Taming Moduli Problems in Algebraic Geometry

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From last time: If X is a variety such that $X \to \operatorname{Spec}(\mathcal{O}_X)$ is projective (equivalently, X is a closed G-equivariant subvariety of $\mathbb{P}^n \times \mathbb{A}^m$). Let $\mathcal{L} = \mathcal{O}(1)$. For any invariant global section $f \in \Gamma(X, \mathcal{L}^n)^G$, the set $X_f = \{x \in X | f(x) \neq 0\}$ is a G-equivariant affine. Moreover, the map

$$q : \bigcup_f X_f/G \to \operatorname{Proj}\left(\bigoplus_{n \geqslant 0} \Gamma(\mathcal{L}^n)^G\right)$$

is a good moduli space. This GMS is projective over $\operatorname{Spec}(\Gamma(\mathcal{O}_X)^G)$. Moreover, the Hilbert-Mumford criterion still holds as stated. (To see this, one notes that X/G is a substack of $\operatorname{Spec}(\oplus_{n\geqslant 0}\Gamma(\mathcal{L}^n))/G \times \mathbb{C}^*$.)

Remark 1.1. The quotient X/G itself does not necessarily have a good moduli space because for example G-equivariant sheaves could have higher cohomology.

Some immediate consequences of the Hilbert-Mumford criterion include.

(i) $X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}^n)$ for $n > 0$. As a result, one can consider GIT for any G-linearized ample $\mathcal{L}$. Additionally, stability is well-defined with respect to $\mathcal{L} \in \operatorname{Pic}(X/G) \otimes \mathbb{Q}$.

(ii) $X^{ss}(\mathcal{L})$ depends only on $c_1(\mathcal{L}) \in H^2_G(X, \mathbb{Q})$. Indeed, the criterion only depends on the weight of a pullback of $\mathcal{L}$ at the origin, which depends only on the cohomology class. (This is useful because the cohomology group is finite-dimensional, whereas the group of line bundles could be very large.)

(iii) perturbation of stability: How does $X^{ss}(\mathcal{L} + \epsilon\mathcal{L}')$ compare to $X^{ss}(\mathcal{L})$ for small $\epsilon \in \mathbb{Q}$? The answer is that $X^{ss}(\mathcal{L} + \epsilon\mathcal{L}') \subset X^{ss}(\mathcal{L})$. The informal idea is the following: For any unstable point $p \in X/G$, there is a map $f : \mathbb{A}^1/\mathbb{G}_m \to X/G$ taking 1 to p such that $\operatorname{wt}(f^*\mathcal{L}|_{\{0\}}) < 0$, but then for small ϵ we still have $\operatorname{wt}(f^*(\mathcal{L} + \epsilon\mathcal{L}')|_{\{0\}}) = \operatorname{wt}(f^*\mathcal{L}|_{\{0\}}) + \epsilon \cdot \operatorname{wt}(f^*\mathcal{L}'|_{\{0\}}) < 0$.

    For example, if X is affine and $\mathcal{L} = \mathcal{O}_X$, then $X^{ss}(\mathcal{O}_X) = X$. We saw that $X^{ss}(\mathcal{O}_X + \epsilon\mathcal{L}) \subset X^{ss}(\mathcal{O}_X)$.

(iv) If $Y/G \to X/G$ is a representable finite map, then $Y^{ss}(\pi^{-1}(\mathcal{L})) = \pi^{-1}(X^{ss}(\mathcal{L}))$. An important special case includes closed immersions. To prove this, the idea is that properness implies that for each $y \in \pi^{-1}(x)$ and each map $f : \mathbb{A}^1/\mathbb{G}_m \to X/G$ satisfying $f(1) = x$, there is a unique lift $\tilde{f} : \mathbb{A}^1/\mathbb{G}_m \to Y/G$ such that $\tilde{f}(1) = y$.

Remark 1.2. An $\mathcal{L} \in \operatorname{Pic}(X^{ss}(\mathcal{L})/G)$ has the property that for each $f : \mathbb{A}^1/\mathbb{G}_m \to X^{ss}(\mathcal{L})/G$, we have $\operatorname{wt}(f^*\mathcal{L}|_{\{0\}}) = 0$. Or for each $x \in X^{ss}(\mathcal{L})$ and $\lambda : \mathbb{G}_m \to G$ fixing x, we have $\operatorname{wt}_\lambda(\mathcal{L}_x) = 0$.

Example 1.3. Let $Y = (\mathbb{P}^1)^n$. There is an action of SL_2 on Y. Let

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(r_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(r_n).$$

All one-parameter subgroups $\lambda : \mathbb{G}_m \to SL_2$ are conjugate to $\mathrm{diag}(t^k, t^{-k})$. This is equivalent to choosing a coordinate system on $\mathbb{P}^1$. The limit point of $t \cdot [\alpha : \beta]$ as $t \to 0$ is

$$\begin{cases} [0:1] & \beta \neq 0 \\ [1:0] & \text{else.} \end{cases}$$

Notice that

$$\mathrm{wt}_\lambda \mathcal{O}_{\mathbb{P}^1}(r_i)|_{[0:1]} = r_i$$
$$\mathrm{wt}_\lambda \mathcal{O}_{\mathbb{P}^1}(r_i)|_{[0:1]} = -r_i.$$

For a point $y = (\ell_1, \ldots, \ell_n) = ([\alpha_1 : \beta_1], \ldots, [\alpha_n : \beta_n])$, we have

$$\mathrm{wt}_\lambda(\mathcal{L}_{\lim_{t\to 0} \lambda(t)\cdot y}) = \sum_i \pm r_i$$

where the sign is $+$ if $\beta_i \neq 0$ and $-$ otherwise. Note that this weight is $\geqslant 0$ if and only if

$$\sum_{\ell_i = [1:0]} r_i \leqslant \sum_{\ell_i \neq [1:0]} r_i.$$

In general, a point $y = (\ell_1, \ldots, \ell_n)$ is $\mathcal{L}$ semi-stable if and only if for all $\ell \in \mathbb{P}^1$, we have

$$\sum_{\ell_i = \ell} r_i \leqslant \sum_{\ell_i \neq \ell} r_i.$$

Remark 1.4. If $r_1 = \cdots = r_n$ in the example above, then $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$ under the map $(\mathbb{P}^1)^n \to \mathbb{P}(\mathrm{Sym}^n(k^2))$. It follows that a point $\varphi(x, y) \in \mathrm{Sym}^n(k^2)$ is semistable if and only if there is no linear factor of multiplicity $> n/2$.

Remark 1.5. Can ask how $Y^{ss}(\mathcal{L})$ varies as $(r_1, \ldots, r_n)$ $(\mathbb{Q}_{>0})^n$. The condition $y \in Y^{ss}(\mathcal{L})$ amounts to a finite set of linear inequalities on $(r_1, \ldots, r_n)$. This example is a first stepping stone into the theory of variation of GIT quotients.

Another consequence of the Hilbert-Mumford criterion is the following. Fix a maximal torus $T \subset G$. A point $x \in X$ is G-semistable if and only if for each $g \in G$, the point $g \cdot x$ is T-semistable. Indeed, the implication T-unstable $\implies$ G-unstable is immediate. If X is G-unstable, then there is a point (x, λ) with $\mu(x, \lambda) < 0$ and up to conjugation, λ belongs to T, i.e., we can find a $g \in G$ such that $(gx, g\lambda g^{-1})$ is a

T-destabilizing datum.

Example 1.6. Let SL_3 act on $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^3)$, the space of degree 3 curves in $\mathbb{P}^2$. What are the semistable points?

First, consider T-semistability. A maximal torus is isomorphic to $\mathbb{G}_m^2$. One such torus is $\mathrm{diag}(t_1, t_1^{-1}t_2, t_2^{-1})$.

Consider the character of the T-representation $\mathrm{Sym}^3\mathbb{C}^3$. [There is a diagram of points corresponding to the action on the standard eigenbasis for $\mathrm{Sym}\mathbb{C}^3$ in $M_\mathbb{R}$ which I could not draw in real time.] A map $\lambda : \mathbb{G}_m \to T$ can be thought of as a co-direction in the diagram. Take a point p and consider its coordinates $(\alpha_1, \ldots, \alpha_n)$ with respect to the eigenbasis. Then for λ, the limit point $\lim_{t\to 0} \lambda(t) \cdot p$ is the projection of p onto the lowest weight eigenspace in which p has a connzero coefficient. Define the subset $\mathrm{St}(p)$ to be the convex hull in $M_\mathbb{R}$ of weights for which $\alpha_i \neq 0$. It follows that p is T- semistable if and only if $\mathrm{St}(p)$ contains the origin.