Taming Moduli Problems in Algebraic Geometry

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Contents

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The topic of the course is "Taming Moduli Problems in Algebraic Geometry." The goal of the course is to give a working knowledge of stacks, which are usually abstract, and to give a survey of some nice results.

Remark 1.1. There are two types of exercises: those that can be solved easily using tools from the course or prerequisite tools and those that are more involved and for which a solution consists of reading a solution in some book or reference.

1.1 Moduli problems

There is a guiding meta-problem in mathematics: the classification of mathematical objects. A famous example is simple Lie algebras over $\mathbb{C}$, whose classification is known in terms of Dynkin diagrams. There is a known classification which is discrete: there are $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ types.

Classifying objects in algebraic geometry often involves finitely many continuous (or non-discrete) parameters. For example, consider the moduli of Riemann surfaces (compact, complex manifolds of complex dimension 1). Riemann originally suggested that it takes $3g-3$ complex parameters to specify a complex structure on a smooth surface of genus g. Ahlfors and Bers confirmed that the first-order deformations of a complex structure are classified by $H^1(X, TX)$ for a general complex manifold X, which can be identified using Serre duality with $H^0(C, K^{\otimes 2})^*$, when X is a curve C. By Riemann-Roch, the dimension of this vector space is $\deg(2K) + 1 - g = 3(g-1)$.

Remark 1.2. There are several ways to see the classification of first-order deformations given above. One way is from the so-called Kodaira-Spencer map. Another way is to give a direct identification between first-oder deformations of integral almost complex structures and Dolbeault cohomology cycles in $H^1(X, TX)$.

There is also a differential geometric perspective. If S is a smooth surface of genus $g > 1$, define the Teichmüller space $T(S)$ to be the quotient space $H(S)/\mathrm{Diff}_0$, where $H(S)$ is the space of Riemannian metrics of constant curvature -1 and Diff_0 is the group of diffeomorphisms isotopic to the identity map. This quotient space can be identified with $\mathbb{R}^{6(g-1)}$, which has a canonical complex structure. Moreover, if we let $MCG = \mathrm{Diff}/\mathrm{Diff}_0$ be the mapping class group, then $T(S)/MCG$ can be identified with $\mathcal{M}_g$, where $\mathcal{M}_g$ is the set of Riemann surface structures on S. One can show that $\mathcal{M}_g$ inherits a topology, and is homeomorphic to a quasi-projective variety over $\mathbb{C}$. This understanding of $\mathcal{M}_g$ is very useful because questions about metrics on S can become questions about a quasi-projective surface, whose properties we understand.

Our goal is to have a general framework for studying moduli problems in algebraic geometry and for finding and constructing "moduli spaces."

1.2 Equivariant geometry

This approach will be our most concrete method of constructing moduli spaces.

Let G be a reductive group over $\mathbb{C}$. This means that G is the complexification of a compact (real) Lie group. Suppose we have a linear action of G on projective space $\mathbb{P}^n$. (In fact, every algebraic action is linear, because the automorphism group of $\mathbb{P}^n$ can be identified with PGL_{n+1}, which can be seen from a functor-of-points definition of $\mathbb{P}^n$.) Let $X \hookrightarrow \mathbb{P}^n$ be a locally closed quasi-projective variety, equivariant for the action of G.

In this course, we wish to discuss equivariant cohomology, equivariant K-theory, equivariant coherent sheaves. The guiding principle is that any equivariant construction should not depend on the quotient construction X/G. This is because you want equivariant geometry to be an extension of usual geometry. For example, if G acts freely in a suitable sense, then there should be a space X/G parameterizing G-orbits, and for example, we want $H^*_G(X) \simeq H^*(X/G)$. However, the existence of such a space X/G is not always possible in general, as the following exercise demonstrates.

Exercise 1.3. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ by scaling. Then any $\mathbb{C}^*$-invariant map to a scheme $\varphi : \mathbb{C}^n \to X$ factors through $\mathbb{C}^n \to \mathrm{pt}$. (Hint: This follows from the non-existence of invariant functions on $\mathbb{C}^n$.) It follows that there can be no orbit space. The issue is the origin, because once it is removed, there is a space parametrizing orbits.

What we will do is think of X/G as a geometric object in its own right, namely as a quotient stack (as a "functor of points").

Exercise 1.4. There is a quasi-projective scheme $X_{g,d,n}$, constructed using Hilbert schemes parametrizing $C \hookrightarrow \mathbb{P}^n$ such that the action of PGL_{n+1} extends to X and $\mathcal{M}_g$ is an orbit space for the action of PGL_{n+1} on $X_{g,d,n}$. (This done in Mumford's book on GIT.)

1.3 Moduli of vector bundles on a curve

One can also consider the moduli of vector bundles over a curve, which will be an integral example for us. In fact, it is of fundamental interest in the geometric Langlands program. However, we will mostly study it because it is a beautiful example exhibiting much pathology yet much structure. In particular, it is highly non-separated, and there are too many vector bundles to be parametrized by a single scheme.

One can do a similar calculation as above to see that the first order deformations of a bundle ξ are classified by $\mathrm{Ext}^1(\xi, \xi)$ which has dimension $(n^2 - 1)(g - 1)$ for generic ξ.

There is an algebraic stack $\mathcal{M}_{n,d}(C)$ parametrizing rank n degree d vector bundles over C. There are several descriptions

(i) a functor of points description

(ii) a local quotient description

(iii) a global quotient description using infinite Grassmannians

(iv) a global quotient description due to Atiyah-Bott in mathematical gauge theory.

Understanding the equivalence of these construction is a fruitful way to understand this moduli problem.

The stack $\mathcal{M}_{n,d}$ has pathologies, but has special stratification (due to Harder-Narasimhan-Shatz), given by

$$\mathcal{M}_{n,d} = \mathcal{M}^{ss}_{n,d} \cup \bigcup_\alpha S_\alpha$$

where α ranges over 2-by-k matrices

$$\alpha = \begin{bmatrix} d_1 & \cdots & d_k \\ n_1 & \cdots & n_k \end{bmatrix}$$

of integers such that $n_i > 0$, $d_1 + \cdots + d_k = d$, $n_1 + \cdots + n_k = n$, and $d_1/n_1 < \cdots < d_k/n_k$. The stratification has the properties that

- $\mathcal{M}^{ss}_{n,d}$ has a projective "good moduli space" $\underline{\mathcal{M}}^{ss}_{n,d}$, whose points parametrize "semistable bundles" up to "S-equivalence"
- The strata S_α deformation retracts onto $\mathcal{M}^{ss}_{n_1,d_1} \times \cdots \times \mathcal{M}^{ss}_{n_k,d_k}$ in a suitable sense, and the latter has a projective good moduli space as well.

Classically, the good moduli space $\underline{\mathcal{M}}^{ss}_{n,d}$ was studied because as a projective scheme, it is a bit more tractable. But thinking about $\mathcal{M}_{n,d}$ and the HNS stratification is the key to many results.

1.4 Striking results

If we restrict our attention to $\mathcal{M}_{2,d}$, then

(i) We have the Atiyah-Bott formula: If

$$P_t(-) = \sum_{i \geqslant 0} t^i \dim H^i(-, \mathbb{Q})$$

is the Poincaré polynomial, then

$$\begin{aligned} P_t(\mathcal{M}_{2,d}^{ss}) &= P_t(\mathcal{M}_{2,d}) - \sum_{k > d/2} t^{\#_k} P_t(\mathcal{M}_{1,k}^{ss}) P_t(\mathcal{M}_{1,d-k}^{ss}) \\ &= \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \sum_{k>d/2} t^{\#_k} \left(\frac{(1+t)^{2g}}{1-t^2} \right)^2 \end{aligned}$$

where $\#_k = 2k - d + g - 1$. The amazing fact is that when d is odd, we have that $P_t(\mathcal{M}_{2,d}^{ss})(1-t^2)$ is a polynomial.

(ii) Verlinde formula: There is a "unique" positive generator L of $\mathrm{Pic}(\mathcal{M}_{2,0}^{ss})$, and the Verlinde formula says that $H^i(\mathcal{M}_{2,0}^{ss}, L^{\otimes k}) = 0$ for $i > 0$ and

$$\dim H^0(\mathcal{M}_{2,0}^{ss}, L^{\otimes k}) = \left(\frac{k+2}{2} \right)^{g-1} \sum_{j=1}^{k+1} (\sin(\pi j/(k+2)))^{2-2g}$$