TAMING MODULI PROBLEMS IN ALGEBRAIC GEOMETRY: EXERCISES

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1. Equivariant geometry

1.1. **Bialynicki-Birula strata.** Let $X' \to X$ be an equivariant map of schemes with $\mathbb{G}_m$-action, and let Y' and Y be the corresponding Bialynicki-Birula schemes defined in class. Show that there is a canonically induced map $Y' \to Y$ and that

(1) if $X' \hookrightarrow X$ is a closed immersion, then

$$\begin{tikzcd} Y' \arrow[r, hook, "j'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook, "j"] & X \end{tikzcd}$$

is Cartesian, and

(2) if $X' \subset X$ is an open immersion, then

$$\begin{tikzcd} Y' \arrow[r, "\pi'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

is Cartesian.

The maps π and j are the canonical projection from the BB stratum to the fixed locus, and the canonical immersion from the BB stratum into the ambient variety which we defined in lecture.

1.2. **On equivariant line bundles.** This is a warmup for the next exercise. Let X be a G-scheme over an algebraically closed field k. Let $\mathrm{Pic}_G(X)$ be the group of invertible sheaves on X/G. Then pullback along the map $X \to X/G$ corresponds to the forgetful map $\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$. Likewise elements of $\mathrm{Pic}(\mathrm{pt}/G)$ are one-dimensional representations of G, i.e. characters of G, and the pullback functor $\mathrm{Pic}(\mathrm{pt}/G) \to \mathrm{Pic}(X/G)$ is given by $\chi \mapsto \mathcal{O}_X \otimes \chi$. Determine conditions under which

$$0 \to \mathrm{Pic}(\mathrm{pt}/G) \to \mathrm{Pic}(X/G) \to \mathrm{Pic}(X)$$

is exact.

1.3. **Normal projective G-varieties are G-projective.** The goal is to prove that if G is a connected reductive group, then every normal projective variety X admits a G-linearized ample bundle. Familiarize yourself with the proof of this fact from Mumford, Kirwan, & Fogarty's GIT book (this is part of the exercise – understand the proof to the extent that you can explain it to a fellow student).

Step 1 Show that every line bundle fixed, as a point in $\mathrm{Pic}(X/k)$, by the action of G admits a G-equivariant structure after raising it to a suitably high power

Step 2 Using the fact that the underlying reduced subschemes of the components of $\mathrm{Pic}(X/k)$ are abelian varieties, because X is normal and projective, show that G must act trivially on $\mathrm{Pic}(X)$.

I claim that you have enough information to prove the key claim in Step 2, on the rationality of G, at least assuming $k = \bar{k}$:

Let G be a connected reductive group over an algebraically closed field. Consider the fiber bundles $G \to G/T \to G/B$. In the case of $G = GL_3$, can you use this to show that G is uni-rational, i.e. dominated by a rational variety. Of course, you already know that GL_3 is rational, but can you adapt the argument to prove the same for arbitrary reductive groups G? If I told you that any principal B-bundle on a scheme is Zariski-locally trivial, can you use this to show that G is in fact rational?

1.4. **Descent.** First some terminology:

- The *banal groupoid* associated to a map of schemes $f : X \to Y$ is the groupoid $X \times_Y X \rightrightarrows X$ (we often omit the identity section and the multiplication map from the data of a groupoid, only indicating the source and target maps in the notation).
- The *simplicial nerve* of a groupoid is the simplicial scheme $X_\bullet$ where X_n consists of "composable n-tuples of arrows."
- The *Cech nerve* of a map $f : X \to Y$ is the simplicial nerve of the banal groupoid of f, so $X_n = X \times_Y X \times_Y \cdots \times_Y X$ with $n+1$-factors.

For descent of quasi-coherent sheaves, it suffices to consider the truncation of the simplicial nerve, by which we mean the piece of the simplicial diagram in schemes consisting of only X_0, X_1, and X_2.

The exercise:

(1) Verify the main claim in the proof of Morita invariance for the category of quasi-coherent sheaves on a groupoid: that the diagram $W_{\bullet,\bullet}$ whose columns are the truncations of the Cech nerve of the map $W_{00} \to Y_0$ and its base change to Y_1, Y_2 coincides with the diagram formed by takind the truncated Cech nerve of the map $W_{00} \to X_0$ and its base change to X_1, X_2.
(2) Use the method from class (see also Behrend's notes on cohomology of stacks) to deduce that quasi-coherent sheaves satisfy fppf descent using as an input the fact that modules satisfy fppf descent for maps of rings.
(3) Use the method from class, and the fact that smooth maps admit sections étale locally on the base, to deduce that any category fibered in groupoids over the category of schemes which has descent for surjective étale maps also has descent for smooth surjective maps.

1.5. **A pathological example of a quotient scheme.** Show that $\mathbb{C}(1)^n \oplus \mathbb{C}(-1)^m \ \{0\}$ has a quotient scheme by $\mathbb{G}_m$ which is not separated.

1.6. **Some facts about quotient stacks.**

(1) Prove the claim from class: if $f : X \to Y$ is a map from a G-scheme to an H-scheme which is equivariant with respect to a homomorphism $G \to H$, then the natural square

$$\begin{tikzcd} H \times_G X \arrow[r, "\mathrm{pr}_X"] \arrow[d, "(h,x)\mapsto h\cdot f(x)"'] & X/G \arrow[d] \\ Y \arrow[r] & Y/H \end{tikzcd}$$

is Cartesian.
(2) Let X be a G-scheme and let $\mathcal{Y} \to X/G$ be a map of stacks which is representable by spaces (respectively schemes). Then there is an algebraic space (respectively scheme) Y with a G action and an isomorphism $Y/G \times \mathcal{Y}$ under which the composition $Y/G \to X/G$ is equivalent to the map induced by a G-equivariant map $Y \to X$.

Remark 1.1. The second exercise suggests the following definition: a *G-action* on a stack $\mathcal{X}$ is the data of a Cartesian diagram of stacks

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{pt}/G \end{tikzcd} .$$

This is a reasonable definition because a G action on a scheme X can be canonically recovered from the corresponding diagram of this kind, where $\mathcal{X} = X$ and $\mathcal{X}' = X/G$.

1.7. **The stack of coherent sheaves.** We will need the following theorem of Grothendieck: if X is a projective k-scheme and E is a coherent sheaf on X, then there is a projective k-scheme $\mathrm{Quot}(E)$ (pronounced "quote") such that a map $T \to \mathrm{Quot}(E)$ is a T-flat coherent sheaf $\mathcal{F}$ on $T \times X$ along with a surjection $\mathcal{O}_T \boxtimes E \twoheadrightarrow \mathcal{F}$.

(1) Show that there is an open subscheme of $U_{n,m} \subset \mathrm{Quot}(\mathcal{O}_X(-n)^{\oplus m})$ classifying families of quotients $\mathcal{O}_X(-n)^{\oplus m} \twoheadrightarrow F$ such that $H^i(X, F(n)) = 0$ for all $i > 0$.
(2) By definition the scheme $U_{n,m}$ comes with a universal coherent sheaf $\mathcal{F}$ on $U_{n,m} \times X$, forgetting the surjection $\mathcal{O}_X(-n)^{\oplus m} \twoheadrightarrow \mathcal{F}$. This defines a map $U_{n,m} \to \underline{\mathrm{Coh}}(X)$, where the latter is the stack parameterizing flat families of coherent sheaves (discussed in lecture). Given any map $T \to \underline{\mathrm{Coh}}(X)$, describe the fiber product $\mathcal{X}$

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & U_{n,m} \arrow[d] \\ T \arrow[r] & \underline{\mathrm{Coh}}(X) \end{tikzcd} ,$$

and show that $\mathcal{X}$ is an open subscheme of a vector bundle over T.
(3) Show that the map from the disjoint union $\bigsqcup_{n,m} U_{n,m} \to \underline{\mathrm{Coh}}(X)$ is surjective and smooth (where we take the definition of a smooth map to only require the map to be locally finitely presented, not finitely presented).

1.8. **Rings of invariants.** Here we walk through some basic facts about the ring of invariants for a $\mathbb{G}_m$-equivariant algebra R (all over a field k). The key fact you should use is the equivalence between the category of $\mathbb{G}_m$-modules and graded vector spaces, where the invariant subspace is the degree 0 piece. This is a good exercise to help you understand the general discussion in MFK's GIT book, Chapter 1, "the affine case" (where $\mathbb{G}_m$ is replaced with a linearly reductive group).

(1) Show that any map $\mathrm{Spec}(R)/G \to \mathrm{Spec}(A)$ factors uniquely through the projection $\mathrm{Spec}(R)/G \to \mathrm{Spec}(R^G)$.
(2) Show that the formation of ring of invariants commutes with base change along an arbitrary map $\mathrm{Spec}(S) \to \mathrm{Spec}(R^G)$, i.e. $(S \otimes_{R^G} R)^G \simeq S$.
(3) Use (2) to show that for any separated scheme X, any map $\mathrm{Spec}(R)/G \to X$ factors uniquely through the map $\mathrm{Spec}(R)/G \to \mathrm{Spec}(R^G)$.
(4) Use (2) to show that for any ideal $I \subset R^G$, we have $(R \cdot I) \cap R^G = I$, and use this to conclude that R^G is Noetherian if R is Noetherian.
(5) Show that a nonnegatively graded ring $R = \bigoplus_{n \geq 0} R_n$ with $R_0 = k$ is finitely generated if and only if it is Noetherian. Use this to show that for an arbitrary finitely generated G-equivariant algebra R^G is finitely generated (Hint: consider an equivariant surjection $\mathrm{Sym}(V) \twoheadrightarrow R$ for some linear G-representation V.
(6) Show that for any two closed equivariant subvarieties $X, Y \subset \mathrm{Spec}(R)$ which do not intersect, there is an invariant function $f \in R^G$ such that $f|_X = 1$ and $g|_Y = 0$.

(7) Use (6) to show that every fiber of $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ contains a unique closed orbit.

1.9. **Hilbert-Mumford criterion.** Let V be a linear representation of a torus T, and let $v \in V$ be a vector. Let N and M denote the cocharacter and character lattice of T respectively, and fix an inner product $|\bullet|$ on $N_{\mathbb{R}}$. We shall evaluate the Hilbert-Mumford criterion for the point $p = [v] \in \mathbb{P}(V)$. Let $\mathrm{St}(p) \subset M_{\mathbb{R}} = M \otimes \mathbb{R}$ denote the "state polytope" of p, defined in lecture as the convex hull of the characters $\chi \in M$ for which the projection of v onto the χ-eigenspace of v is non-zero.

(1) Show that
$$\nu(\lambda, p) := \max_{\chi \in \mathrm{St}(p)} \left\langle -\frac{\lambda}{|\lambda|}, \chi \right\rangle,$$
regarded as a function of $\lambda \in N_{\mathbb{R}} \setminus \{0\}$, is minimized if and only if λ lies on the unique ray connecting the origin to the point on $\mathrm{St}(p)$ closest to the origin with respect to the inner product $|\bullet|$.

(2) Show by example that the point $p_0 = \lim_{t \to 0} \lambda(t) \cdot p \in \mathbb{P}(V)$, where λ is the unique minimizer from (1), depends on the inner product $|\bullet|$, even though the question of whether p is semi-stable does not depend on $|\bullet|$.

1.10. **Generalized filtrations.**

1.10.1. *Vector bundles on $\mathbb{A}^1/\mathbb{G}_m$.*

(1) Show that the category of vector bundles on $\mathbb{A}^1/\mathbb{G}_m$ and linear maps between them (not just isomorphisms) is equivalent to the category of finite dimensional vector spaces (over the fixed ground field k) along with a weighted descending filtration $\cdots V^{p+1} \subset V^p \subset \cdots \subset V$ such that $V^p = 0$ for $p \gg 0$, where the maps in the latter category are linear maps $V \to W$ mapping $V^p \to W^p$ for all p.

(2) Show that every vector bundle on $\mathbb{A}^1/\mathbb{G}_m$ is isomorphic to $\mathcal{O}_{\mathbb{A}^1} \otimes V$ where V is some finite dimensional representation of $\mathbb{G}_m$.

(3) Establish an analogous description of the category of vector bundles on $\mathbb{A}^2/\mathbb{G}_m^2$ in terms of $\mathbb{Z}^2$-weighted filtrations.

(4) Use this description to prove the following key fact, used in the proof of Kempf's theorem: Given a $\mathbb{G}_m^2$-equivariant vector bundle on $\mathbb{A}^2 \setminus \{(0,0)\}$, there is an extension to an equivariant bundle on $\mathbb{A}^2$ which is unique up to unique isomorphism.