Taming Moduli Problems in Algebraic Geometry

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**Proposition 1.1.** *Suppose that $f : X_\bullet \to Y_\bullet$ is a Morita morphism of groupoid schemes. Then the pullback functor $f^* : QCoh(Y_\bullet) \to QCoh(X_\bullet)$ is an equivalence of categories.*

*Proof.* We sketch a proof here.

*Step 0.* Cook up a bit of machinery and notation. There is a cleaner way to think of the category $\mathrm{QCoh}(X_\bullet)$. The idea is to define a category QCoh with objects being pairs $(X, E)$ where $X$ is a scheme and $E$ is a quasi-coherent sheaf on $X$, and maps consist of maps of schemes $f : X \to Y$ together with maps of sheaves $F \to f_*E$ over $Y$. One can say that a map in QCoh is **Cartesian** if the induced map $f^*F \to E$ is an isomorphism.

We think of $X_\bullet$ as a diagram in the category of schemes. In particular, there is a small category $\mathcal{G}$ such that each groupoid scheme $X_\bullet$ is a functor $\mathcal{G} \to \mathrm{Sch}$. There is a projection functor $\mathrm{QCoh} \to \mathrm{Sch}$. With this terminology, the category $\mathrm{QCoh}(X_\bullet)$ is the category of sections $\mathcal{G} \to \mathrm{QCoh}$ all of whose arrows are Cartesian. (In fact, for any diagram $X_\bullet$ of schemes, one can define the category $\mathrm{QCoh}(X_\bullet)$ as the category of Cartesian sections.)

*Step 1.* Consider the banal groupoid for an fppf map. For a map $U_0 \to X$, form the groupoid $U_1 = U_0 \times_X U_0 \rightrightarrows U_0 \to X$, called the banal groupoid. We claim that given an fppf map $U_0 \to X$, the pullback functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(U_\bullet)$ is an equivalence. This is called "fppf descent for quasi-coherent sheaves." Using the formal methods sketched below, we can reduce this to the case of a faithfully flat surjective map of affine schemes (or rings) $A \to B$. The statement becomes that the category $A$-Mod is equivalent to the category $B$-Mod along with an isomorphism $B \otimes_A M \simeq M \otimes_A B$ as $(B \otimes_A B)$-modules satisfying a cocycle condition.

*Step 2.* The Morita morphism $f$ gives a map $X_0 \to Y_0$ which is fppf and that the groupoid structure on $X_0$ is induced from $Y_\bullet$. We perform the following construction. Form the fibred product $W_{00}$ as

$$\begin{array}{ccccc} W_{00} & \longrightarrow & Y_1 & \xrightarrow{t} & Y_0\ , \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ X_0 & \xrightarrow{f_0} & Y_0 & & \end{array}$$

with all maps being fppf. The claim is that the rows and columns of the following diagram are all banal groupoids

$$
\begin{array}{ccccccccc}
W_{22} & \longrightarrow & W_{12} & \longrightarrow & W_{02} & \longrightarrow & & \longrightarrow & Y_2 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
W_{21} & \longrightarrow & W_{11} & \longrightarrow & W_{01} & \longrightarrow & & \longrightarrow & Y_1 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
W_{20} & \longrightarrow & W_{10} & \longrightarrow & W_{00} & \longrightarrow & Y_1 & \xrightarrow{t} & Y_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s} & & \\
X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \xrightarrow{f_0} & Y_0 & &
\end{array}
$$

Here, $W_{10}$ is the banal groupoid associated to the map $W_{00} \to Y_0$.

*Step 3.* Use step 2 to show that the pullback functors are equivalences

$$\mathrm{QCoh}(X_\bullet) \to \mathrm{QCoh}(W_{\bullet\bullet}) \leftarrow \mathrm{QCoh}(Y_\bullet)$$

by step 1. □

Secretly we have seen a stack, namely, QCoh $\to$ Sch. Intuitively, a stack is a functor $\mathrm{Sch}^{op} \to \mathrm{Gpd}$ from schemes to groupoids. However, notice that Gpd is a 2-category, so one has to be careful what one means by a functor of 2-categories, and indeed, this is difficult to formalize. A better way is to consider categories fibered in groupoids.

**Definition 1.2.** A **category fibered in groupoids** consists of a category $\mathcal{F}$ together with a functor $b : \mathcal{F} \to \mathrm{Sch}$ such that

(i) For any diagram

$$
\begin{array}{ccc}
 & & \xi \\
 & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{f} & Y
\end{array}
$$

for schemes $X, Y$, there is a "Cartesian" arrow

$$
\begin{array}{ccc}
\xi' & \longrightarrow & \xi \\
\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

(ii) (Existence of pullbacks) The fiber $\mathcal{F}(U) = b^{-1}(U)$ over any scheme $U$ is a groupoid.

**Remark 1.3.** The notion of "Cartesian arrow" is determined by a formal universal property generalizing the property for pullbacks in QCoh.

**Definition 1.4.** A **stack** is a category $\mathcal{F}$ fibered in groupoids such that for any surjective étale map of schemes $U_0 \to X$, the pullback functor is an equivalence of categories $\mathcal{F}(X) \to \mathcal{F}(U_1 \rightrightarrows U_0)$.

**Example 1.5.** We call this stack "point mod $G$," denoted $\cdot/G$. Here, the category $\mathcal{F}$ consists of pairs $(X, E)$ where $E \to X$ is a principal $G$-bundle, and maps are simply $G$-equivariant maps $E_0 \to E_1$ and the functor $\mathcal{F} \to \mathrm{Sch}$ is just the forgetful functor.

**Example 1.6.** Quotient stacks $Y/G$. We are given an action of $G$ on a scheme $Y$. The category $\mathcal{F}$ consists of triples $(X, E, f)$ where $X$ is a scheme, $E$ is a principal $G$-bundle over $X$, and $f : E \to Y$ is a $G$-equivariant map.