Taming Moduli Problems in Algebraic Geometry

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Recall the set-up from last time.

Example 1.1. Let SL_3 act on $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^3)$, the space of degree 3 curves in $\mathbb{P}^2$. What are the semistable points?

First, consider T-semistability. A maximal torus is isomorphic to $\mathbb{G}_m^2$. One such torus is $\mathrm{diag}(t_1, t_1^{-1}t_2, t_2^{-1})$.

Consider the character of the T-representation $\mathrm{Sym}^3\mathbb{C}^3$. [There is a diagram of points corresponding to the action on the standard eigenbasis for $\mathrm{Sym}\mathbb{C}^3$ in $M_\mathbb{R}$ which I could not draw in real time.] A map $\lambda : \mathbb{G}_m \to T$ can be thought of as a co-direction in the diagram. Take a point p and consider its coordinates $(\alpha_1, \ldots, \alpha_n)$ with respect to the eigenbasis. Then for λ, the limit point $\lim_{t\to 0} \lambda(t) \cdot p$ is the projection of p onto the lowest weight eigenspace in which p has a connzero coefficient. Define the subset $\mathrm{St}(p)$ to be the convex hull in $M_\mathbb{R}$ of weights for which $\alpha_i \neq 0$. It follows that p is T- semistable if and only if $\mathrm{St}(p)$ contains the origin.

The Hilbert Mumford criterion implies that

(i) p is T-semistable if and only if the state polytope $\mathrm{St}(p) \subset M_\mathbb{R}$ contains the origin.

(ii) p is G-semistable if and only if $g \cdot p$ is T semistable for all $g \in G$. Indeed, we saw this last time.

Moreover, more is true, after fixing a Weyl-invariant inner product $|\cdot|$ (e.g. the Killing form) on the cocharacter lattice N. Given a state polytope $\mathrm{St}(p) \subset M_\mathbb{R}$, there is a unique $\lambda \in N_\mathbb{R}$ (up to scale) which minimizes

$$\nu(p,\lambda) = \frac{1}{|\lambda|}\mu(p,\lambda) = \max_{\chi \in \mathrm{St}(p)} \langle -\frac{\lambda}{|\lambda|}, \chi\rangle.$$

The function

$$\lambda \mapsto \max_{\chi \in \mathrm{St}(p)} \langle \lambda, \chi\rangle$$

is convex upward on $N_\mathbb{R}$. If one restricts to the unit sphere in $N_\mathbb{R}$, the function is continuous and strictly convex upward.

Lemma 1.2. *There is a unique closest point $\chi \in St(p)$ to the origin, and the λ which minimizes $\nu(p,\lambda)$ is dual to χ under $|\cdot|$.*

Proof. Exercise. □

As a result, there is a unique $\lambda \in N_\mathbb{R}$ which minimizes

$$\nu(p,\lambda) = \frac{1}{|\lambda|}\mathrm{wt}_\lambda(\mathcal{O}(1)_{p_0}).$$

Note that this quantity does not depend on the choice of torus T, only depending on $|\cdot|$ and $\mathcal{O}(1)$.

Kempf's theorem says the following: There is a λ, unique up to conjugation by $g \in P_\lambda$, which minimizes $\nu(p,\lambda)$ for any unstable p.

Remark 1.3. Recall that maps $\mathbb{A}^1/\mathbb{G}_m \to X/G$ up to isomorphism are equivalent to test data (p,λ) up to equivalence

Theorem 1.4 (Kempf)**.** *Let G be a reductive group. Let X be a projective-over-affine variety over k with linearized G action. Let $\mathcal{L} \in NS_G(X)$ be a line bundle. Fix a Weyl-invariant inner product on $N = cochar(T)$. Assume that $\mathcal{L}|_X$ is NEF. Then*

(i) For each unstable point $p \in X^{us}$, there is a unique map $f : \mathbb{A}^1/\mathbb{G}_m \to X/G$ with an isomorphism $f(1) \simeq p$ which minimizes $\nu(f) = \nu(p,\lambda)$, up to ramified covering $(p,\lambda) \mapsto (p,\lambda^n)$.

(ii) If p specializes to q, then $M(q) \leqslant M(p)$, where $M(p) = \min_{(p,\lambda)} \nu(p,\lambda)$. (That is, the set of points $\{p \in X : M(p) \leqslant c\}$ is closed for any $c < 0$.)

(iii) Up to conjugation, only finitely many λ appear as optimal destabilizers in (i).

Remark 1.5. Recall that maps $\mathbb{A}^1/\mathbb{G}_m \to X/G$ up to isomorphism are equivalent to test data (p,λ) up to equivalence $(p,\lambda) \sim (p, g\lambda g^{-1})$ for $g \in P_\lambda$ and $(p,\lambda) \sim (g \cdot p, g\lambda g^{-1})$ for any $g \in G$.

We will prove the theorem in the case where X is affine. The non-affine case when $\mathcal{L}$ is G-ample can be reduced to the affine case.

Proof. The idea of the proof is to use an object known as the "spherical building" $\mathrm{Sph}(G)$ of G. The object $\mathrm{Sph}(G)$ is a large topological space constructed in the following manner

(1) For any maximal torus T, let S_T denote the unit sphere in $\mathrm{cochar}(T)_\mathbb{R}$.

(2) For any Borel subgroup $T \subset B \subset G$, we get a top dimensional cone (Weyl chamber) in $\mathrm{cochar}(T)_\mathbb{R}$. Intersecting this cone with the unit sphere gives a polyhedral sector $\Delta_B \subset S_T$. (One can do the same procedure for other parabolic subgroups $T \subset P \subset G$ and the cone will be lower dimensional.)

(3) We glue S_T to $S_{T'}$ along Δ_B for each borel $B \supset T, T'$.

Then $\mathrm{Sph}(G)$ is the resulting glued space. (One can also think of the spherical building as all sectors Δ_P glued together along the inclusions $\Delta_{P'} \subset \Delta_P \supset \Delta_{P''}$ for $P'' \supset P \subset P'$. We can think of Δ_P as parameterizing dominant $\lambda : \mathbb{G}_m \to P$ up to conjugation and ramification $\lambda \mapsto \lambda^n$.)

With these conventions, the function

$$\nu(p,\lambda) = \frac{1}{|\lambda|}\mathrm{wt}_\lambda(\mathcal{L}_{p_0}), \qquad p_0 = \lim_{t\to 0} \lambda(t)p$$

extends to a continuous function on a subset $\mathrm{Deg}(p)$ of $\mathrm{Sph}(G)$, which parameterizes $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot p$ exists. Kempf's theorem says that there is a unique minimizer of $\nu : \mathrm{Deg}(p) \to \mathbb{R}$. ☐