Taming Moduli Problems in Algebraic Geometry Daniel Halpern-Leistner

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Recall from last time that we described a stack X/G where the category  $\mathcal{F}$  consists of triples (U, E, f) where U is a scheme, E is a principal G-bundle over U, and  $f: E \to X$  is a G-equivariant map.

Another way of thinking is as follows. To any groupoid scheme  $X_{\bullet}$  we can consider the category fibered in groupoids  $\underline{X}_{\bullet}$ , where the objects consist of pairs  $(U,\xi)$  where U is a scheme and  $\xi \in X_0(U)$ . The morphisms consist of pairs  $(f,\gamma): (U,\xi') \to (V,\xi)$  where  $\gamma \in X_1(U)$  satisfies  $t(\gamma) = \xi'$  and  $s(\gamma) = f^*(\xi)$ . The fiber over  $U \in$  Sch is the groupoid  $X_1(U) \rightrightarrows X_0(U)$ .

For any fibered category  $\mathcal{F}$ , there is a canonical stackification  $\mathcal{F} \to \mathcal{F}^a$ , where  $\mathcal{F}^a$  is a stack and the map is universal with respect to maps from  $\mathcal{F}$  to a stack. In our example, the stackification satisfies  $(X_{\bullet})^a = (X/G)$ .

We claim that  $\underline{X}_{\bullet}$  does not satisfy descent in general. Indeed, consider the example  $\cdot/G$ , namely the stackification  $G \Rightarrow$  pt. For each scheme U, the functor  $\cdot/G$  maps U to the set of G-bundles over U. However, the functor  $G \Rightarrow$  pt maps U to a single object groupoid with automorphism group G(U). There is a base-preserving functor

$$(G \rightrightarrows \operatorname{pt})(U) \to (\cdot/G)(U)$$
  
 $\operatorname{pt} \mapsto U \times G$ 

In general, there is a functor

$$(G \times X \rightrightarrows X)(U) \to (X/G)(U)$$
  
 $(f: U \to X) \mapsto (G \times U \xrightarrow{g \cdot f(U)} X)$ 

## **1.1** Fiber products

For groupoids  $C_1, C_2$  over D, the homotopy fiber product is universal with respect to diagrams of the following kind

$$\begin{array}{c} A \longrightarrow C_2 \\ \downarrow & \qquad \downarrow f_2 \\ C_1 \xrightarrow{} f_1 \xrightarrow{} D \end{array}$$

where the diagram commutes up to natural transformation. The objects of  $C_1 \times_D C_2$  consist of pairs (X, Y)of objects  $X \in C_1$ ,  $Y \in C_2$  together with an isomorphism  $f_1(X) \to f_2(Y)$ . Morphisms are maps  $X_1 \to X_2$ and  $Y_1 \to Y_2$  which commute with all necessary maps. We claim that for stacks  $\mathfrak{X}_1, \mathfrak{X}_2$  over  $\mathfrak{Y}$ , the fiber product is still a stack.

There is a 2-Yoneda Lemma which states the following. For  $X \in \text{Sch}$ , can regard this as a category  $\underline{X}$  fibered in groupoids. Indeed, the objects are pairs (U, f) where  $f : U \to X$ . And morphisms are maps which commute with the maps to X. Then there is an equivalence of categories  $\text{Map}_{\text{Sch}}(\underline{X}, \mathcal{F}) \simeq \mathcal{F}(X)$  for a category fibered in groupoids in  $\mathcal{F}$ .

The previous paragraph justifies referring to a stack as representable. We can also define the notion of representable map: Say that  $f: \mathfrak{X} \to \mathfrak{Y}$  is representable if for any map  $\underline{U} \to \mathfrak{Y}$ , the pullback  $\mathcal{F}$ 



is representable. For any property of morphisms of schemes which is local on the target, we can define such a property in the same way for spaces.

**Theorem 1.1.** The following are equivalent for a stack  $\mathfrak{X}$ .

- (i)  $\mathfrak{X} \simeq (X_{\bullet})^a$  for a smooth groupoid scheme
- (ii) The diagonal functor  $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$  is representable by algebraic spaces, and there is a smooth surjection from a scheme  $U \to \mathfrak{X}$
- (iii) There is a representable smooth surjection  $\underline{U} \to \mathfrak{X}$  from a scheme.

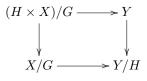
Furthermore, these are equivalent to conditions 1,2,3 with "fppf" replacing smooth.

**Definition 1.2.** Any stack  $\mathfrak{X}$  satisfying one of these three equivalent conditions is called **algebraic**.

**Remark 1.3.** Given  $U \to \mathfrak{X}$ , we get a groupoid  $U_0 \times_{\mathfrak{X}} U_0 = U_1 \to U_0 = U$ .

## 1.2 Constructing maps between stacks

Let  $\psi: G \to H$  be a group homomorphism, let X be a G-scheme and Y an H-scheme. Then an equivariant map  $f: X \to Y$  induces a functor of groupoid schemes  $(G \times X \rightrightarrows X) \to (H \times Y \rightrightarrows)$ , which in turn induces a map of stacks  $f: X/G \to Y/H$ . In particular, f is representable by algebraic spaces if and only if the fiber product



is an algebraic space, which is equivalent to saying that G acts freely on  $H \times X$ . For example, if G is a subgroup of H, then f is representable and  $X/G \simeq (H \times_G X)/H$  (by Shapiro's lemma). The notation  $H \times_G X$  means  $(H \times X)/G$  where G acts as  $g \cdot (h, x) = (hg^{-1}, gx)$ . One can show that X/G is equivalent to  $(G \times X)/(G \times G)$  and also that  $X/G \times X/G$  is equivalent to  $(X \times X)/(G \times G)$ . And the diagonal map  $X/G \to (X/G) \times (X/G)$  corresponds to the action map  $(G \times X)/(G \times G) \to (X \times X)/(G \times G)$  described by  $(g, x) \mapsto (x, g \cdot x)$ .