Taming Moduli Problems in Algebraic Geometry

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Remark 1.1. There was a mistake in one of the arguments presented last time, but one can invoke a bigger fact to make the arguments work.

1.1 Final remarks on BB schemes

Lemma 1.2.

(i) If $X \hookrightarrow X'$ is a closed $\mathbb{G}_m$-equivariant immersion, then there is an induced map on BB schemes $Y \to Y'$ and

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{j'} & X' \end{array}$$

is Cartesian.

(ii) If $X \subset X'$ is an open $\mathbb{G}_m$-equivariant subscheme, then

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X^{\mathbb{G}_m} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\pi'} & (X')^{\mathbb{G}_m} \end{array}$$

is Cartesian.

Proof. Exercise. □

Remark 1.3. These two lemmas are good practice in using the functor of points construction.

Remark 1.4. One way of thinking about the BB stratum is as the set of points which have a limit as $t \to 0$.

Example 1.5. Let $\mathbb{A}^2 = \mathbb{A}^1(1) \times \mathbb{A}^1(0)$, where the numbers indicate the action of $\mathbb{G}_m$. Can think of $\mathbb{A}^2$ as $\mathrm{Spec}(k[x, y])$ where x has weight -1 and y has weight 0. Then the BB stratum in $\mathbb{A}^2$ is $\mathrm{Spec}(R/R \cdot R_+) = \mathrm{Spec}(R)$, since the ideal $R \cdot R_+$ is the zero ideal.

If we remove the origin $X = \mathbb{A}^2 \setminus \{0\}$, then the BB stratum can be found using (ii) above. The stratum is not the same. Not every point has a limit under the action of $\mathbb{G}_m$. Remember the action is $t \cdot (x, y) = (tx, y)$. So such a point has a limit as $t \to 0$ if and only if $y \neq 0$. Thus $Y = \{(x, y) : y \neq 0\}$.

Example 1.6. A group G itself has an action via conjugation, and for any one parameter subgroup $\lambda : \mathbb{G}_m \to G$, we can define the following. Informally, let P_λ be the elements $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists. Formally, let P_λ be the BB subscheme associated to this $\mathbb{G}_m$-action by conjugation.

Then P_λ is a parabolic subgroup, i.e., (G/P_λ) is proper. Moreover, any parabolic subgroup in any connected reductive group arises in this way. Concretely, if we embed $G \hookrightarrow GL(V)$, then the diagram

$$\begin{array}{ccc} G & \longrightarrow & GL(V) \\ \downarrow & & \downarrow \\ P_\lambda & \longrightarrow & P_{GL(V),\lambda} \end{array}$$

is Cartesian. One can show that the matrices that belong to $P_{GL(V),\lambda}$ are block upper triangular. (Do this by choosing a weight decomposition $V = \oplus_\alpha V_\alpha$.) Note that $G^{\lambda(\mathbb{G}_m)}$ is the centralizer of $\lambda(\mathbb{G}_m)$.

1.2 Quotients

The quotients G/H exist and are quasi-projective schemes. Indeed $(G/H)(R)$ is the sheafification (in the étale topology) of the presheaf $R \mapsto G(R)/H(R)$.

Theorem 1.7 (Chevalley)**.** *If $H \subset G$ is a linear algebraic subgroup, then there is a representation V and a line $L \subset V$ such that $H = Stab(L)$ (as a point in $\mathbb{P}(V)$). Then the orbit is the quotient G/H.*

Proof. The algebra $k[G]$ is a linear representation of H. Find a finite-dimensional sub H-representation $V \subset I_H$ which generates as an ideal. Then $H = \mathrm{Stab}(I_H) = \mathrm{Stab}(V)$.

Then find a G-representation $V' \supset V$ such that G acts faithfully on V' (i.e., the map $G \to GL(V')$ is a closed immersion).

Use the Plücker embedding. The space of Grassmannians $\mathrm{Gr}(\dim(V), V')$ has a G-action. There is a G-equivariant embedding of this Grassmannian into $\mathbb{P}(\Lambda^{\dim(V)}(V'))$. The stabilizer of the image of V is H. □

We don't quite have the technology to show the general existence of quotients, but we state the theorem regardless.

Theorem 1.8. *Let G be a linear algebraic group acting on a scheme X, and suppose that the map $G \times X \to X \times X$ is a monomorphism. Then X/G is an "algebraic space."*

Remark 1.9. The map $G \times X \to X \times X$ is a monomorphism if and only if $G(R)$ acts freely on $X(R)$ for each R.

This theorem is the motivation for introducing algebraic spaces, which we do now.

Recall the definition of a representable map $F \to G$ of sheaves.

Definition 1.10. An **algebraic space** is a sheaf $F : \mathrm{Ring} \to \mathrm{Set}$ such that

(i) the diagonal $F \to F \times F$ is representable by schemes

(ii) there is a surjective étale map $U \to F$ where U is a scheme.

Remark 1.11.

(1) (i) implies that any map $U \to F$ is representable. (Didn't follow this argument.)

(2) For an fppf map of schemes f, we have that f is surjective as a map of schemes if and only if f is surjective as a map of sheaves. This means that there is no ambiguity int he meaning of (ii).

Remark 1.12. There is a notion of equivalence relation in schemes $R \hookrightarrow X \times X$ such that $R(A) \to X(A) \times X(A)$ is an equivalence relation for each A. We say that an equivalence relation is **étale** if $p_1 : R \to X$ is étale.

For example, if G is a finite group acting freely on X, then $G \times X \to X \times X$ is an étale equivalence relation. This map is still an equivalence relation even if G is not finite, but in such a case, it is not étale.

For any equivalence relation in schemes $R \to U \times U$, one can form a sheaf U/R obtained by sheafification.

Theorem 1.13 (Tag 04S5)**.** *The following are equivalent for a sheaf $F : Ring \to Set$.*

(i) F is an algebraic space.

(ii) There is a representable étale surjection $U \to F$ with U a scheme

(iii) There is an étale equivalence relation $R \to U \times U$ such that F is isomorphic to U/R.

Moreover, these are equivalent to the same statements where the word étale is replaced with the word(s) fppf. In particular (i') is

(i') The diagonal $F \to F \times F$ is representable and there is a surjective fppf map $U \to F$.

This is the approach to prove (ii) $\implies$ (iii). Suppose that there is a surjective étale representable map $U \to F$. Then define R to be the fiber product

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & F \end{array}$$

We get a canonical map of presheaves $U/R \to F$. The argument works in the other direction too.

Example 1.14. Let us consider the example again of a free action. Then $G \times X \to X \times X$ is an fppf equivalence relation. So this implies that X/G is an algebraic space.

Another way of phrasing the theorem is to say that any fppf equivalence relation is "equivalent" to an étale equivalence relation.

One can say that fppf equivalence relations are equivalent if there is a map of equivalence relations $W_\bullet \to V_\bullet$ such that $W_0/W_1 \to V_0/V_1$ is an equivalence after sheafification. (Remember that an equivalence relation consists of V_0, V_1 together with a map $V_1 \to V_0 \times V_0$ satisfying some condition.)

Remark 1.15. Can think of algebraic spaces as the category of equivalence relations where those which are equivant are invertible.