Taming Moduli Problems in Algebraic Geometry

Daniel Halpern-Leistner

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The topic for today is linear algebraic groups.

We will work mainly over an arbitrary field k (for the whole course).

Definition 1.1. A *linear algebraic group* is a smooth affine group scheme over k. This means that it is a scheme, and a group object in the category of schemes. In particular, there are morphisms

$$G \times G \xrightarrow{\mu} G$$
$$G \times G \xrightarrow{i} G \times G$$
$$\mathrm{pt} \xrightarrow{e} G$$

called multiplication, inverse, and identity, respectively. These morphisms satisfy the group axioms, which consist of certain commuting diagrams. The axioms include associativity, inverses, and identity.

Another way to think is to think of G is as a functor $G : \mathsf{Rings} \to \mathsf{Grp}$.

Another useful perspective is to think of G as a commutative Hopf algebra structure on the ring

$$k[G] = \{\text{algebraic functions on } G\}.$$

This means that there are maps which are dual to the maps above

$$k[G] \xrightarrow{\Delta} k[G] \otimes k[G]$$
$$k[G] \otimes k[G] \xrightarrow{a} k[G] \otimes k[G]$$
$$k[G] \xrightarrow{\epsilon} k[G]$$

Remark 1.2. Sometimes we talk about group schemes.

Example 1.3. The general linear group $GL_n = \mathrm{Spec}(k[x_{ij}][1/\det])$

Example 1.4. The multiplicative group $\mathbb{G}_m = \text{Spec}k[t^{\pm 1}]$ with

$$\Delta : k[t^{\pm 1}] \to k[t_t^{\pm 1}] \otimes k[t_2^{\pm 1}]$$
$$t \mapsto t_1 \otimes t_2$$

From a functor of points description, we can view $\mathbb{G}_m$ as the assignment $\mathbb{G}_m : k\text{-}\mathsf{Alg} \to \mathsf{Grp}$ described by

$$\mathbb{G}_m(R) = R^\times$$

for any k-algebra R.

Example 1.5. A split torus $T \simeq (\mathbb{G}_m)^n$.

Notation 1.6. Let X be a scheme. Remember the functor of points description: This says that X can be reconstructed uniquely from the functor $k\text{-}\mathsf{Alg} \to \mathsf{Set}$ which assigns to any k-algebra R, the set $\text{Map}(\text{Spec}(R), X)$. We denote the set of R points of X by $X(R) = \text{Map}(\text{Spec}(R), X)$. In the case above, where $X = \mathbb{G}_m$, one can check that $\text{Map}(\text{Spec}(R), \mathbb{G}_m)$ can be identified with $R^\times$. Indeed, we can identify $\text{Map}(\text{Spec}(R), \mathbb{G}_m)$ with $\text{Map}_k(k[t^{\pm 1}], R)$, which is then identified with $R^\times$ by sending a morphism $\varphi : k[t^{\pm 1}] \to R$ to the image $\varphi(t) \in R^\times$.

Notation 1.7. Let X be a scheme over k and let R be a k-algebra. It is customary to denote $\text{Spec}(R) \times_{\text{Spec}(k)} X$ by X_R, called the base change of X, which is a scheme over $\text{Spec}(R)$ via the first projection. For example, if A is a k-algebra and $X = \text{Spec}(A)$, then $X_R = \text{Spec}(A \otimes_k R)$.

Definition 1.8. A **torus** is a linear algebraic group scheme such that $T_{\bar{k}} \simeq (\mathbb{G}_m)^n_{\bar{k}}$. In particular, the **Deligne torus** is constructed as follows: Take $(\mathbb{G}_m)_\mathbb{C}$, and perform the "Weil restriction" (see next remark) along the map $\pi : \text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})$ to obtain $\mathbb{S}$ over $\mathbb{R}$. We find that the $\mathbb{R}$-points are given by $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, and $\mathbb{S}_\mathbb{C} \simeq (\mathbb{G}_m)_\mathbb{C} \times (\mathbb{G}_m)_\mathbb{C}$ and also that $k[\mathbb{S}] = (\mathbb{C}[z^{\pm 1}, \bar{z}^{\pm 1}])^{\mathbb{Z}/2}$.

Remark 1.9. The "Weil restriction" is that if $\pi : X \to Y$ is a finite, flat morphism and we are given W/X, we can construct $\pi_*(W)/Y$ where $\pi_*(W)(R) = W(X \times_Y R)$. For example if W is the total space of a locally free sheaf, then $\pi_*(W)$ is the total space of the pushforward sheaf. This construction preserves group schemes. Moreover, Weil restriction commutes with base change in the sense that

$$\begin{tikzcd}
\text{pt}_\mathbb{C} \cup \text{pt}_\mathbb{C} \arrow[r] \arrow[d] & \text{Spec}(\mathbb{C}) \arrow[d, "\pi_*"'] \\
\text{Spec}(\mathbb{C}) \arrow[r] & \text{Spec}(\mathbb{R})
\end{tikzcd}$$

commutes.

Example 1.10. Let A be a geometrically reduced finite k-algebra. This means that when one performs a base change to $\bar{k}$, then one obtains a product of copies of $\bar{k}$. Then perform the Weill restriction of $\mathbb{G}_m$ along

$\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ to obtain a torus T. In this way, we obtain $(T)_{\bar{k}} \simeq (\mathbb{G}_m)_{\bar{k}}^{\dim(A)}$. Morally speaking, we can view $T = A^\times$. In fact, the multiplication action of A on itself gives an embedding $T \hookrightarrow GL(A) \simeq GL_{\dim(A)}$. This construction gives examples of "maximal" tori in GL_n which are not split; there are in fact many such tori.

1.1 Representations of G

Definition 1.11. There are several equivalent definitions of a representation of G:

(i) A representation of G is a homomorphism of groups $G \to GL(V)$ for a finite dimensional vector space V.

(ii) Equivalently, a representation is a map $G \times V \to V$ which satisfies certain properties, such that it is linear and there should be associativity.

(iii) Equivalently, we can view a representation as a functor $V : \mathrm{Ring}/k \to \mathrm{Set}$ which lifts to k-vector spaces.

(iv) A useful description is to think of a representation as a comodule over $k[G]$, because this works for infinite-dimensional representations. A **comodule** is a map $\rho : V^* \to V^* \otimes k[G]$ which satisfies the axioms

 (a) identity map is given by $V^* \to V^* \otimes k[G] \xrightarrow{1 \otimes \xi} V^*$

 (b) there is associativity

$$\begin{tikzcd} V^* \arrow[r, "\rho"] \arrow[d, "\rho"'] & V^* \otimes k[G] \arrow[d, "1\otimes\Delta"] \\ V^* \otimes k[G] \arrow[r, "\rho\otimes 1"] & V^* \otimes k[G] \otimes k[G] \end{tikzcd}$$

We will use this fourth description as our definition of a representation.

To go from (i) to (iv), the map $G \times V \to V$ gives $V^* \hookrightarrow \mathrm{Sym}(V^*) \to \mathrm{Sym}(V^*) \otimes k[G]$, which must then lift to a map $V^* \to V^* \otimes k[G]$.

Example 1.12. This is an important example showing that the category of $\mathbb{G}_m$ representations is equivalent to the category of $\mathbb{Z}$-graded vector spaces. In fact, given a map

$$\begin{aligned} V &\to V \otimes_k k[t^\pm 1] \\ v &\mapsto \sum_i v_i t^i, \end{aligned}$$

then V decomposes as $V = \oplus_i V_i$ where V_i is the span of all elements of the form $\rho(v)_i$ for $v \in V$. It remains to show that V_i is a sub comodule.

Example 1.13. Let T be a split torus. Consider the group $\mathrm{Hom}_{\mathrm{grp}}(T, \mathbb{G}_m)$, which is isomorphic to the character lattice M. Any representation V of T splits canonically as a direct sum $\oplus_{\chi \in M} V_\chi$. On this space V_χ, an element $t \in T$ acts via $\chi(t) \cdot (-)$.

Example 1.14. There is a homomorphism $w : \mathbb{S} \to \mathbb{G}_m$ induced by the map

$$(\mathbb{C}[z^{\pm 1}, \bar{z}^{\pm 1}])^{\mathbb{Z}/2} \to \mathbb{R}[t^{\pm 1}]$$
$$z, \bar{z} \mapsto t.$$

This implies that a representation of $\mathbb{S}$ is an $\mathbb{R}$-vector space V with a grading $V = \oplus_w V_w$ along with the data of a direct sum decomposition

$$(V_w)_{\mathbb{C}} \simeq \bigoplus_{a+b=w} (V_w)_{\mathbb{C}}^{a,b}$$

such that $(V_w)_{\mathbb{C}}^{a,b} = \overline{(V_w)_{\mathbb{C}}^{b,a}}$. This implies that representations of the Deligne-torus $\mathbb{S}$ are real Hodge structures.

Proposition 1.15. *Any representation V of G is a union of finite-dimensional sub co-modules.*

Proof. Let $v \in V$. (We want to show that v lies in some finite-dimensional sub co-module.) Choose a basis $\{e_i\}$ for $k[G]$, as a vector space. Using this basis, write

$$\rho(v) = \sum_i v_i \otimes e_i \in V \otimes k[G]$$

for some $v_i \in V$. The claim is that the linear span of v_i is a sub co-module.

Indeed, using the co-multiplication, write $\Delta(e_i) = \sum_{i,j,k} r_{ijk} e_j \otimes e_k$ for some constants r_{ijk} in the field. Then the associativity axiom applied to $\rho(v)$ implies that

$$\sum_{i,j,k} r_{ijk}(v_i \otimes e_j \otimes e_k) = \sum_k \rho(v_k) \otimes e_k.$$

This means that $\rho(e_k) = \sum_{i,j} r_{ijk}(v_i \otimes e_j)$. $\square$

1.2 Main structure theorems

Theorem 1.16. *There is an n such that there is an embedding $G \hookrightarrow GL_n$.*

Remark 1.17. To prove this, find a finite sub-representation V^* of $k[G]$ containing a set of generators, which will lead to an embedding $G \hookrightarrow GL(V)$.

Theorem 1.18 (Jordan decomposition)**.** *For any $g \in G(\bar{k})$, there is a unique way to write $g = g_{ss} \cdot g_u$, where g_{ss} is semi-simple in some representation (meaning that it is diagonalizable in some faithful representation) and g_u is unipotent (meaning that it is unipotent in some representation) and such that g_{ss}, g_u commute.*

Theorem 1.19. *There exist maximal connected solvable subgroups $B \hookrightarrow G$ over $\bar{k}$ which are unique up to conjugation. Among connected, solvable subgroups, the condition of being maximal is equivalent to the fact that G/B is projective.*

Remark 1.20. The condition that G/B is projective is the definition of a **parabolic subgroup**. In fact, all parabolic subgroups contain a Borel B.

Theorem 1.21. *There is a maximal torus $T \hookrightarrow G$ such that $T_{\bar{k}} \to G_{\bar{k}}$ is maximal, and over $\bar{k}$, this torus is unique up to conjugation.*

Theorem 1.22. *There is a unique maximal normal unipotent connected subgroup $R_u(G) \hookrightarrow G \to H$.*

Definition 1.23. We say that G is **reductive** if $R_u(G) = 1$.

Theorem 1.24. *If the characteristic of k is zero, then the condition of being reductive is equivalent to **linearly reductive**, which means that the category of co-modules of $k[G]$ has no higher Ext's, or equivalently, if $V \to W$ is a surjective of G-representations, then there is a splitting $\sigma : W \to V$. (Moreover, Nagata showed that in characteristic p, the condition of being linearly reductive is equivalent to the connected component of the identity being $G_0 = ((\mathbb{G})_m)^n$ and $|G/G_0|$ is coprime to the characteristic of k.)*