Taming Moduli Problems in Algebraic Geometry

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There are some really nice sets of notes:

(i) Bialynicki-Birula, Actions of algebraic groups on schemes

(ii) M. Brion, Actions of algebraic groups on schemes

(iii) Milne has an incomplete book on algebraic groups which works over a general field and works with modern algebraic language

(iv) B. Conrad has notes on SGA

1.1 Functor of points

We first recall some points about the functor of points formalism.

The Yoneda lemma implies that there is an embedding

$$\mathrm{Sch}/k \hookrightarrow \mathrm{Fun}((\mathrm{Sch}/k)^{op}, \mathrm{Set})$$

described by sending a scheme X to the set of morphisms $\mathrm{Map}(-, X)$.

A **site** is a category with a class of "coverings" $\{U_\alpha \to X\}_{\alpha \in I}$ which satisfies certain axioms. We always assume that the category has fibre products. The axioms are supposed to generalize the notion of covering by open sets in a topological space.

Example 1.1. Open sets in a topological space form a site on the category of open sets of a fixed topological space.

Example 1.2. If the category is all topological spaces and covering families are given by jointly surjective collections of local homeomorphisms $U_\alpha \to X$, then this forms a site. One can also use the category of C^∞ manifolds, or complex analytic spaces, etc.

Example 1.3. The category of schemes over k with the Zariski topology, or étale topology, or fppf maps. One can also use the category of $(\mathrm{Ring}/k)^{op}$ with the same topologies. Or one can consider only the full subcategory of finite-type objects.

Given a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$, we say that it satisfies **descent with respect to a covering** $\{U_\alpha \to X\}$ if the diagram

$$F(X) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

is an equalizer. We say that F is a **sheaf** if it satisfies descent with respect to all covering families. The category of sheaves of sets is denoted $\mathrm{Sh}(\mathcal{C})$.

Lemma 1.4. *The restriction functor*

$$Sh((Sch/k)_\tau) \to Sh((Ring/k)^{op}_\tau)$$

is an equivalence for τ the Zariski topology, the etale topology, or the fppf topology. One way of saying this is that the "topos associated to the big site of schemes is equivalent to the topos associated to the big site of rilngs."

Remember the following notation: If X is a scheme and R is a ring, then $X(R) = \mathrm{Map}(\mathrm{Spec} R, X)$.

Example 1.5. $\mathbb{G}_m(R) = R^\times$
$GL_n(R) = \mathrm{End}_R(R^{\oplus n})^\times$.

Definition 1.6. Let $F : \mathrm{Ring} \to \mathrm{Set}$ be a functor. We say that F is **locally finitely presented** (lfp) if for all filtered systems of rings R_i, then the functor

$$\mathrm{colim}_i F(R_i) \to F(\mathrm{colim}_i R_i)$$

is an equivalence.

Note 1.7. A scheme is lfp as a scheme if and only if its functor of points is lfp. In the affine case, this says that R is finitely presented as an algebra if and only if $\mathrm{Map}(R, -)$ commutes with filtered colimits of rings.

Note 1.8. There is a left-adjoint that is fully faithful to $F : \mathrm{Sh}((\mathrm{Ring}/k)^{op}) \to \mathrm{Sh}((\mathrm{Ring})^{op}_{ft})$ if and only if F is lfp. One way to interpret this is to say that "lfp sheaves are functorially determined by their values on finite type rings."

1.2 Group actions on schemes

A **group action** on a scheme is a map

$$\sigma : G \times X \to X$$

satisfying certain axioms

(i) identity axiom: the composition $X \xrightarrow{e\times 1} G \times X \xrightarrow{\sigma} X$ is the identity map

(ii) associativity axiom:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{1\times\sigma} & G \times X \\ \downarrow{\scriptstyle \mu\times 1} & & \downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

From a functor of points definition, we can view G as a group valued functor $G : \mathrm{Ring} \to \mathrm{Grp}$ and a lifting

$$\begin{array}{ccc} \mathrm{Ring} & \longrightarrow & \mathrm{Set} \\ & \searrow & \uparrow \\ & & G\text{-Set} \end{array} \quad .$$

Some basic constructions are as follows.

There is a map

$$\psi : G \times X \to X \times X$$

described by $\psi(g, x) = (\sigma(g, x), x)$. More generally, for an S-point $S \to X$, we can form

$$\psi_S : G \times S \to X \times S.$$

For example, if $S = \mathrm{Spec}(k) \hookrightarrow X$, then ψ_S is the orbit map $G \to X$ given by $g \mapsto g \cdot x$. For any S-point $f : S \to X$, we can consider the stabilizer of f, denoted Stab_f, which is a scheme over S and which is the fibre product

$$\begin{array}{ccc} \mathrm{Stab}_f & \longrightarrow & G \times S \\ \downarrow & & \downarrow{\scriptstyle \psi_S} \\ S & \xrightarrow{\mathrm{diag}} & X \times S. \end{array}$$

In other words $\mathrm{Stab}_f \subset G \times S$ is a subscheme which is a group subscheme. The geometric fibres of Stab_f/S are the stabilizers (in the usual sense) of the corresponding points of X.

Note 1.9. The fiber dimension of $\mathrm{Stab}_f \to S$ is upper semi-continuous. For a general finite type map $p : X \to Y$, then $x \mapsto \dim_x f^{-1}(f(x))$ is upper semi-continuous.

Example 1.10. Let $\mathbb{G}_m$ act on $\mathbb{A}^1$. Then this corresponds to the usual grading on $k[t]$. What is the stabilizer scheme of the identity $f : \mathbb{A}^1 \to \mathbb{A}^1$? The stabilizer of the generic point is trivial, but with a large fiber over the origin.

Note 1.11. If X is Noetherian of finite Krull dimension, then for any point $x \in X$, we have

$$\dim(G) = \dim(G \cdot x) + \dim(G_x).$$

The closure $\overline{G \cdot x}$ is a union of lower-dimensional orbits, and if an orbit there has minimal dimension then it must be closed.

1.3 Actions on affine schemes

A G action on $\mathrm{Spec}(R)$ is equivalent to giving a G-module structure on R compatible with the ring structure maps, or to giving a $k[G]$-comodule structure on R such that $\rho : A \to A \otimes k[G]$ is a map of algebras.

Example 1.12. Let T be a split torus. Then an action is the same as an M-grading on R, where M is the character lattice ($R = \oplus_{\chi \in M} R_\chi$). It is useful to note that every invariant ideal is generated by homogenous elements.

Lemma 1.13. *If G acts on Spec(R) and R is of finite type, then there is an equivariant embedding Spec$(R) \hookrightarrow B$ for some linear representation V.*

Proof. We know that R is a G-module, or a $k[G]$ co-module. We can find a finite-dimensional G-submodule V^*, which generates as an algebra. □

Remark 1.14 (Matsushima)**.** If G is reductive and $H \subset G$ is a closed subgroup, then G/H is affine if and only if H is reductive as well.

Note 1.15. We can always choose an n such that $G \hookrightarrow GL_n$. If G is reductive, then GL_n/G is affine. By the lemma, there is a GL_n-equivariant embedding $GL_n/G \to V$. Write v for the image of 1 under this map. Then $G = \mathrm{Stab}_{GL_n}(v)$.

A linear representation of G acting on V gives an action of G on $\mathbb{P}(V)$. We consider the schemes $X \hookrightarrow \mathbb{P}(V)$ which are G-equivariant, which means that that map $G \times X \to \mathbb{P}(V)$ factors through X, in which case, it does so uniquely and induces an action on X.

1.4 Torus actions

The main simplification is Sumihira's theorem:

Theorem 1.16. *Let T be a torus, and let X be a T-quasi-projective scheme. Then X is covered by T-equivariant open affines.*

Proof. We give a proof next time. □