D} \cdot L' - rL' \cdot L' = b_* \tilde{D} \cdot b_* L' + r = r$$

and therefore $\tilde{D} = b^* b_* \tilde{C}$ is the pull-back of the divisor class $D' := b_* \tilde{D}$ on X'. Since all $(-1)-$curves on X' are images of $(-1)-$curves on X, by induction we know that D' is nef, and thus $\tilde{D}$ is nef. Hence $D = \tilde{D} + n(-K_X)$ is a non-negative linear combination of nef divisors, and thus D is nef. $\square$

From this proposition we deduce immediately the following corollary.

Corollary 0.3 *Let X_δ be a del Pezzo surface of degree $9 - \delta \leq 8$. Let $D \in \mathrm{Pic}(X_\delta)$ be a nef divisor. Then we can find*

- *non-negative integers $n_2, \dots, n_\delta$;*
- *a sequence of contraction of $(-1)-$curves*

$$X_\delta \longrightarrow X_{\delta-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \ ;$$

- *a nef divisor $D' \in \mathrm{Pic}(X_1)$;*

such that

$$D = n_\delta(-K_{X_\delta}) + n_{\delta-1}(-K_{X_{\delta-1}}) + \ldots + n_2(-K_{X_2}) + D'$$

Proof. We proceed by induction on δ. If $\delta \leq 1$, there is nothing to prove.

Suppose that $\delta \geq 2$ and let $n := \min\{L \cdot D \mid L \subset X \text{ a } (-1)-\text{curve}\}$. By assumption we have $n \geq 0$. Let $\bar{D} := D + nK_{X_\delta}$; for every $(-1)-$curve $L \subset X_\delta$ we have

$$\bar{D} \cdot L = D \cdot L + nK_{X_\delta} \cdot L \geq n - n = 0$$

Thus thanks to the previous Proposition, $\bar{D}$ is nef. By construction there is a $(-1)-$curve $L_0 \subset X$ such that $\bar{D} \cdot L_0 = 0$. Thus $\bar{D}$ is the pull-back of a nef divisor on the del Pezzo surface $X_{\delta-1}$ obtained by contracting L_0. By induction, we have a sequence of contractions

$$X_{\delta-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \ ,$$

non-negative integers $n_2, \ldots, n_{\delta-1}$ and a nef divisor D' on X_1 such that we may write $\bar{D} = n_{\delta-1}(-K_{X_{\delta-1}}) + \ldots + n_2(-K_{X_2}) + D'$. Let $n_\delta := n$; with this notation we have

$$D = n_\delta(-K_{X_\delta}) + \bar{D}' = n_\delta(-K_{X_\delta}) + \ldots + n_2(-K_{X_2}) + D'$$

and a sequence of contractions as in the statement of the corollary. This concludes the proof. □

Theorem 0.4 *Let X be a del Pezzo surface of degree d.*

1. *A divisor class $D \in \mathrm{Pic}(X)$ is represented by a free curve if and only if D is nef and $-K_X \cdot D \geq 2$.*

2. *A divisor class $D \in \mathrm{Pic}(X)$ is represented by a very free curve if and only if D is nef, $-K_X \cdot D \geq 3$ and $D^2 \neq 0$.*

Proof. We start proving 1. Suppose that D is represented by a free curve and let $f : \mathbb{P}^1 \to X$ be a morphism such that $f_*[\mathbb{P}^1] = D$ and $f^*\mathcal{T}_X$ is semi-positive. Since the images of the deformations of f cover X, it follows that D is nef and that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X) \geq 4$. The dimension of every irreducible component of the space of morphisms at $[f]$ is at least $-K_X \cdot D + 2$ and this space is smooth at $[f]$ ([Ko] Theorem II.1.2), thus $-K_X \cdot D \geq 2$.

Conversely, if D is nef and $-K_X \cdot D \geq 2$ using Corollary 0.3 we can write

$$D = n_\delta(-K_{X_\delta}) + n_{\delta-1}(-K_{X_{\delta-1}}) + \ldots + n_2(-K_{X_2}) + D'$$

where $X = X_\delta$ and

$$-K_X \cdot D = (9-\delta)n_\delta + (8-\delta)n_{\delta-1} + \ldots + 7n_2 - K_X \cdot D' \geq 2$$

This inequality only excludes the possibility that $D = -K_{X_8}$.

Note that $-K_{X_i}$ is represented by a free curve if $i < 8$. Together with an easy application of the smoothing result II.7.6 in [Ko] we deduce that D is represented by a free curve, if $n_8 = 0$.

If $D = 2(-K_{X_8})$, let $\nu_i : C_i \to X$, $i \in \{1, 2\}$ be two morphisms such that $C_i \simeq \mathbb{P}^1$, $(\nu_i)_*[C_i] = -K_{X_8}$ and $\nu_1(C_1) \neq \nu_2(C_2)$. Since $(-K_{X_8})^2 = 1$, it follows that the images of C_1 and C_2 meet transversely at a unique point $p \in X$. We construct a curve C by attaching C_1 and C_2 with a node at the point $\nu_i^{-1}(p)$ on C_i. Let $\nu : C \to X$ be the morphism coinciding with ν_i on C_i. The dimension of every irreducible component of $\overline{\mathcal{M}}_{0,0}(X_8, -2K_{X_8})$ is at least 1, and since the rational curves $\nu_i(C_i)$ do not deform, it follows that

ν can be deformed to a morphism with irreducible domain $f : \mathbb{P}^1 \to X_8$. Note that the deformations of f cover X_8. Using Proposition II.3.10 of [Ko] and generic smoothness, it follows that we may assume that f is free, i.e. that $f^*\mathcal{T}_{X_8}$ is semi-positive.

Analogously, if $D = -K_{X_8} + \tilde{D}$, where $\tilde{D}$ is represented by a free curve, then D is also represented by a free curve, using the smoothing result II.7.6 in [Ko]. This concludes the proof of 1.

We now proceed to prove 2. Suppose that D is represented by a very free curve and let $f : \mathbb{P}^1 \to X$ be a morphism such that $f_*[\mathbb{P}^1] = D$ and $f^*\mathcal{T}_X$ is ample. Factoring f as a degree e cover $\mathbb{P}^1 \to \mathbb{P}^1$ followed by a morphism which is birational to its image, it is clear that it is enough to treat the case in which f birational to its image. Let $p \in \mathbb{P}^1$ be a closed point such that $f(p)$ is a smooth point of $f(\mathbb{P}^1)$. Let X' be the blow up of X at $f(p)$ and let $f' : \mathbb{P}^1 \to X'$ be the morphism lifting f. We easily see that $(f')^*\mathcal{T}_{X'} \simeq f^*\mathcal{T}_X(-p)$. Thus $(f')^*\mathcal{T}_{X'}$ is semi-positive and from the previous argument $2 \leq -K_{X'} \cdot f'_*[\mathbb{P}^1] = -K_X \cdot f_*[\mathbb{P}^1] - 1$. Thus indeed $-K_X \cdot f_*[\mathbb{P}^1] \geq 3$. To see that $D^2 \neq 0$, note that the image of f admits deformations keeping a point fixed, thus $D^2 \geq 1$.

Conversely, let as before

$$D = n_\delta(-K_{X_\delta}) + n_{\delta-1}(-K_{X_{\delta-1}}) + \ldots + n_2(-K_{X_2}) + D'$$

where $X = X_\delta$, $D^2 \neq 0$ and

$$-K_X \cdot D = (9-\delta)n_\delta + (8-\delta)n_{\delta-1} + \ldots + 7n_2 - K_X \cdot D' \geq 3$$

The first of these two inequalities excludes the possibility that $D = D'$, with $D^2 = 0$, while the second of the two inequalities excludes the possibilities that $D = -K_{X_8}$, $-2K_{X_8}$ and $-K_{X_7}$.

If $D = -3K_{X_8}$ we may find a morphism $\nu : C_1 \cup C_2 \to X_8$ such that $\nu|_{C_1}$ is free, $\nu_*[C_1] = -2K_{X_8}$, $\nu_*[C_2] = -K_{X_8}$ and $C_i \simeq \mathbb{P}^1$. Proceeding as before, we may smooth this morphism to a morphism with irreducible domain, $f : \mathbb{P}^1 \to X_8$ which is birational to its image (since the initial morphism ν had this property). Using complement II.3.14.4 of [Ko], we may further assume that f is an immersion. Consider the short exact sequence of sheaves on $\mathbb{P}^1$, defining the sheaf $\mathcal{N}_f$:

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^1} \longrightarrow f^*\mathcal{T}_X \longrightarrow \mathcal{N}_f \longrightarrow 0$$

By assumption, $\mathcal{N}_f$ is locally free, and its degree is $-K_X \cdot f_*[\mathbb{P}^1] - 2 = 1$. Thus $f^*\mathcal{T}_X$ is an extension of ample sheaves, and it is therefore ample: D is represented by a very free curve.

Similar arguments allow us to conclude the proof of the theorem. □

Note that if D is a nef divisor on X_δ of anti-canonical degree 1, then X_δ has degree 1 (that is $\delta = 8$) and $D = -K_{X_8}$. A nef divisor D on X_δ of anti-canonical degree 2 is one of $-2K_{X_8}$, $-K_{X_7}$ or the class of a conic of a conic bundle structure on X_δ. Moreover, the only classes D of irreducible curves on X_δ such that $D^2 = 0$ are the non-negative multiples of the conics of a conic bundle (the corresponding curves are non-reduced unless D is a conic).

References

[Ko] J. Kollár, *Rational curves on algebraic varieties*, Springer-Verlag, 1996.