Taming Moduli Problems in Algebraic Geometry

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Recall from last time that we described a stack $X/G$ where the category $\mathcal{F}$ consists of triples $(U, E, f)$ where $U$ is a scheme, $E$ is a principal $G$-bundle over $U$, and $f : E \to X$ is a $G$-equivariant map.

Another way of thinking is as follows. To any groupoid scheme $X_\bullet$ we can consider the category fibered in groupoids $\underline{X_\bullet}$, where the objects consist of pairs $(U, \xi)$ where $U$ is a scheme and $\xi \in X_0(U)$. The morphisms consist of pairs $(f, \gamma) : (U, \xi') \to (V, \xi)$ where $\gamma \in X_1(U)$ satisfies $t(\gamma) = \xi'$ and $s(\gamma) = f^*(\xi)$. The fiber over $U \in \mathrm{Sch}$ is the groupoid $X_1(U) \rightrightarrows X_0(U)$.

For any fibered category $\mathcal{F}$, there is a canonical stackification $\mathcal{F} \to \mathcal{F}^a$, where $\mathcal{F}^a$ is a stack and the map is universal with respect to maps from $\mathcal{F}$ to a stack. In our example, the stackification satisfies $(X_\bullet)^a = (X/G)$.

We claim that $\underline{X_\bullet}$ does not satisfy descent in general. Indeed, consider the example $\cdot/G$, namely the stackification $G \rightrightarrows \mathrm{pt}$. For each scheme $U$, the functor $\cdot/G$ maps $U$ to the set of $G$-bundles over $U$. However, the functor $G \rightrightarrows \mathrm{pt}$ maps $U$ to a single object groupoid with automorphism group $G(U)$. There is a base-preserving functor

$$
\begin{aligned}
(G \rightrightarrows \mathrm{pt})(U) &\to (\cdot/G)(U) \\
\mathrm{pt} &\mapsto U \times G
\end{aligned}
$$

In general, there is a functor

$$
\begin{aligned}
(G \times X \rightrightarrows X)(U) &\to (X/G)(U) \\
(f : U \to X) &\mapsto (G \times U \xrightarrow{g \cdot f(U)} X)
\end{aligned}
$$

## 1.1 Fiber products

For groupoids $C_1, C_2$ over $D$, the homotopy fiber product is universal with respect to diagrams of the following kind

$$
\begin{array}{ccc}
A & \longrightarrow & C_2 \\
\big\downarrow & & \big\downarrow {\scriptstyle f_2} \\
C_1 & \underset{f_1}{\longrightarrow} & D
\end{array}
$$

where the diagram commutes up to natural transformation. The objects of $C_1 \times_D C_2$ consist of pairs $(X, Y)$ of objects $X \in C_1$, $Y \in C_2$ together with an isomorphism $f_1(X) \to f_2(Y)$. Morphisms are maps $X_1 \to X_2$ and $Y_1 \to Y_2$ which commute with all necessary maps.

We claim that for stacks $\mathfrak{X}_1, \mathfrak{X}_2$ over $\mathfrak{Y}$, the fiber product is still a stack.

There is a 2-Yoneda Lemma which states the following. For $X \in \mathrm{Sch}$, can regard this as a category $\underline{X}$ fibered in groupoids. Indeed, the objects are pairs $(U, f)$ where $f : U \to X$. And morphisms are maps which commute with the maps to $X$. Then there is an equivalence of categories $\mathrm{Map}_{\mathrm{Sch}}(\underline{X}, \mathcal{F}) \simeq \mathcal{F}(X)$ for a category fibered in groupoids in $\mathcal{F}$.

The previous paragraph justifies referring to a stack as representable. We can also define the notion of representable map: Say that $f : \mathfrak{X} \to \mathfrak{Y}$ is representable if for any map $\underline{U} \to \mathfrak{Y}$, the pullback $\mathcal{F}$

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \underline{U} & \longrightarrow & \mathfrak{Y} \end{array}$$

is representable. For any property of morphisms of schemes which is local on the target, we can define such a property in the same way for spaces.

**Theorem 1.1.** *The following are equivalent for a stack $\mathfrak{X}$.*

*(i) $\mathfrak{X} \simeq (\underline{X_\bullet})^a$ for a smooth groupoid scheme*

*(ii) The diagonal functor $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable by algebraic spaces, and there is a smooth surjection from a scheme $U \to \mathfrak{X}$*

*(iii) There is a representable smooth surjection $\underline{U} \to \mathfrak{X}$ from a scheme.*

*Furthermore, these are equivalent to conditions 1,2,3 with "fppf" replacing smooth.*

**Definition 1.2.** Any stack $\mathfrak{X}$ satisfying one of these three equivalent conditions is called **algebraic**.

**Remark 1.3.** Given $U \to \mathfrak{X}$, we get a groupoid $U_0 \times_{\mathfrak{X}} U_0 = U_1 \to U_0 = U$.

## 1.2 Constructing maps between stacks

Let $\psi : G \to H$ be a group homomorphism, let $X$ be a $G$-scheme and $Y$ an $H$-scheme. Then an equivariant map $f : X \to Y$ induces a functor of groupoid schemes $(G \times X \rightrightarrows X) \to (H \times Y \rightrightarrows)$, which in turn induces a map of stacks $f : X/G \to Y/H$. In particular, $f$ is representable by algebraic spaces if and only if the fiber product

$$\begin{array}{ccc} (H \times X)/G & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & Y/H \end{array}$$

is an algebraic space, which is equivalent to saying that $G$ acts freely on $H \times X$. For example, if $G$ is a subgroup of $H$, then $f$ is representable and $X/G \simeq (H \times_G X)/H$ (by Shapiro's lemma). The notation $H \times_G X$ means $(H \times X)/G$ where $G$ acts as $g \cdot (h, x) = (hg^{-1}, gx)$.

One can show that $X/G$ is equivalent to $(G \times X)/(G \times G)$ and also that $X/G \times X/G$ is equivalent to $(X \times X)/(G \times G)$. And the diagonal map $X/G \to (X/G) \times (X/G)$ corresponds to the action map $(G \times X)/(G \times G) \to (X \times X)/(G \times G)$ described by $(g, x) \mapsto (x, g \cdot x)$.