Taming Moduli Problems in Algebraic Geometry

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1.1 Geometric invariant theory

There is a classic book by Mumford-Fogarty-Kirwan, which we will *not* follow too closely. We will follow more closely a work by Alper, in which it is noticed that many properties of GIT quotients are consequences of simple axioms involving $\mathrm{QCoh}(\mathfrak{X})$, which we discuss now.

Lemma 1.1. *Let $\mathfrak{X}$ be a geometric stack. Then every $F \in QCoh(\mathfrak{X})$ is a union of its coherent subsheaves.*

This lemma is a consequence of the fact that a geometric stack has a presentation $X_\bullet$ where X_0 and X_1 are both affine. This comes from the fact that we may chose an affine scheme $X_0 = \mathrm{Spec}(R)$ and an fppf affine map $X_0 \to X$, and taking the fiber product $X_1 = X_0 \times_{\mathfrak{X}} X_0$ is also affine.

Moreover, for a geometric stack $\mathfrak{X}$ we have $\mathrm{QCoh}(\mathfrak{X}) = \mathrm{Ind}(\mathrm{Coh}(\mathfrak{X}))$. This means that

(i) coherent sheaves are finitely presented objects and $\mathrm{Hom}(S, -)$ commutes with filtered colimits for a coherent sheaf S.

(ii) $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Fun}(\mathrm{Coh}(\mathfrak{X})^{op}, \mathrm{Ab})$ is an equivalence of categories.

For any algebraic stack $\mathfrak{X}$, one can show that $\mathrm{QCoh}(\mathfrak{X})$ is a "Grothendieck abelian category."

Definition 1.2. Say that a category is a **Grothendieck abelian category** if it has arbitrary direct sums and filtered colimits, filtered colimits are exact, and there is a generating object U, meaning that for all $M \subset N$, there is a map $U \to N$ which doesn't factor through M.

Theorem 1.3. *In a Grothendieck abelian category there is enough injective objects, and has enough K-injective complexes. (A K-complex is a special type of complex which plays the role of an injective resolution when forming the unbounded derived category.)*

Remark 1.4. There are other definitions of bounded and unbounded derived categories, but they all agree for geometric stacks.

Any map of stacks $f : \mathfrak{X} \to \mathfrak{Y}$ can be modeled as a map of groupoids

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ \uparrow & & \uparrow \\ V_0 & \longrightarrow & U_0 \\ \uparrow & & \uparrow \\ V_1 & \longrightarrow & U_1 \end{array}$$

This implies that there is a pullback functor $f^* : \mathrm{QCoh}(\mathfrak{Y}) \simeq \mathrm{QCoh}(V_\bullet) \to \mathrm{QCoh}(U_\bullet) \simeq \mathrm{QCoh}(\mathfrak{X})$, which is independent of the choices. One can define a pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ as the right adjoint of f^*. We will sometimes think about the derived functor of pushforward Rf_* using injective resolutions or K-injective resolutions.

Example 1.5. Suppose that $f : X/G \to Y$. Let $\tilde{f} : X \to Y$ be a corresponding lift. Given $E \in \mathrm{QCoh}(X/G)$, then $\tilde{f}_*(E|_X) \in \mathrm{QCoh}(Y)$ canonically belongs to $\mathrm{QCoh}(Y \times (\cdot/G))$. The map f factors

$$\begin{array}{ccccc} X/G & \xrightarrow{p} & Y \times (\cdot/G) & \xrightarrow{q} & Y \\ \uparrow & & \uparrow & & \\ X & \longrightarrow & Y & & \end{array}$$

as $q \circ p$. Then $p_*(E)$ arises from $p'_*(E|_X)$ via smooth descent. Moreover q_* is taking invariants under G.

Theorem 1.6. *If G is linearly reductive, then $R\Gamma^i(X/G, E) \simeq R\Gamma^i(X, E|_X)$.*

1.2 Good moduli spaces

Definition 1.7. Let $q : \mathfrak{X} \to Y$ be a map from an algebraic stack $\mathfrak{X}$ to an algebraic space Y. We say that q is a **good moduli space (GMS)** if

(i) $q_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(Y)$ is exact

(ii) $\mathcal{O}_Y \to q_*\mathcal{O}_{\mathfrak{X}}$ is an isomorphism.

Example 1.8. Let G be linearly reductive. Let $X = \mathrm{Spec}(R)$. Then the map

$$\mathrm{Spec}(R)/G \to \mathrm{Spec}(R^G)$$

is a GMS.

We study now the main properties of a GMS $q : \mathfrak{X} \to Y$.

(i) q is surjective, universally closed, universally submersive

(ii) If k is algebraically closed and $x_1, x_2 \in \mathfrak{X}(\bar{k})$, then $q(x_1) = q(x_2)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \varnothing$ in $\mathfrak{X} \times_Z \mathrm{Spec}(k)$.

(iii) The property of being a GMS is stable under base change along $Y' \to Y$ and fppc local on Y.

(iv) If $\mathfrak{X}$ is locally Noetherean, then Y is locally Noetherian. If $\mathfrak{X}$ is finite type over k, then Y is finite type over k.

Example 1.9. Let $\mathbb{C}^2 = \mathbb{C}(1) \oplus \mathbb{C}(-1)$ with the $\mathbb{C}^*$ action indicated by the 1 and -1. We can instead consider the scheme associated to the ring $R = \mathbb{C}[x, y]$ where x has weight one and y has weight -1. The ring of invariants is $R^G = \mathbb{C}[xy]$. There are three types of orbits.

(i) hyperbolas $xy = c \neq 0$ where $\mathbb{C}^*$ acts freely.

(ii) the axes

(iii) the origin

The origin is the intersection of the closures of the axes.

Example 1.10. This is a nonexample. Blow up $\mathbb{C}^2/\mathbb{C}^*$ at the origin. This is isomorphic to the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^1$ with $\mathbb{C}^*$ acting with weight 2 on $\mathbb{P}^1$. We have a map to $\operatorname{Spec}(\mathbb{C}[xy])$, but the corresponding pushforward map will not be exact.