Taming Moduli Problems in Algebraic Geometry

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Remark 1.1. To any category C, one can associate the *simplicial set*. This assignment is called the *nerve*. Remember that a simplicial set is a sequence of sets $X_0, X_1, X_2, \ldots$ with various maps between them. For a category C, one lets X_0 be the set of objects, X_1 the set of arrows, and X_2 the set of composable arrows, etc.

Given a groupoid scheme $X_\bullet = X_1 \rightrightarrows X_0$ with source and target maps s and t, one can define a category $\mathrm{QCoh}(X_\bullet)$ as follows.

- The objects consist of

  (i) A quasi-coherent sheaf E on X_0

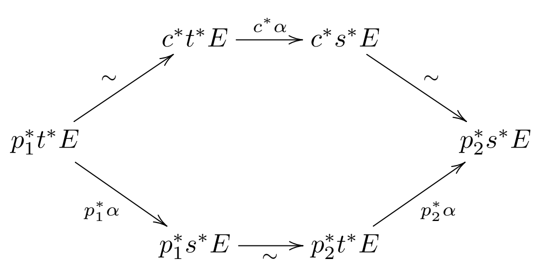
  (ii) An isomorphism $\alpha : t^*E \to s^*E$

  satisfying the following conditions.

  (a) There are three maps $p_1, p_2, c : X_1 \times_{X_0} X_1 \to X_1$ defined as follows

$$
\begin{aligned}
p_1(\gamma_0, \gamma_1) &= \gamma_0 \\
p_2(\gamma_0, \gamma_1) &= \gamma_1 \\
c(\gamma_0, \gamma_1) &= \gamma_0\gamma_1.
\end{aligned}
$$

  With this notation, the following hexagon commutes



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  (b) The morphism $e^*\alpha : e^*t^*(E) \to e^*s^*(E)$ is the identity map.

- The morphsims are morphisms $E \to E'$ in $\mathrm{QCoh}(X_0)$ which commute with α and α'.

Example 1.2. For the groupoid scheme $X_\bullet = G \rightrightarrows \{\mathrm{pt}\}$, we have that $\mathrm{QCoh}(X_\bullet) \simeq \mathrm{Rep}(G)$.

Remark 1.3. For $G \times X \rightrightarrows X$ associated to a group action, this is a notion of an "equivariant sheaf." In formally, on geometric fibers of E over X, for each pair (g, x), we have an isomorphism $E_x \to E_{g \cdot x}$, in such a way that these isomorphisms are compatible with the group operation and composition of maps (stated more precisely in the cocycle condition).

Example 1.4. We work over $\mathbb{P}(V)$. Consider the tautological line bundle $\mathcal{O}(1)$. We wish to know whether this line bundle is equivariant with respect $PGL(V)$. We claim that it is *not*. Indeed, if we fix a point in $\mathbb{P}(V)$, there is an orbit map $PGL(V) \times \{x\} \to \mathbb{P}(V)$ which is surjective and which factors through the map to $\{x\}$. The pullback of $\mathcal{O}(1)$ under this map has order $\dim(V)$ in $\mathrm{Pic}(PGL(V))$. (However, $\mathcal{O}(\dim V)$ does have an equivariant structure.)

On the other hand, $\mathcal{O}(1)$ is linearizable for the action of $GL(V)$. One can see this algebraically or geometrically. For the geometric perspective, note that locally free sheaves over X are equivalent to vector bundles, and under this equivalence, specifying an equivariant structure is the same as giving an action on the total space of the vector bundle which is linear in the fibers. In our case, the total space of $\mathcal{O}(1)$ consists of pairs (ℓ, α) where ℓ is a line in V and α is a linear functional in $\ell^\vee$. This set has a $GL(V)$-action which does not descend to $PGL(V)$, which is given by $g \cdot (\ell, \alpha) = (g \cdot \ell, \alpha \circ g^{-1})$.

Note 1.5. If $X_1 \rightrightarrows X_0$ are flat maps, then $\mathrm{QCoh}(X_\bullet)$ is abelian, with kernels and cokernels formed on X_0. For example, there are morphisms of $GL(V)$-equivariant sheaves of the form

$$\bigoplus_i \mathcal{O}(n_i) \to \bigoplus_j \mathcal{O}(n_j)$$

over $\mathbb{P}(V)$, and in fact, every equivariant coherent sheaf arises as the cokernel of such a morphism.

Remark 1.6. The category $\mathbf{QCoh}(X_\bullet)$ has enough coherent sheaves, which means that ...

Note 1.7. To say X is G-quasi-projective is equivalent to saying that X admits an equivariant invertible sheaf which is ample (forgetting the equivariant structure).

Theorem 1.8. *If X is a normal, projective k-scheme with a G-action and G is connected, then X is G-quasi-projective.*

The idea of the proof is that there is a scheme $\mathrm{Pic}(X/k)$, and one can show that $\mathrm{Pic}(X/k)$ has a fixed point L under the G-action such that some power of L is G-linearizable. Moreover, if X is projective and normal, then the reduced components of $\mathrm{Pic}(X/k)$ are abelian varieties. The group $G_{\bar{k}}$ is rational, so the action on $\mathrm{Pic}(X/k)$ must be trivial. The whole argument can be found in Mumford's book.

Our goal is the proposition

Proposition 1.9. *If $f : Y_\bullet \to X_\bullet$ is a Morita morphism, then the pullback map $QCoh(X_\bullet) \to QCoh(Y_\bullet)$ is an equivalence of categories.*