Taming Moduli Problems in Algebraic Geometry

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Last time, we considered the case of $\mathfrak{X} = X/G$ with X a G-quasi-projective scheme. We considered the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. One can show that X/G is equivalent to $(G \times X)/(G \times G)$ and also that $X/G \times X/G$ is equivalent to $(X \times X)/(G \times G)$. And the diagonal map $X/G \to (X/G) \times (X/G)$ corresponds to the action map $(G \times X)/(G \times G) \to (X \times X)/(G \times G)$ described by $(g, x) \mapsto (x, g \cdot x)$.

We claim that $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine. It suffices to check after base change to a smooth cover. By the above discussion, it suffices to show that the orbit map is affine. One can consider the restricted orbit map for an open set $\mathrm{Spec}(R) \to X \times X$:

$$\begin{array}{ccc} G \times \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(R) \times X \\ \uparrow & & \uparrow \Gamma \\ Y & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

The fiber product Y is affine because X is separated and hence the right map is a closed immersion (and thus so is the left map).

Note 1.1. If $\mathfrak{X} \to Y/G$ is representable, then X is equivalent to X/G for some space X with a G-equivariant map $X \to Y$.

Definition 1.2. A **geometric stack** is an algebraic stack which is quasi-compact and $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine.

Consider $\mathfrak{X} = \cdot/G$. Let ξ_1, ξ_2 be principal G-bundles. Let T be a scheme with a morphism $T \to \mathfrak{X} \times \mathfrak{X}$. (I got lost here ... Need to check the notes.)

Remark 1.3. "Linear" objects tend to have affine automorphism groups.

Example 1.4. Let X/k be a proper connected scheme. Then consider the stack $\mathrm{Coh}(X/k)$ whose fiber over a scheme U is the collection of U-flat coherent sheaves on $U \times X$. There is a substack $\mathrm{Bun}(X/k)$ whose fiber over a scheme U is the collection of vector bundles over $U \times X$. There is also the substack $\mathrm{Pic}(X/k)$ whose fiber over any scheme U is the collection of locally free sheaves on $U \times X$.

If X is also projective, then one can consider the substack $\mathrm{Coh}(X/k)_P$ consisting of families with Hilbert polynomial P.

If E_1, E_2 are two U-flat coherent sheaves on $U \times X$, then $\mathrm{Isom}_U(E_1, E_2) \subset \pi_*(\mathrm{Hom}_{U \times X}(E_1, E_2))$ where $\pi : U \times X \to U$ is the projection map. The map $\mathrm{Isom}_U(E_1, E_2) \to U$ is quasi-affine, and in many cases, actually affine. This means that the stack of coherent sheaves has an affine diagonal.

Example 1.5. Let C be a smooth curve over k, and consider the stack whose fiber over U consists of all vector bundlers over $C \times U$. In this case, the connected components are given by the rank and the degree.

Example 1.6. Not all algebraic stacks are geometric. For example, consider the stack of flat families of curves with geometrically connected and reduced fibers and arithmetic genus equal to 1. In particular, a scheme U maps to the groupoid of families of curves over U.

Theorem 1.7. *Let $\mathfrak{X}$ be a geometric stack. If $\mathfrak{X}$ is normal and every coherent sheaf is a quotient of a vector bundle, then $\mathfrak{X} \simeq X/GL_n$ for some quasi-affine scheme X. Furthermore, over k, then $\mathfrak{X} \simeq Spec(R)/G$ for some affine group over k.*