Taming Moduli Problems in Algebraic Geometry

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Theorem 1.1 (Sumihiro). *Let $X \hookrightarrow \mathbb{P}(V)$ be a T-quasi-projective scheme. Then there is a cover of X by equivariant open affines.*

Proof. We first consider the case where k is algebraically closed and $X \hookrightarrow \mathbb{P}(V)$ is closed. If $f \in V^*$ is an eigenvector for the action of the torus, then the subset $\{f \neq 0\}$ is T-equivariant.

For the general case where k is algebraically closed, consider the closure $\bar{X} \hookrightarrow \mathbb{P}(V)$. By the previous case, we can reduce to the situation of a T-equivariant open subvariety $U \subset \mathrm{Spec}(A)$. We claim that for any point in U, we can find an eigenvector $f \in A$ which does not vanish at p, and vanishes on $\mathrm{Spec}(A) \setminus U$. (This will imply that $p \in \mathrm{Spec}(A_f) \subset U$.)

For the case of general k, we use that X is separated, and hence the intersection of open affines is affine. We use the fact that an affine open $U \subset X_{\bar{k}}$ can be modeled over $X_{k'}$ for some finite Galois extension k'. Then $U \subset X_{k'}$ will be T-equivariant. If we intersect all Galois conjugates of U, then we obtain a T-equivariant subset of $X_{k'}$ which comes from a T-equivariant subset of X, which will still be affine. □

Theorem 1.2 (Existence of fixed points). *Let X be a T-quasi-projective scheme. Then $X^T \hookrightarrow X$ is a closed subscheme, which is smooth if X is smooth. (Here, the notation X^T means $X^T(R) = Map_T(Spec(R), X)$, where we use the trivial T action on $Spec(R)$.)*

Proof. Similarly to before, one can reduce to the case of k algebraically closed. The previous theorem allows one to reduce to the affine case.

Suppose $X = \mathrm{Spec}(A)$, where A is an M-graded algebra. Then define $B = A/(A \cdot \oplus_{\chi \neq 0} A_\chi)$. Then B represents X^T in this case because a T-invariant map must annihilate $\oplus_{\chi \neq 0} A_\chi$. For the smoothness claim, one can show that for $x \in Z = \mathrm{Spec}(B)$, the tangent space $T_x(Z) = (T_x X)^T = ((\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*)^T$. We observe that if X is smooth at x, then we can lift any nonzero eigenvector in $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ to an eigenvector in $\mathfrak{m}_{X,x}$, and these functions will cut out $Z \hookrightarrow X$ transversally in a neighborhood of x. □

Remark 1.3. One can also show that X^T is smooth by showing that its functor of points is "formally smooth."

1.1 Reducing to the affine case

Definition 1.4. If $F, G : \text{Ring} \to \text{Set}$ are sheaves, then a map $f : F \to G$ is called **representable** if for each map $\text{Spec}(A) \to G$, we can form the fibred product[1]

$$\begin{array}{ccc} F' & \longrightarrow & F \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \text{Spec}(A) & \longrightarrow & G \end{array}$$

and F' is a scheme. For representable maps, one can define any property associated to a map of schemes as long as it is stable under base change and local on the target. If P is such a property, we say that f has property P if for all $\text{Spec}(A) \to G$, the base change f' has property P. (As an example of such a property one could take "open immersion.")

Remark 1.5. There are different topologies

$$\text{Zariski} \subset \text{etale} \subset \text{fppf}.$$

For Zariski, there are two few coverings. For fppf, it is harder to prove descent. The etale topology is just right, and always the topology we use on Ring^{op}.

Lemma 1.6. *A sheaf X is a scheme if and only if there is a surjection of sheaves $\sqcup_\alpha Spec(A_\alpha) \to X$ such that $Spec(A_\alpha) \to X$ is a representable open immersion.*

The reduction of the fixed point theorem to the affine case via Sumihiro uses the fact that if $U \subset X$ is an equivariant affine open, then the map of functors $U^T \to X^T$ is also a representable open subfunctor. This is because the diagram

$$\begin{array}{ccc} U^T & \longrightarrow & U \\ \downarrow & & \downarrow \\ X^T & \longrightarrow & X \end{array}$$

is Cartesian. The fact that U^T is an affine scheme implies that X^T admits an open cover, that is, is a scheme.

Theorem 1.7 (Bialynicki-Birula). *If $X \hookrightarrow \mathbb{P}(V)$ is $\mathbb{G}_m$-quasi-projective, then*

(i) The functor $Y(R) = Map(\mathbb{A}^1 \times Spec(R), X)$ is a scheme.

(ii) The restriction map $Y(R) \xrightarrow{i} Map(\{1\} \times Spec(R), X)$ is a local immersion, and $X^T(R) \xrightarrow{\sigma} Y(R)$ is a closed embedding (this map is composition with $\mathbb{A}^1 \times Spec(R) \to Spec(R)$), and the map $\pi : Y(R) \to X^T(R)$ induced by restriction along the $\mathbb{G}_m$-equivariant map $\{0\} \times Spec(R) \hookrightarrow \mathbb{A}^1 \times Spec(R)$ is affine.

[1] The forgetful functor $\text{Sh}(\text{Ring}^{et}/k) \to \text{PSh}(\text{Ring}^{et}/k)$ has a left adjoint (sheafification) which commutes with all limits. Thus $F'(R) = \text{Map}(\text{Spec}(R), \text{Spec}(A)) \times_{G(R)} F(R)$.

(iii) If X is smooth, then so is Y and $\pi : Y \to X^T$ is an etale locally trivial bundle of affine spaces $\mathbb{A}^n$.

Proof. The proof again uses a reduction to the affine case, but there are some issues, including that equivariant open subsets of Y are in bijective correspondence with open subsets of X^T under π.

In the affine case, $X = \mathrm{Spec}(A)$, the $\mathbb{G}_m$-action decomposes $A = \oplus_{n\in\mathbb{Z}} A_n$. The claim is that i is a closed immersion and $Y = \mathrm{Spec}(A/A \cdot \oplus_{n>0} A_n)$. □

Example 1.8. Let $\mathbb{G}_m$ act on V (a linear representation). Then choose an eigenbasis so that $t \cdot [z_0, \ldots, z_n] = [t^{a_0} z_0, \ldots, t^{a_n} z_n]$ with $a_0 \leqslant \cdots \leqslant a_n$. Then

$$\mathbb{P}(V)^{\mathbb{G}_m} = \bigsqcup_{a_\alpha} \{[0, \ldots, 0, z_i, \ldots, z_{i+k}, 0, \ldots, 0]\}$$

where a_α ranges over all nonzero eigenvalues of the $\mathbb{G}_m$ action. And

$$Y = \bigsqcup_{a_\alpha} \{[0, \ldots, 0, 1, *, \ldots, *]\}$$

(Here we have written the case of a one-dimensional eigenspace.)