Taming Moduli Problems in Algebraic Geometry

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We will now complete the proof of the Hilbert Mumford criterion in the affine case:

Theorem 1.1 (Hilbert-Mumford). *Let $X = Spec(R)$ and let G act on X.*

(1) $x \in X^{ss}(\mathcal{L})$ if and only if $\mu(x, \lambda) \geqslant 0$ for all λ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists.

(2) $x \in X^s_{(0)}(\mathcal{L})$ if and only if $\mu(x, \lambda) > 0$ for all such λ. Here the notation $X^s_{(0)}$ beams the set of $x \in X^s$ such that the stabilizer G_x is finite.

We will in fact only prove (1), leaving it as an exercise for the reader to adapt the argument on [**?**, page 53] to our setting.

Proof. First, reformulate semistability. Let

$$V = \mathrm{Spec}\left(\bigoplus_n \mathcal{L}^n\right) = \mathrm{Tot}_X(\mathcal{L}^\vee).$$

Then we claim that $x \in X$ is $\mathcal{L}$-semistable if and only if for some point x^* in $V \setminus \{0\}$ over $x \in X$, the closure of $G \cdot x^*$ in V does not intersect the zero section. Indeed, notice that an $f \in \Gamma(X, \mathcal{L}^n)^G$ such that $f(x) \neq 0$ determines a function on V with $f(x^*) \neq 0$, but $f(\{0\}) = 0$. For the converse, use the fact that any two disjoint closed subsets of V which are G-equivariant can be separated by $F \in \Gamma(\mathcal{O}_V)^G$.

We want to show that if the closure of the orbit of x^* meets the zero section, then there is a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot x^* \in \{\text{zero section}\}$ (and for such a λ, one will have $\mu(x, \lambda) < 0$.) If the closure meets the zero section, then we can find a diagram of the form

$$\begin{array}{ccccc}
\mathrm{Spec}(k((t^{1/m}))) & \longrightarrow & \mathrm{Spec}(k((t))) & \longrightarrow & \{G \cdot x^*\} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(k[|t^{1/m}|]) & \longrightarrow & \mathrm{Spec}(k[|t|]) & \xrightarrow{f} & \overline{\{G \cdot x^*\}}
\end{array}$$

such that 0 in $\mathrm{Spec}(k[|t|])$ mapsto a point in the zero section. After passing to an etale extension, we can find a lift

$$\mathrm{Spec}(k((t^{1/m}))) \xrightarrow{\phi} G \to \{G \cdot x^*\}$$

of the top map above. We are trying to product a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$. There is a structure theorem for the group $G(k((t)))$ due to Iwahori asserting the following: For any reductive G, there is a correspondence of sets

$$\mathrm{Hom}(\mathbb{G}_m, G)/_{conj}G \longleftrightarrow G(k[|t|]) \setminus G(k((t)))/G(k[|t|]),$$

which is described by [didn't quite catch this part]. As a consequence, there are $g_1, g_2 \in G(k[|t|])$ such that

(i) $g_1|_{k((t))} \cdot \phi \cdot g_2|_{k((t))} = \langle\lambda\rangle$

(ii) the left-hand side extends to $g_1 \cdot f \cdot g_2$

□

We now state a short version of the main theorem of GIT.

Theorem 1.2. *Let $q : \mathfrak{X} \to Y$ be a finite type GMS map, with Y a quasi-compact algebraic space over an algebraically closed field of characteristic zero, and let $\mathcal{L} \in Pic(\mathfrak{X})$. Say $x \in \mathfrak{X}(k)$ is semistable if for all $f : \mathbb{A}^1/\mathbb{G}_m \to \mathfrak{X}$ with $f(1) \simeq x$, we have*

$$wt_{\{0\}}(f^*(\mathcal{L})) \geqslant 0.$$

Then $\mathfrak{X}^{ss}(\mathcal{L}) \subset \mathfrak{X}$ is open and the map

$$\mathfrak{X}^{ss}(\mathcal{L}) \to \underline{\mathrm{Proj}}_Y \left(\bigoplus_{n \geqslant 0} q_*(\mathcal{L}^n) \right)$$

is a GMS, and $\mathcal{L}^N$ descends to a relatively ample bundle for $N >> 0$.

This theorem follows from a result of Alper-Hall-Rydh. They prove informally the following: near any point in any geometric stack $\mathfrak{X}$ whose automorphism group is linearly reductive, we can find an etale neighborhood of the form $\mathrm{Spec}(R)/G$.

Remark 1.3. If Y is not a scheme, then it's not clear that this notion of semistability agrees with the notion from Alper. However, this notion is slightly more general.

Step 1. Any map $\mathbb{A}^1/\mathbb{G}_m \to \mathfrak{X}$ corresponds to a point $y \in Y(k)$ and a map to the fiber $\mathfrak{X}_y$. This implies that the notion of semistability above is etale local over Y.

Step 2. Does this Hilbert-Mumford criterion agree with the one we proved? Is every map $\mathbb{A}^1/\mathbb{G}_m \to \mathrm{Spec}(R)/G$ induced by an equivariant map $\mathbb{A}^1 \to \mathrm{Spec}(R)$? The answer is yes. The case for $G = GL_n$ will be an exercise in the homework.

Proposition 1.4. *There is a correspondence between isomorphism classes of maps $\mathbb{A}^1/\mathbb{G}_m \to X/G$ with pairs (λ, x) such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists modulo the relation $(\lambda, x) \sim (\lambda, px) \sim (g\lambda g^{-1}, g \cdot x)$ where $g \in G$ and $p \in P_\lambda = \{$block upper triangular matrices w.r.t. eigenspaces of $\lambda\}$.*

Theorem 1.5. *If $\mathfrak{X}$ is finite type with affine diagonal and has a GMS $q : \mathfrak{X} \to Y$, then there is a linearly reductive group G and a cartesian diagram of the form*

$$\begin{array}{ccc} Spec(R)/G & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow q \\ Spec(R^G) & \xrightarrow[\text{étale surj.}]{} & Y \end{array}$$

Suppose in particular that $X \hookrightarrow \mathbb{P}^n$ is a subvariety which is acted upon by G, then define the semistable points in the following manner. Let $X^{ss}(\mathcal{O}_X(1))$ be the set of points x for which there is an $n > 0$ and an $f \in \Gamma(X, \mathcal{O}_X(n))^G$ such that $f(x) \neq 0$. Each of these X_f's is a G-equivariant affine scheme, which cover X^{ss}. The theorem is the following.

Theorem 1.6. *The map*

$$X^{ss}/G \to Proj\left(\bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n)^G)\right)$$

is a good moduli space.

This is a special case of the affine GIT, because we can consider the affine cone $C(X) \subset \mathbb{A}^{n+1}$ over X with an action of $G \times \mathbb{G}_m$. Can choose the line bundle $\mathcal{L} = \mathcal{O}_{C(X)} \otimes \mathbb{C}(1)$. With this choice,

$$C(X)^{ss}(\mathcal{L})/G \times \mathbb{G}_m \simeq X^{ss}/G.$$

In fact, the same type of reasoning works for X which is even projective-over-affine. The Hilbert-Mumford criterion is identical to that of the affine case in this case as well.