Taming Moduli Problems in Algebraic Geometry

Daniel Halpern-Leistner

1 September 22, 2016

1.1 Remarks on separation axioms

If $f : X \to Y$ is a map of algebraic spaces, then there is a map $\Delta_f : X \to X \times_Y X$, which is representable, locally finite type, a monomorphism, separated, and locally quasi-finite.

There are certain "separation axioms" which impose nice conditions on the diagonal.

- Say that f is **separated** if Δ_f is a closed immersion
- Say that f is **quasi-separated** if Δ_f is quasi-compact.

This latter condition tends to be the weakest condition under which intuition works.

Example 1.1. This is a terrible example. Let $k = \mathbb{Q}$. Then $\mathbb{A}^1(R) = R$ and define $\underline{\mathbb{Z}}(R) = \mathrm{Map}(\pi_0(\mathrm{Spec}R), \mathbb{Z})$, which is a group scheme, which acts freely on $\mathbb{A}^1$ by translation. One can form the quotient $\mathbb{A}^1/\underline{\mathbb{Z}}$ is not quasi-separated. Thus this example provides a counterexample to the following theorem.

Theorem 1.2. *Let X be a quasi-separated, quasi-compact algebraic space over k. Then there is a dense open subscheme $X' \subset X$*

Remark 1.3. This theorem gives a perspective on algebraic spaces. In fact, the simplest constructions of algebraic spaces come from constructions in birational geometry, namely, flips and flops of 3-folds over $\mathbb{C}$.

Corollary 1.4. *If G acts freely on a quasi-compact scheme X, then there is a dense open G-invariant subscheme $U \subset X$ such that U/G is a scheme.*

Proof. In this case, the diagonal is quasi-compact (because G is quasi-compact). We have a map

$$G \times X \rightrightarrows X \to Y := X/G$$

and take the fiber product

$$\begin{array}{ccc} U & \longrightarrow & Y^0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

which will give G-equivariant U by commutativity. $\square$

Definition 1.5. Let X be a scheme. A **principal G-bundle** on X is an algebraic space $\pi : Y \to X$ along with a G-action such that $\pi : Y \to X$ is G-invariant and such that

(i) $G \times Y \to Y \times_X Y$ is an equivalence (this roughly means that G acts freely on the fibers of Y)

(ii) étale locally, π admits a section

Another formulation views Y as a sheaf of sets on the site of X which satisfies further properties.

If π admits a section, can use this section and (i) to construct an isomorphism $Y \simeq X \times G$.

Remark 1.6. In fact, any sheaf of sets on the big étale site of schemes over X satisfying (i), (ii) is an algebraic space.

Lemma 1.7. *Because G is affine, any principal G bundle is a scheme and in fact affine over X.*

Proof. The idea of the proof is as follows. For $U \subset X$, there is an isomorphism $\pi^{-1}(U) \simeq U \times G$. In this case, $G \times U = \underline{\mathrm{Spec}}_U \mathcal{A}$ for some quasi-coherent sheaf of algebras $\mathcal{A}$, which descends to some $\mathcal{A}_X$ on X. Then $Y = \underline{\mathrm{Spec}}_X(\mathcal{A}_X)$. □

Remark 1.8. For certain groups, which are called "special" groups, principal G-bundles are always Zariski locally trivial (when they are usually étale locally trivial).

For example, for GL_n there is an equivalence of categories (or of groupoids) between the category of GL_n-bundles over X and the category of vector bundles over X and isomorphisms between them. The equivalence sends a vector bundle E to its frame bundle $\mathrm{Fr}(E)$, which is the sheaf mapping U to $\mathrm{Isom}(\mathcal{O}_U^{\oplus n}, E|_U)$. One can map a GL_n-bundle Y to the space $\mathbb{A}^n \times_{GL_n} Y$, which is a vector bundle.

As another example, if $B \subset G$ is Borel, then B is "special".

1.2 Return to quotients

Remember that for an action of G on a scheme, there is a subscheme $(X/G)^\circ \subset (X/G)$ which is itself a scheme. So there is a $U \subset X$ such that G/U is a scheme. In fact, formally, the map $U \mapsto U/G$ is a principal G-bundle. Because G is affine, this implies that the map is affine.

We conclude that if X is a scheme with a free G action (or separated algebraic space), then there is a dense open subscheme of X covered by G-equivariant open affines U_α such that U_α/G is also affine. Moreover, the quotient X/G is a scheme if and only if X admits an open affine cover of this form.

1.3 Stacks

The question that stacks answer is this: What if the G action is not free? This means that $G \times X \to X \times X$ is not a monomorphism. In other words, the fibers of the map do not just consist of single points. However, the map $G \times X \to X \times X$ is still a groupoid, that is, a category in which all arrows are invertible.

Definition 1.9. A **groupoid scheme** consists of two schemes X_1, X_2 together with five maps

$$\begin{aligned} s, t : X_1 &\to X_0 \\ e : X_0 &\to X_1 \\ m : X_1 \times_{X_0,s,t} X_1 &\to X_1 \\ i : X_1 &\to X_1 \end{aligned}$$

such that $s \circ e$ and $t \circ e$ are the identity morphisms, and $s \circ m = s \circ p_1$, $t \circ m = t \circ p_2$ and other obvious conditions that generalize the axioms of group action. In practice, it is usually written as

$$X_1 \rightrightarrows X_0.$$

Maps between groups are level-wise maps commuting with the structure maps.

Definition 1.10. A **Morita morphism** is a map of groupoid schemes $f_\bullet : X_\bullet \to Y_\bullet$ such that

(i) $f_0 : X_0 \to Y_0$ is fppf (this is essentially a very strong kind of essential surjectivity)

(ii) The following diagram is cartesian

$$\begin{array}{ccc} X_1 & \longrightarrow & Y_1 \\ {\scriptstyle s,t}\downarrow & & \downarrow{\scriptstyle s,t} \\ X_0 \times X_0 & \xrightarrow[f_0, f_0]{} & Y_0 \times Y_0 \end{array}$$

Remark 1.11. Groupoid schemes form a 2-category. A natural transformation between groupoid schemes $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ is a map $\eta : X_0 \to Y_1$ such that the resulting map on R-points is a natural transformation.

Note 1.12. If $f_\bullet : X_\bullet \to Y_\bullet$ is a Morita morphism with a section σ of $X_0 \to Y_0$, then there is a functor $\sigma_\bullet : Y_\bullet \to X_\bullet$ such that $f_\bullet \circ \sigma_\bullet$ is the identity. There is also a natural transformation from $\sigma_\bullet \circ f_\bullet$ to the identity $\mathrm{id}_{X_\bullet}$. Therefore, this notion of Morita morphism together with a section is an appropriate notion of equivalence. (We will think of all Morita morphisms as "local equivalences.")

Example 1.13. Equivalence relations. There is a Banal groupoid: for an fppf map of schemes $X_0 \to Y$, and we let $X_0 \times_Y X_0$.

If G acts on X and $H \subset G$ is a subgroup, then X/G should be equivalent to $(H \times_G X)/H$. This can be made more precise using Morita morphisms.