Taming Moduli Problems in Algebraic Geometry

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Remember last time we had the following definition.

Definition 1.1. Let $q : \mathfrak{X} \to Y$ be a map from an algebraic stack $\mathfrak{X}$ to an algebraic space Y. We say that q is a **good moduli space (GMS)** if

(i) $q_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(Y)$ is exact

(ii) $\mathcal{O}_Y \to q_*\mathcal{O}_{\mathfrak{X}}$ is an isomorphism.

We now state some properties of GMS maps.

Proposition 1.2. *(1) Given a Cartesian diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{X} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow{f} & Y \end{array}$$

(a) if f is fppf and q' is GMS, then q is GMS

(b) for f arbitrary, if q is GMS, then q' is GMS.

(2) If $q : \mathfrak{X} \to Y$ is GMS, then the canonical map $F \to q_*q^*(F)$ is an isomorphism.

Proof sketch. (1a): The proof amounts to flat base change. We want to show that $R^iq_*(E) = 0$ for $i > 0$ and $E \in \mathrm{QCoh}(\mathfrak{X})$. It suffices by fpqc descent to show that $f_*R^iq_*(E) = 0$. There is a base change theorem which says that a flat map gives a base change equivalence. Thus $f_*R^iq_*(E) \simeq R^i(q')_*((f')^*(E))$.

(1b): Assume that f is affine. It is a fact that $R^if_*(E) = 0$ for $i > 0$ when f is representable and affine. Because we desire to show that $R^i(q'_*)(E) = 0$, it suffices to show, since $f_* : \mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y)$ is faithful, that $f_*(Ri(q')_*(E)) = 0$. But we have the equivalences $f_*(R^i(q')_*(E)) \simeq R^i(f \circ q')_*(E) \simeq R^iq_*((f')_*(E))$.

(2): First reduce to the case where Y is affine (using flat base change). (Choose an fpqc map $\cup\mathrm{Spec}(A_i) \to Y$. The formation of $q_*q^*(F)$ commutes with flat base change, as is the map $F \to q_*q^*F$.)

If Y is affine, we can find a presentation

$$\mathcal{O}_Y^{\oplus I} \to \mathcal{O}_Y^{\oplus J} \to F \to 0$$

where F is isomorphic to the cokernel of a map of free modules. This implies that

$$\mathcal{O}_{\mathfrak{X}}^{\oplus I} \to \mathcal{O}_{\mathfrak{X}}^{\oplus J} \to q^*F \to 0$$

is exact. Since q_* is exact and $q_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_Y$, we obtain exact

$$\mathcal{O}_Y^{\oplus I} \to \mathcal{O}_Y^{\oplus J} \to q_*q^*F \to 0.$$

As a result, we have the following.

Corollary 1.3. *(i) If $I \subset \mathcal{O}_{\mathfrak{X}}$ is an ideal sheaf for a closed substack, then $q_*(\mathcal{O}_{\mathfrak{X}}/I) \simeq \mathcal{O}_Y/q_*I$.*

(ii) $q_*(I_1)+q_*(I_2) = q_*(I_1+I_2)$. (This follows from the fact that q_* is an exact functor of abelian categories.)

(iii) If $J \subset \mathcal{O}_Y$ is an ideal sheaf and $I \subset \mathcal{O}_{\mathfrak{X}}$ is the preimage ideal sheaf, then the map $J \to q_*I$ is an isomorphism.

These statements about abelian categories have geometric consequences.

Note 1.4. For example, (ii) says that if $Z_1, Z_2 \hookrightarrow \mathfrak{X}$ are closed substacks, then $\mathrm{im}(Z_1)\cap\mathrm{im}(Z_2) = \mathrm{im}(Z_1\cap Z_2)$. This leads to an S-equivalence relation on geometric points, by saying that two geometric points map to the same points of Y if and only if their closures intersect.

As a consequence of (iii), if $\mathfrak{X}$ is Noetherian, then Y is Noetherian. Indeed, given an ascending chain of ideal sheaves on Y

$$J_1 \subset J_2 \subset \cdots \subset \mathcal{O}_Y,$$

we can take the preimages to obtain an ascending chain of ideal sheaves on X

$$I_1 \subset I_2 \subset \cdots \subset \mathcal{O}_{\mathfrak{X}},$$

which stabilizes when X is Noetherian. The pushforwards by q_* must also stabilize eventually, which by (iii), implies that $J_n = q_*I_n$ stabilize as well.

Corollary 1.5 (Hilbert 14)**.** *If R is a finitely generated G-equivariant k-algebra and G is linearly reductive, then R^G is finitely generated.*

Proof. Reduce to the case of a linear action $R = k[V]$ for some representation V of G. (More precisely, there is a surjection $k[V] \to R$ which implies that $k[V]^G \to R^G$ is surjective because G is linearly reductive, from which it follows that if $k[V]^G$ is finitely generated, then so is R^G.)

We have seen that $\mathrm{Spec}(k[V])/G \to \mathrm{Spec}(k[V]^G)$ is GMS. This implies that $k[V]^G$ is Noetherian.

The proof is complete from the fact that a graded ring $A = k \oplus \bigoplus_{n>0} A_n$ is finitely generated if and only if it is Noetherian, which is left as an exercise. □

Remark 1.6. One idea for finding good moduli spaces is the following: Cover a stack $\mathfrak{X}$ by open substacks which have good moduli spaces themselves.

Example 1.7. The map $\mathrm{Spec}(R)/G \to \mathrm{Spec}(R^G)$ is always a GMS for any ring R and any linearly reductive G. One does not even need R to be finitely generated. One way to find an open substack is the following: for $f \in R^G$, then $\{f \neq 0\}$ is G-equivariant. Another way is the following: if $\chi : G \to \mathbb{G}_m$ is a character and $f \in R$ is such that $g \circ f = \chi(g)f$ for each $g \in G$, then $\{f \neq 0\}$ is also G-equivariant and affine; such f is called **semi-invariant**. In fact, given $\chi : G \to \mathbb{G}_m$, can define $\mathrm{Spec}(R)^{\chi-ss}$ to be the set of points $x \in \mathrm{Spec}(R)$ such that there is a χ^n semi-invariant f with $f(x) \neq 0$ for some $n > 0$. One can show that $\mathrm{Spec}(R)^{\chi-ss}$ is the union

$$\bigcup_{f \in \chi\text{-semi-invariant}} \mathrm{Spec}(R[f^{-1}]).$$

Moreover, $\mathrm{Spec}(R)^{\chi-ss}/G$ has a GMS given by the map from Proj of the ring of χ^n-semi-invariants to $\mathrm{Spec}(R^G)$.