EXISTENCE OF MODULI SPACES FOR ALGEBRAIC STACKS

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Abstract. We provide necessary and sufficient conditions for when an algebraic stack admits a good moduli space. This theorem provides a generalization of the Keel–Mori theorem to moduli problems whose objects have positive dimensional automorphism groups. We also prove a semistable reduction theorem for points of algebraic stacks equipped with a theta-stratification. Using these results we find conditions for the good moduli space to be separated or proper. To illustrate our method, we apply these results to construct proper moduli spaces parameterizing semistable $\mathcal{G}$-bundles on curves.

Contents

1. Introduction

In the study of moduli problems the construction of moduli spaces is a recurring problem. Classically, most moduli spaces have been constructed via geometric invariant theory, but this requires a description of the underlying moduli stack as a global quotient, which can be difficult to find.

For algebraic stacks with finite automorphism groups the Keel-Mori theorem finally gave a satisfactory existence result. This can often be used to construct coarse moduli spaces of stable objects. Many examples have semistable object which are not stable, however, and therefore can not have a coarse moduli space.

Our main result, which gives necessary and sufficient conditions under which an algebraic stack admits a good moduli space in the sense of [Alp13] is the following.

Theorem. *Theorem 4.1 Let $\mathcal{X}$ be an algebraic stack, of finite type over a Noetherian algebraic space S, with affine diagonal. Then $\mathcal{X}$ admits a good moduli space if and only if*

(1) $\mathcal{X}$ is locally linearly reductive (Definition 2.1);
(2) $\mathcal{X}$ is Θ-reductive (Definition 3.9); and
(3) $\mathcal{X}$ has unpunctured inertia (Definition 3.54).

The coarse moduli space X is separated if and only if $\mathcal{X}$ is S-complete (Definition 3.37).

Let us give an informal explanation of the above conditions. The first condition is that closed points of $\mathcal{X}$ have linearly reductive stabilizers. In the language of geometric invariant theory this would amount to the condition that the automorphism groups of polystable objects are (linearly) reductive. The second condition is the geometric analog of the statement that filtrations by semistable objects extend under specialization. This is formulated in terms of maps from the stack $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ into $\mathcal{X}$. The third condition is an analog of the condition in the Keel-Mori theorem, it roughly states that the connected components of stabilizer groups extend to closed points. In particular this condition is automatic if all stabilizer groups are connected (which happens for example for moduli of coherent sheaves). In Section 6 we provide some tools to verify this condition.

Finally, S-completeness is a geometric property that is reminiscent of properties of the classical notion of S-equivalence.

The condition of linear reductivity is very strong in positive characteristic and it arises here, through the recent local structure theorems on algebraic stacks from [AHR15],[AHR] which are a key ingredient in our proof. However, we are careful to prove intermediate results that do not require this condition.

For example we find an analog of Langton's semistable reduction theorem for moduli of bundles, that works for a large class of algebraic stacks equipped with a notion of stability that induces a Θ-stratification, a geometric analog of the notion of Harder-Narasimhan-Shatz stratifications. As in Langton's theorem, the statement is that if a family of objects parametrized by a discrete valuation ring specializes to a point that is more unstable than the generic fiber of the family, then one can modify the family along the closed point to get a family that has the same stability properties as the generic fiber. Surprisingly the existence of modifications can be obtained from the local geometry of Θ-stratifications. The formal statement is the following.

Theorem (Theorem 5.3)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a a Noetherian algebraic space S, and let $\mathcal{S} \hookrightarrow \mathcal{X}$ be a Θ-stratum (Definition 5.1). Let R be a discrete valuation ring with fraction field K and residue field k. Let $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ be an R-point such that the generic point ξ_K is not mapped to $\mathcal{S}$, but the special point ξ_k is mapped to $\mathcal{S}$:*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, hook] \arrow[d, "\xi_K"] & \operatorname{Spec} R \arrow[d, "\xi_R"] & \operatorname{Spec} k \arrow[l, hook'] \arrow[d, "\xi_k"] \\ \mathcal{X} - \mathcal{S} \arrow[r, hook, "j"] & \mathcal{X} & \mathcal{S}. \arrow[l, hook', "\iota"'] \end{tikzcd}$$

Then there exists an elementary modification of ξ such that $\xi' : \operatorname{Spec}(R') \to \mathcal{X}$ lands in $\mathcal{X} - \mathcal{S}$.

To illustrate our results we give include some applications that may be of independent interest. First, we use the semistable reduction theorem to give a proof that the Hitchin fibration for semistable G-Higgs bundles is proper if the characteristic of the ground field is not too small. This result is of course expected, but it doesn't seem to appear in the literature.

Second we apply our existence theorem to provide some new coarse moduli spaces. Namely we construct proper coarse moduli spaces for semistable $\mathcal{G}$-bundles on curves in characteristic 0, generalizing work of Balaji and Seshadri.

Let us mention that in positive characteristics we would expect to have variants of the above results. For the existence of coarse moduli spaces the main obstacle to proving that adequate moduli spaces exist for stacks for which closed points have geometrically reductive stabilizers instead of linearly reductive stabilizers is the lack of an analog of the local structure theorem for such stacks. Similarly we expect that in the semistable reduction theorem weak Θ-strata (that only require canonical filtrations to exist after a purely inseparable extension) should be sufficient, which are available in greater generality in positive characteristic. Again, the main obstruction for this generalization is a version of the local structure theorem that allows to replace the embedding by a radicial map.

2. Preliminaries

As our arguments build on the one hand on local structure theorems and on the other hand on notions that came up in the study of notions of stability on algebraic stacks, we briefly recall these results in this section.

Throughout we will fix a base S that will be a quasi-separated algebraic space, but of course the most interesting case for most readers will be that $S = \operatorname{Spec}(k)$ is the spectrum of a field.

2.1. **Reminder on local structure theorems for algebraic stacks.** For ease of notation let us introduce the following terminology.

Definition 2.1. An algebraic stack $\mathcal{X}$ with affine stabilizers is *locally linearly reductive* if the closed points are dense in $\mathcal{X}$ and every closed point of $\mathcal{X}$ has a linearly reductive stabilizer.

Note that in the case of a quasi-compact quotient stack $\mathcal{X} = [X/G]$ the closed points correspond to closed orbits of G on X, so in this case the above condition only requires that points in the closed orbits have a linearly reductive stabilizer. In particular, a locally linearly reductive stack will often have geometric points with non-reductive stabilizers.

Definition 2.2. If $\mathcal{X}$ is an algebraic stack and $x \in |\mathcal{X}|$ is a point with residual gerbe $\mathcal{G}_x$, we call an étale and affine pointed morphism $f: (\mathcal{W}, w) \to (\mathcal{X}, x)$ of algebraic stacks a *local quotient presentation around* x if (1) $\mathcal{W} \cong [\operatorname{Spec}(A)/\operatorname{GL}_N]$ for some N and (2) $f|_{f^{-1}(\mathcal{G}_x)}$ is an isomorphism.

Theorem 2.3. [AHR, Thm. 1.1] *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation over S, with affine diagonal. If $x \in |\mathcal{X}|$ is a point with image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and the stabilizer of x is linearly reductive, then there exists a local quotient presentation $f: (\mathcal{W}, w) \to (\mathcal{X}, x)$ around x.*

In particular, if in addition $\mathcal{X}$ is locally linearly reductive, then there exist local quotient presentations around any closed point.

Remark 2.4. If S is the spectrum of an algebraically closed field, this follows from [AHR15, Thm. 1.2]. In this case, one can arrange that there is a local quotient presentation $(\mathcal{W}, w) \to (\mathcal{X}, x)$ with $\mathcal{W} \cong [\operatorname{Spec} A/G_x]$, the quotient of an affine scheme by the stabilizer G_x of x.

Remark 2.5. While GL_N is linearly reductive in characteristic 0, it is not linearly reductive in positive or mixed characteristic. For the same reason, the morphism $[\operatorname{Spec}(A)/\operatorname{GL}_n] \to \operatorname{Spec}(A^{\operatorname{GL}_N})$ will only be an adequate moduli space (and not a good moduli space) in general.

To prove the semistable reduction theorem, we will need a relative version of the above local structure theorem where we fix a subgroup isomorphic to the multiplicative group $\mathbb{G}_m$ of the stabilizer G_x, but do not assume G_x to be linearly reductive. A very general result of this form is the following theorem.

Theorem 2.6. [Hal14, Theorem B.1] *Let $\mathcal{X}$ be a quasi-compact quasi-separated algebraic stack, finitely presented over a noetherian base S with affine stabilizer groups. Let $\mathcal{Y} \subset \mathcal{X}$ be a closed substack and $p_Y: [Y/\mathbb{G}_m^r] \to \mathcal{Y}$ a representable, smooth surjective map with Y affine over S. Then there exists a $\mathbb{G}_m^r$-scheme U that affine over S together with a representable smooth map $p_U: [U/\mathbb{G}_m^r] \to \mathcal{X}$ such that there exists a factorization $[U/\mathbb{G}_m^r] \times_{\mathcal{X}} \mathcal{Y} \xrightarrow{f} [Y/\mathbb{G}_m^r] \xrightarrow{p_Y} \mathcal{Y}$ where f is representable étale and surjective.*

Remark 2.7. As the proof of the result has not yet appeared let us recall a special case, which will be sufficient for us if $S = \operatorname{Spec}(k)$ is the spectrum of a field and all stabilizer groups of $\mathcal{X}$ are smooth (a condition that is automatic in characteristic 0). Namely, if $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field, $x \in \mathcal{X}(k)$ with smooth automorphism group G_x, $\mathcal{Y} = BG_x \subset \mathcal{X}$ is the canonical inclusion, $\mathbb{G}_m^r \subset G_x$ is a subgroup and $Y = [\operatorname{Spec} k/\mathbb{G}_m^r]$ the above result is a special case of [AHR15, Theorem 1.2].

2.2. **Reminder on mapping stacks and filtrations.** As in [Hal14] we will denote by $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ the quotient stack defined by the standard contracting action of the multiplicative group on the affine line and by $B\mathbb{G}_m = [\mathrm{pt}/\mathbb{G}_m]$ the classifying stack of the group $\mathbb{G}_m$. Both stacks are defined over $\mathrm{Spec}(\mathbb{Z})$ and therefore pull back to any base S. Note that since $\mathbb{G}_m$ is a linearly reductive group, the structure morphism $\Theta \to \mathrm{Spec}(\mathbb{Z})$ and $B\mathbb{G}_m \to \mathrm{Spec}(\mathbb{Z})$ are good moduli spaces.

Maps from Θ into a stack are the key ingredient to define stability notions on algebraic stacks ([Hal14],[Hei17]) and we need to recall some of their properties.

By definition for any stack $\mathcal{X}$ and point $\mathrm{Spec}\, k \to S$ a map $B\mathbb{G}_{m,k} \to \mathcal{X}$ is a point $x \in \mathcal{X}(k)$ together with a cocharacter $\mathbb{G}_{m,k} \to \mathrm{Aut}_{\mathcal{X}}(x)$. As the action of $\mathbb{G}_m$ on a vector space is the same as a grading on the vector space, we often think of a morphism $B\mathbb{G}_m \to \mathcal{X}$ as a point of $\mathcal{X}$ equipped with a grading.

Similarly, a vector bundle on $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ is the same as a $\mathbb{G}_m$ equivariant bundle on $\mathbb{A}^1$ and these are the same as vector spaces equipped with a filtration. So we think of morphisms $f\colon \Theta_k \to \mathcal{X}$ as an object of $x_1 \in \mathcal{X}(k)$ (the object $f(1)$) together with a grading of x_1 and as $f(0) = x_0$ as the associated graded object.

In examples it is often easy to see that once one has found that some moduli problem is described by an algebraic stack, the stacks of filtered or graded objects are again algebraic. This turns out to be a general phenomenon, which we recall next. For algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ over S, we denote by

$$\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$$

the stack over S parameterizing S-morphisms $\mathcal{Y} \to \mathcal{X}$. If $\mathcal{Y}$ is defined over $\mathrm{Spec}(\mathbb{Z})$, we will the convention that $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$ denotes the mapping stack $\underline{\mathrm{Map}}_S(\mathcal{Y} \times S, \mathcal{X})$.

That these mapping stacks are again algebraic if $\mathcal{Y} = \Theta$ or $\mathcal{Y} = B\mathbb{G}_m$ for quite general $\mathcal{X}$ follows from a general result established in [AHR] and [HLP14, Thm. 1.6]: if $\mathcal{X}$ is locally of finite presentation and quasi-separated over S with affine stabilizers, and $\mathcal{Y}$ is of finite presentation over S with affine diagonal such that $\mathcal{Y} \to S$ is flat and a good moduli space, then $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$ is an algebraic stack, locally of finite presentation over S, with quasi-separated diagonal. Moreover, if $\mathcal{X} \to S$ has affine (resp. quasi-affine, resp. separated) diagonal, then so does $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$.

2.3. **The example of quotient stacks.** In case that $\mathcal{X} = [X/G]$ is a quotient stack, where G is a smooth algebraic group acting on a quasi-separated agebraic space X, these mapping stacks have a classical interpretation ([Hal14, Thm. 1.49]). To state this recall that given $\lambda\colon \mathbb{G}_m \to G$, one defines

$$L_\lambda = \{l \in G \,|\, l = \lambda(t) l \lambda(t)^{-1}\ \forall t\} \quad \text{and} \quad P_\lambda^+ = \{p \in G \,|\, \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists}\}.$$

If G is geometrically reductive, then $P_\lambda^+ \subset G$ is a parabolic subgroup. There is a surjective homomorphism $P_\lambda^+ \to L_\lambda$, defined by $p \mapsto \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1}$.

Similarly, one defines the functors:

$$\begin{aligned} X_\lambda^0 &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathrm{Spec}(k), X) && \text{(the fixed locus)} \\ X_\lambda^+ &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathbb{A}^1, X) && \text{(the attractor)} \\ \widetilde{X}_\lambda &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}_{\mathbb{A}^1}(\mathbb{A}^2, X \times \mathbb{A}^1) && \text{(the interpolator)} \end{aligned}$$

here $\mathbb{G}_m$ acts on $\mathbb{A}^1$ with weight 1 and on $\mathbb{A}^2$ with weights $(1, -1)$. By [Dri13, Thm. 1.4.2], these functors are representable by algebraic spaces. Moreover, there are the following natural morphisms: a closed immersion $X^0_\lambda \hookrightarrow X$, an unramifed morphism $X^0_\lambda \to X$ (given by evaluation at 1) and an affine morphism $X^+_\lambda \to X^0_\lambda$ (given by evaluation at 0). If X is separated, then $X^0_\lambda \to X$ is a monomorphism.

The k-points of X^0_λ are simply the λ-fixed points, and if X is separated, the k-points of X^+_λ are the points $x \in X(k)$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists. The generic fiber of $\widetilde{X}_\lambda \to \mathbb{A}^1$ is X while the special fiber over 0 is $X^+_\lambda \times_{X^0_\lambda} X^-_\lambda$, where $X^-_\lambda := X^+_{\lambda^{-1}}$ is the "repeller." There is a morphism $\widetilde{X}_\lambda \to X \times X \times \mathbb{A}^1$ where the two maps to X are obtained from restricting along the two maps $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 1)$ and $x \mapsto (1, x)$.

The algebraic space X^0_λ inherits an action of L_λ and X^+_λ inherits an action of P^+_λ such that the evaluation map $\mathrm{ev}_0 : X^+_\lambda \to X^0_\lambda$ is equivariant with respect to the surjection $P^+_\lambda \to L_\lambda$.

We can now recall the description of our mapping stacks for quotient stacks:

Proposition 2.8. [Hal14, Thm. 1.49] *Let X be a quasi-separated algebraic space over an algebraically closed field k equipped with an action of a smooth algebraic group G over k. Let Λ be a complete set of conjugacy classes of one-parameter subgroups $\mathbb{G}_m \to G$. Then there are isomorphisms*

$$\underline{\mathrm{Map}}_k(B\mathbb{G}_m, [X/G]) \cong \bigsqcup_{\lambda\in\Lambda} [X^0_\lambda / L_\lambda];$$

$$\underline{\mathrm{Map}}_k(\Theta, [X/G]) \cong \bigsqcup_{\lambda\in\Lambda} [X^+_\lambda / P^+_\lambda].$$

Moreover, the morphism $\mathrm{ev}_1 : \underline{\mathrm{Map}}_k(\Theta, [X/G]) \to [X/G]$ is induced by the $(P^+_\lambda \to G)$-equivariant morphism $X^+_\lambda \to X$. The morphism $\mathrm{ev}_0 : \underline{\mathrm{Map}}_k(\Theta, [X/G]) \to \underline{\mathrm{Map}}(B\mathbb{G}_m, [X/G])$ is induced by the $(P^+_\lambda \to L_\lambda)$-equivariant morphism $X^+_\lambda \to X^0_\lambda$.

Remark 2.9. In [Hal14, Thm. 1.49], it is assumed that the action is 'locally affine' but one can check that the statement is valid without this hypothesis.

3. Specialization properties on stacks

For our argument it will be important to understand the behavior of the stacks $\mathrm{Map}(\Theta, \mathcal{X})$ under morphisms $\mathcal{X} \to \mathcal{Y}$, i.e., study the behavior of filtrations on objects under morphisms.

3.1. **Properties of morphisms preserved under passage to stacks of filtrations.**

We will need that some properties of morphisms of algebraic stacks pass to the induced morphism on mapping stacks. Most of these are not hard, but the last point requires an extra argument.

Lemma 3.1. *Let S be a quasi-separated algebraic space. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation and quasi-separated over S, with affine stabilizers. Suppose f satisfies one of the following properties*

(a) representable;
(b) monomorphism;
(c) separated;
(d) unramifed; or
(e) étale,
(f) representable, étale and surjective,

then $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ has the same property.

Proof. Properties (a) and (b) are clear. Property (c) follows from the valuative criterion and descent. Properties (d) and (e) follow from the formal lifting criterion and descent. For (f), it remains to show that $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is surjective. Let $h\colon \Theta_k \to \mathcal{Y}_s$ be a morphism over a geometric point $s\colon \operatorname{Spec}(k) \to S$. We will use Tannaka duality to construct a lift to $\mathcal{X}$. As any étale representable cover of $B\mathbb{G}_{m,k}$ admits a section, we may choose a lift $B\mathbb{G}_{m,k} \to \mathcal{X}_s$ of $B\mathbb{G}_{m,k} \hookrightarrow \Theta_k \xrightarrow{h} \mathcal{Y}_s$. Let $\Theta_k^{[n]} = [\operatorname{Spec}(k[x]/x^{n+1})/\mathbb{G}_m]$ be the nth nilpotent thickening of $B\mathbb{G}_m \hookrightarrow \Theta$. Since f is étale, there exist compatible lifts $\Theta_k^{[n]} \to \mathcal{X}_s$ of $\Theta_k^{[n]} \hookrightarrow \Theta_k \xrightarrow{h} \mathcal{Y}_s$. Since Θ_k is coherently complete along $B\mathbb{G}_{m,k}$, by [AHR15, Cor. 3.6], there is an equivalence of categories $\operatorname{Hom}_k(\Theta_k, \mathcal{X}_s) = \varprojlim_n \operatorname{Hom}_k(\Theta_k^{[n]}, \mathcal{X}_s)$. This constructs the desired lift $\Theta_k \to \mathcal{X}_s$ of h. $\square$

Remark 3.2. Property (f) is not preserved if the representability hypothesis is dropped. For instance, if $\mathcal{X} = B\mathbb{G}_m \to B\mathbb{G}_m = \mathcal{Y}$ is induced by $\mathbb{G}_m \to \mathbb{G}_m, t \to t^d$ for $d > 1$, then $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is not surjective.

3.2. **Property Θ-$\mathcal{P}$.** If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks over an algebraic space S, we denote by $\mathrm{ev}(f)_1$ the induced morphism of stacks

$$\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}, \mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}), \qquad \lambda \mapsto (\mathrm{ev}_1(\lambda), f \circ \lambda),$$

i.e., this morphism takes an object together with a filtration in $\mathcal{X}$ and remembers the object together with the induced filtration on the image in $\mathcal{Y}$. It sits in a diagram:

$$\begin{tikzcd}
\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \arrow[dr, "\mathrm{ev}(f)_1"] \arrow[drr, bend left] \arrow[ddr, bend right, "\mathrm{ev}_1"'] & & \\
& \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[r, "p_2"] \arrow[d, "p_1"] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[d, "\mathrm{ev}_1"] \\
& \mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd} \tag{1}$$

Definition 3.3. Let $\mathcal{P}$ be a property of morphisms of algebraic stacks over a quasi-separated algebraic space S. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, locally of finite presentation and quasi-separated over S with affine stabilizers, has *property* Θ-$\mathcal{P}$ if $\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}, \mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ has property $\mathcal{P}$. We say that $\mathcal{X}$ has *property* Θ-$\mathcal{P}$ if $\mathcal{X} \to S$ does.

For example a morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ is Θ-*surjective* if on can lift filtrations on any point $f(x)$ to filtrations on x.

The assignment $f \mapsto \mathrm{ev}(f)_1$ behaves well with respect to compositions and base change. Namely, given a composition $g \circ f\colon \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ of morphisms of algebraic stacks over S, then

$$\begin{aligned}
\mathrm{ev}(g \circ f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) &\xrightarrow{\mathrm{ev}(f)_1} \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \\
&\xrightarrow{\mathrm{id} \times \mathrm{ev}(g)_1} \mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{Z}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Z})) \cong \mathcal{X} \times_{\mathcal{Z}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Z}),
\end{aligned}$$

and if

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

is a Cartesian diagram of algebraic stacks over S, then

(2)
$$\begin{tikzcd} \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}') \arrow[r, "\mathrm{ev}(f')_1"] \arrow[d] & \mathcal{X}' \times_{\mathcal{Y}'} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \arrow[d] \\ \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \arrow[r, "\mathrm{ev}(f)_1"] & \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[r] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \end{tikzcd}$$

is Cartesian. We conclude:

Proposition 3.4. *Let $\mathcal{P}$ be a property of morphisms of algebraic stacks. If $\mathcal{P}$ is stable under composition and base change, then so is the property Θ-$\mathcal{P}$. If $\mathcal{P}$ is stable under fppf (respśmooth, resp. étale) descent, then Θ-$\mathcal{P}$ is stable under descent by morphisms $\mathcal{Y}' \to \mathcal{Y}$ such that $\underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is fppf (resp. smooth and surjective, resp. étale and surjective).* □

Lemma 3.5. *Let $\mathcal{P}$ be a property of representable morphisms of algebraic stacks. If $\mathcal{P}$ is stable under étale descent, then Θ-$\mathcal{P}$ is stable under descent by representable, étale and surjective morphisms.*

Proof. This follows immediately from Proposition 3.4 and Lemma 3.1(f). □

Lemma 3.6. *Let S be a quasi-separated algebraic space. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation and quasi-separated over S, with affine stabilizers. Assume $\mathcal{X}$ has separated diagonal.*

(1) *The morphism $\mathrm{ev}(f)_1$ is representable.*
(2) *If f is separated, then so is $\mathrm{ev}(f)_1$.*
(3) *If f is representable and separated, then $\mathrm{ev}(f)_1$ is a monomorphism.*
(4) *If $f\colon \mathcal{X} \to S$ is a Deligne-Mumford stack, then $\mathrm{ev}(f)_1$ is an isomorphism.*
(5) *If f is étale, then so is $\mathrm{ev}(f)_1$.*
(6) *If f is representable, étale, and separated, then $\mathrm{ev}(f)_1$ is an open immersion.*

Proof. For (1), by Diagram 1, it suffices to show that $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X}$ is representable. Let k be a field and $\lambda \in \underline{\mathrm{Map}}_S(\Theta, \mathcal{X})(k)$ be a k-valued point corresponding to a morphism $\lambda\colon \Theta_k \to \mathcal{X}$. Suppose that α_1, α_2 are two automorphisms of λ whose restrictions to $\mathrm{ev}_1(\lambda)$ agree. This gives a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d] & I_{\mathcal{X}} \arrow[d] \\ \Theta_k \arrow[r, "\lambda"'] \arrow[ur, shift left, "\alpha_1"] \arrow[ur, shift right, "\alpha_2"'] & \mathcal{X}. \end{tikzcd}$$

Since $I_{\mathcal{X}} \to \mathcal{X}$ is separated, $\alpha_1 = \alpha_2$.

Part (2) follows from Lemma 3.1(c).

For (3), to show that $\mathrm{ev}(f)_1$ is a monomorphism, we need to show that for every affine scheme $\mathrm{Spec}(R)$, any commutative diagram of solid arrows

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

can filled in with a dotted arrow. As f is representable and separated, the base change $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \Theta_R \to \Theta_R$ is a closed immersion containing the dense set $\mathrm{Spec}(R)$; it is therefore an isomorphism.

Part (5) follows directly from Lemma 3.1(e) using Diagram 1. Part (6) follows directly from Parts (3) and (5) as étale monomorphisms are open immersions.

For (4), choose an étale cover $f\colon U \to \mathcal{X}$ where U is a scheme. Since any étale, representable cover of Θ_k admits a section by Lemma 3.1(f), $\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, U) \to U \times_{\mathcal{X}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ is surjective. But by Part (6) this means that $\mathrm{ev}(f)_1$ is an fact an isomorphism. As $\underline{\mathrm{Map}}_S(\Theta, U) \to U$ is an isomorphism, by descent $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X}$ is an isomorphism as well.

$\square$

Remark 3.7. The morphism $\mathrm{ev}(f)_1$ is not in general quasi-compact. For an example, if $f\colon B\mathbb{G}_{m,k} \to \mathrm{Spec}(k)$, the morphism $\mathrm{ev}(f)_1$ is the evaluation morphism is $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_S(\Theta, B\mathbb{G}_{m,k}) = \bigsqcup_{n\in\mathbb{Z}} B\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k}$.

Remark 3.8. If f is representable but not separated, then $\mathrm{ev}(f)_1$ is not necessarily a monomomorphism. See Example 3.18.

3.3. **Θ-reductive morphisms.** In this section, we study the class of Θ-reductive morphisms as introduced in [Hal14]. As before, we set $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ defined over $\mathrm{Spec}(\mathbb{Z})$. If R is a DVR with fraction field K, we set $0 \in \Theta_R := \Theta \times \mathrm{Spec}(R)$ to be the unique closed point. Observe that a morphism $\Theta_R \setminus 0 \to \mathcal{X}$ is the data of morphisms $\mathrm{Spec}(R) \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

Definition 3.9. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks is *Θ-reductive* if for every DVR R, any commutative diagram

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{3}$$

of solid arrows can be uniquely filled in.

Remark 3.10. Let S be a Noetherian algebraic space and $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite type and quasi-separated over S, with affine stabilizers. Then f is Θ-reductive if and only if $\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ satisfies the valuative criterion for properness, that is, for every discrete valuation ring R with fraction field K, any diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \arrow[d, "\mathrm{ev}(f)_1"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \end{tikzcd}$$

of solid arrows can be uniquely filled in. Note that the morphism $\mathrm{ev}(f)_1$ is always representable (Lemma 3.6(1)) and locally of finite type. However, the morphism $\mathrm{ev}(f)_1$ is not in general quasi-compact (see Remark 3.7) and therefore $\mathrm{ev}(f)_1$ is not in general proper.

3.4. **Examples illustrating Θ-reductivity.** In the following examples, we work over an algebraically closed field k. The following proposition gives a criterion using the notation from §2.3 for when a quotient stack $[X/G]$ is Θ-reductive.

Proposition 3.11. *Let $\mathcal{X} = [X/G]$ be a quotient stack, where X is a quasi-separated algebraic space locally of finite type over an algebraically closed field k and G is a (smooth but not necessarily connected) reductive algebraic group over k. Then $\mathcal{X}$ is Θ-reductive if and only if for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the morphism $X^+_\lambda \to X$ is proper.*

Remark 3.12. If X is separated, then $X^+_\lambda \to X$ is proper if and only if it is a closed immersion.

Proof. This follows easily from the explicit description of the mapping stack $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ in Proposition 2.8. Indeed, there is a factorization

$$\mathrm{ev}_1\colon [X^+_\lambda/P^+_\lambda] \to [X/P^+_\lambda] \to [X/G]$$

and since G is reductive, each $P^+_\lambda \subset G$ is a parabolic subgroup. Since the quotient G/P^+_λ is projective, the morphism $[X^+_\lambda/P^+_\lambda] \to [X/P^+_\lambda] \to [X/G]$ is proper. Thus properness of $\mathrm{ev}_1|_{[X^+_\lambda/P^+_\lambda]}$ is equivalent to properness of $X^+_\lambda \to X$. □

In order to develop some intuition for Θ-reductivity, we use this result to provide some basic examples and counterexamples of Θ-reductivity. For an integer n, we denote by $\lambda_n\colon \mathbb{G}_m \to \mathbb{G}_m$ the one-parameter subgroup defined by $t \mapsto t^n$; in this way, the integers $\mathbb{Z}$ index the one-parameter subgroups of $\mathbb{G}_m$.

Example 3.13. Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$. Then

$$X^+_{\lambda_n} = \begin{pmatrix} V(y) & \text{if } n > 0 \\ \mathbb{A}^2 & \text{if } n = 0 \\ V(x) & \text{if } n < 0 \end{pmatrix}$$

The evaluation morphism restricted to the component indexed by λ_n is $[X^+_{\lambda_n}/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ which is induced by the inclusion $X^+_{\lambda_n} \to X$. We see directly that $[X/\mathbb{G}_m]$ is Θ-reductive.

Example 3.14. Generalizing Example 3.13, if $X = \mathrm{Spec}(A)$ is an affine scheme of finite type over k with an action of a reductive algebraic group G, then $[X/G]$ is Θ-reductive. Indeed, if $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then A inherits a $\mathbb{Z}$-grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$. If I^-_λ denotes the ideal generated by homogeneous elements of strictly negative degree, then it is easy to see that $X^+_\lambda = V(I^-_\lambda)$; see [Dri13, §1.3.4]. Thus, $X^+_\lambda \to X$ is a closed immersion and the conclusion follows from the characterization in Proposition 3.11.

Example 3.15. Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2 \setminus 0$ via $t \cdot (x, y) = (tx, y)$. Then

$$X^+_{\lambda_n} = \begin{pmatrix} \{y \neq 0\} & \text{if } n > 0 \\ X & \text{if } n = 0 \\ V(x) & \text{if } n < 0 \end{pmatrix}$$

and we see that $[X/\mathbb{G}_m]$ is *not* Θ-reductive as $X^+_{\lambda_n} \to X$ is not proper for $n > 0$. Similarly, for a DVR R, the algebraic stack $\Theta_R \setminus 0$ is not Θ-reductive. These are the prototypical examples of non-Θ-reductive stacks.

Example 3.16. Consider the multiplication action of $\mathbb{G}_m$ on $X = \mathbb{P}^1$ via $t \cdot [x, y] = [tx, y]$. Then

$$X^+_{\lambda_n} = \begin{cases} \mathbb{P}^1 \setminus \{0\} \sqcup \{0\} & \text{if } n > 0 \\ \mathbb{P}^1 & \text{if } n = 0 \\ \mathbb{P}^1 \setminus \{\infty\} \sqcup \{\infty\} & \text{if } n < 0 \end{cases}$$

We see that $[\mathbb{P}^1/\mathbb{G}_m]$ is *not* Θ-reductive.

Example 3.17. Consider the action of $\mathbb{G}_m$ on the nodal cubic $X \subset \mathbb{P}^2$. Let $\pi \colon \mathbb{P}^1 \to X$ be the $\mathbb{G}_m$-equivariant normalization where $\mathbb{G}_m$ acts on $\mathbb{P}^1$ via $t \cdot [x, y] = [tx, y]$. Then

$$X^+_{\lambda_n} = \begin{cases} \mathbb{P}^1 \setminus \{0\} & \text{if } n > 0 \\ C & \text{if } n = 0 \\ \mathbb{P}^1 \setminus \{\infty\} & \text{if } n < 0 \end{cases}$$

and $X^+_{\lambda_n} \to X$ is induced via π. Here $[X/\mathbb{G}_m]$ is *not* Θ-reductive.

Example 3.18. Consider the $\mathbb{G}_m$-action on $X = \mathbb{A}^2 \cup_{\mathbb{A}^2 \setminus 0} \mathbb{A}^2$ via $t \cdot (x, y) = (tx, y)$ on both copies. Then

$$X^+_{\lambda_n} = \begin{cases} \mathbb{A}^2 \cup_{\mathbb{A}^2 \setminus \{y=0\}} \mathbb{A}^2 & \text{if } n > 0 \\ X & \text{if } n = 0 \\ \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k) & \text{if } n < 0 \end{cases}$$

Thus $[X/\mathbb{G}_m]$ is *not* Θ-reductive. The evaluation morphism ev_1 does satisfy the existence part of valuative criterion but not the uniqueness; that is, $[X/\mathbb{G}_m]$ is not Θ-separated.

3.5. **Properties of Θ-reductive morphisms.** We now give a few properties of Θ-reductive morphisms. First observe from Proposition 3.4 and Lemma 3.5 that Θ-reductive morphisms are stable under composition and base change, and that they descend under representable, étale and surjective morphisms. We first show that one can check the lifting criterion of (3) on complete DVRs.

Proposition 3.19. *A morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks is Θ-reductive if and only if for every complete DVR R and any commutative diagram (3) of solid arrows, there exists a unique dotted arrow filling in the diagram.*

Proof. One can adapt the standard argument for schemes as in [Hei17, Rmk. 2.5]. □

Proposition 3.20.

1. *An affine morphism of locally Noetherian algebraic stacks is Θ-reductive.*
2. *Let S be a locally Noetherian scheme. Let $G \to S$ be a geometrically reductive and étale-locally embeddable group scheme acting on a locally Noetherian scheme X affine over S. Then the morphism $[X/G] \to S$ is Θ-reductive.*

(3) A good moduli space $\mathcal{X} \to X$, where $\mathcal{X}$ is a locally Noetherian algebraic stack with affine diagonal, is Θ-reductive.

Remark 3.21. In the case that $S = \operatorname{Spec}(k)$ where k is an algebraically closed field, Part (2) implies that $[\operatorname{Spec}(A)/G]$, where G is a geometrically reductive algebraic group, is Θ-reductive. In the case that G is smooth, then this follows from the explicit calculation in Example 3.14.

Proof. For (1), since $0 \in \Theta_R$ has codimension 2 and Θ_R is regular for a DVR R, we have that $(\Theta_R \setminus 0 \to \Theta_R)_* \mathcal{O}_{\Theta_R \setminus 0} = \mathcal{O}_{\Theta_R}$. Given an affine morphism $f\colon \mathcal{X} \to \mathcal{Y}$, we have canonical isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &\cong \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_* \mathcal{O}_{\mathcal{X}}, (\Theta_R \setminus 0 \to \mathcal{Y})_* \mathcal{O}_{\Theta_R \setminus 0}) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_* \mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_* \mathcal{O}_{\Theta_R}) \\
&\cong \operatorname{Hom}_{\mathcal{Y}}(\Theta_R, \mathcal{X}).
\end{aligned}$$

Part (2), since Θ-reductive morphisms descend under representable, étale and surjective morphisms, we may assume that S is an affine Noetherian scheme and that G is a closed subgroup of $\mathrm{GL}_{N,S}$ for some N. We first show that $B\,\mathrm{GL}_{N,\mathbb{Z}}$ is Θ-reductive, which implies that $B\,\mathrm{GL}_{N,S}$ is also Θ-reductive. A morphism $\Theta_R \setminus 0 \to \mathcal{X}$ corresponds to a vector bundle $\mathcal{E}$ on $\Theta_R \setminus 0$. If $\widetilde{\mathcal{E}}$ is any coherent sheaf on Θ_R extending $\mathcal{E}$, then the double dual $\widetilde{\mathcal{E}}^{\vee\vee}$ is a vector bundle extending $\mathcal{E}$. This provides the desired extension $\Theta_R \to \mathcal{X}$. Since $\mathrm{GL}_{N,S}/G$ is affine ([Alp14, Thm. 9.4.1]), $BG \to B\,\mathrm{GL}_n$ is affine. By Part (1), BG is Θ-reductive. Since X is affine over S, $[X/G] \to BG$ is affine which implies using again Part (1) that $[X/G]$ is Θ-reductive.

For (3), we may assume that X is quasi-compact. By Theorem 2.3 ([AHR, Thm. A.1]), there exists an étale cover $\operatorname{Spec}(B) \to X$ such that $\mathcal{X} \times_X \operatorname{Spec}(B) \cong [\operatorname{Spec}(A)/\mathrm{GL}_N]$ for some N and $B = A^{\mathrm{GL}_N}$. Since Θ-reductive morphisms descend under representable, étale and surjective morphisms, this reduces to the statement that $[\operatorname{Spec}(A)/\mathrm{GL}_N] \to \operatorname{Spec}(A^{\mathrm{GL}_N})$ is Θ-reductive which follows from Part (2). ☐

Proposition 3.22. *A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks, such that $\mathcal{X}$ and $\mathcal{Y}$ both have quasi-finite inertia, is Θ-reductive.*

Proof. Let R be a DVR with fraction field K with residue field k. We first claim that any morphism $\Theta_R \to \mathcal{Y}$ to a Noetherian algebraic stack with quasi-finite inertia factors through $\Theta_R \to \operatorname{Spec}(R)$. By fpqc descent, it suffices to check this for complete DVRs. Let $\mathcal{S}^{[n]} \subset \Theta_R$ be the nth nilpotent thickening of $B\mathbb{G}_{m,k} \subset \Theta_R$. The morphism $\mathcal{S}^{[0]} \to \mathcal{Y}$ necessarily factors as $\mathcal{S}^{[0]} \to \operatorname{Spec}(k) \xrightarrow{y} \mathcal{Y}$. Choose a smooth presentation $U \to \mathcal{Y}$ and a lift $u\colon \operatorname{Spec}(k) \to U$ of y. By the lifting criterion for formally smooth morphisms, there are compatible lifts $\mathcal{S}^{[n]} \to U$ of $\mathcal{S}^{[0]} \to \operatorname{Spec}(k) \xrightarrow{u} U$. As Θ_R is coherently complete along $B\mathbb{G}_{m,k}$, the Tannaka duality implies that $\Theta_R \to \mathcal{Y}$ factors as $\Theta_R \to U \to \mathcal{Y}$ but the former morphism clearly factors through $\Theta_R \to \operatorname{Spec}(R)$.

Similarly, any morphism $\Theta_K \to \mathcal{X}$ factors through $\operatorname{Spec}(K) \to \mathcal{X}$. Therefore, any morphism $\Theta_R \setminus 0 = \operatorname{Spec}(R) \bigcup_{\operatorname{Spec}(K)} \Theta_K \to \mathcal{X}$ factors through $\operatorname{Spec}(R) \to \mathcal{X}$. In the valuative criterion (3), the composition $\Theta_R \to \operatorname{Spec}(R) \to \mathcal{X}$ gives the unique lift of $\Theta_R \to \operatorname{Spec}(R) \to \mathcal{Y}$. ☐

3.6. **Θ-surjective morphisms.** In this section, we study the class of Θ-surjective morphisms; that is, morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ (of algebraic stacks, locally of finite presentation and quasi-separated over a quasi-separated algebraic space S, with affine stabilizers) such that

$$\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}(\Theta, \mathcal{Y})$$

is surjective. We already know from Proposition 3.4 and Lemma 3.5 that Θ-surjective morphisms are stable under composition and base change, and that they descend under representable, étale and surjective morphisms.

Remark 3.23. The condition of Θ-surjectivity translates nicely into the following lifting criterion: For a field k, denote by $i\colon \mathrm{Spec}(k) \hookrightarrow \Theta_k$ the open immersion. Then $f\colon \mathcal{X} \to \mathcal{Y}$ is Θ-surjective if and only if for any algebraically closed field k, any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d, "i"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_k \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{4}$$

of solid arrows can be filled in with a dotted arrow.

Remark 3.24. If f is representable and separated, it follows from Lemma 3.6(3) that there is at most one lift in Diagram 4, that is, f is Θ-injective. This fails for non-separated morphisms; see Example 3.32.

We also note that if f is proper, then the valuative criterion for properness implies that there exists a unique lift in the above diagram. Therefore proper morphisms are Θ-bijective.

Lemma 3.25. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k such that either*

(1) $\mathcal{X}$ is locally linearly reductve; or
(2) $\mathcal{X} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N.

Then any specialization $x \rightsquigarrow x_0$ of k-points where x_0 is a closed point is a realized by a morphism $\Theta_k \to \mathcal{X}$.

Proof. The first case follows from the second by Theorem 2.3 while the second case follows from the Hilbert-Mumford criterion [Kem78, Thm. 4.2]. □

Lemma 3.26. *Let S be a quasi-separated algebraic space and $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation over S. Suppose that $\mathcal{Y}$ is locally linearly reductive and f is Θ-surjective. If $x \in |\mathcal{X}|$ is a point with image $s \in |S|$ such that $x \in |\mathcal{X}_s|$ is closed, then $f(x) \in |\mathcal{Y}_s|$ is closed.*

Proof. We immediately reduce to the case when S is the spectrum of an algebraically closed field k and $x \in |\mathcal{X}|$ is a closed point. If $f(x)$ is not closed, then there exists a specialization $f(x) \rightsquigarrow y_0$ of k-points to a closed point. By Lemma 3.25, there exists a morphism $\Theta_k \to \mathcal{Y}$ realizing $f(x) \rightsquigarrow y_0$. As the diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "x"] \arrow[d, "i"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_k \arrow[r] \arrow[ur, dashed, "h"] & \mathcal{Y} \end{tikzcd}$$

can be filled in with a morphism h and $x \in |\mathcal{X}|$ is closed, $h(0) = h(1)$. It follows that $f(x) = y_0$ is closed. $\square$

Remark 3.27. The converse of Lemma 3.26 is not true; see Example 3.33.

For the construction of coarse moduli spaces we will need a variant of the above properties. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, of finite type over a Noetherian algebraic space S, with affine diagonal, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism. Define $\Sigma_f \subset |\mathcal{X}|$ be the set of points $x \in |\mathcal{X}|$ where f is *not* Θ-surjective at x. Using the notation of Diagram 1, it is easy to see that Σ_f is precisely the image under p_1 of the complement of the image of $\mathrm{ev}(f)_1$, i.e.,

$$\Sigma_f = p_1\Big(\big(\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})\big) \setminus \mathrm{ev}(f)_1(\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}))\Big) \subset |\mathcal{X}|. \tag{5}$$

Lemma 3.28. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, of finite type over a Noetherian algebraic space S, with affine diagonal, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism. Suppose that either*

(1) $\mathcal{Y}$ admits a good moduli space; or
(2) $\mathcal{Y} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N.

Then the locus $\Sigma_f \subset |\mathcal{X}|$ is closed.

Proof. By Zariski's Main Theorem, there exists a factorization $f\colon \mathcal{X} \xrightarrow{i} \widetilde{\mathcal{X}} \xrightarrow{\widetilde{f}} \mathcal{Y}$ where i is an open immersion and $\widetilde{f}$ is a finite morphism. It is easy to check that $\Sigma_i = \Sigma_f$ using that $\widetilde{f}$ is proper. Thus, it suffices to assume that f is an open immersion. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced complement of $\mathcal{Y}$ and let $\pi\colon \mathcal{Y} \to Y$ denote the adequate moduli space. We claim that $\Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{X}|$.

Indeed, the inclusion "$\subset$" is clear: the morphism $\mathcal{Y} \setminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{Y}$ is the base change of the Θ-surjective morphism $Y \setminus \pi(|\mathcal{Z}|) \hookrightarrow Y$ of algebraic spaces. For the inclusion "$\supset$," let $x \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{X}|$ and let $\overline{x}\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a representative of x, where k is algebraically closed, with image $s\colon \mathrm{Spec}(k) \to S$. Let $x_s \in |\mathcal{X}_s|$ be the image of $\mathrm{Spec}\, k \to \mathcal{X}_s$ and $z \in |\mathcal{Z}_s|$ be the unique closed point in the closure of x_s. If $\mathcal{Y}$ admits a good moduli space, it is in particular locally linearly reductive. Therefore, in either case (1) or (2), we may apply Lemma 3.25 to obtain a morphism $\Theta_k \to \mathcal{Y}_s$ realizing the specialization $x_s \rightsquigarrow z$. Since the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\, k \arrow[r, "\overline{x}"] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\Theta_k \arrow[r] \arrow[ur, dashed] & \mathcal{Y}
\end{tikzcd}$$

does not admit a lift, $x \in \Sigma_f$. As $\pi^{-1}(\pi(|\mathcal{Z}|)) \subset |\mathcal{Y}|$ is closed, the conclusion follows. $\square$

Proposition 3.29. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, of finite type over a Noetherian algebraic space S, with affine diagonal, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable, quasi-finite and separated morphism. If $\mathcal{Y}$ is locally linearly reductive, then $\Sigma_f \subset \mathcal{X}$ is constructible.*

Proof. By Theorem 2.3, the hypotheses imply that there exists a representable, étale and surjective morphism $g\colon \mathcal{Y}' \to \mathcal{Y}$, where $\mathcal{Y}' \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N. Let $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ with projections $g'\colon \mathcal{X}' \to \mathcal{X}$ and $f'\colon \mathcal{X}' \to \mathcal{Y}'$. By Lemma 3.1(f), the morphism $\underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is surjective. Therefore by Cartesian

Diagram (2), the complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}')$ in $\mathcal{X}' \times_{\mathcal{Y}'} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}')$ surjects onto the complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ in $\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$. It follows that $\Sigma_f = g'(\Sigma_{f'})$. By Chevalley's Theorem and Lemma 3.28, the locus Σ_f is constructible. $\square$

Let us give some simple examples and non-examples of Θ-surjectivity. In these examples, we work over an algebraically closed field k.

Example 3.30. Consider the open immersion $f\colon \mathrm{Spec}(k) \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$. Then the inclusion $\mathrm{Spec}(k) \hookrightarrow \mathbb{A}^1$ is Θ-reductive but *not* Θ-surjective. Indeed, this is the prototypical example of a morphism that does not send closed points to closed points.

Example 3.31. Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2 \setminus 0$ via $t \cdot (x, y) = (tx, y)$ (as in Example 3.15)) and the inclusion $f\colon \mathbb{A}^1 \hookrightarrow [X/\mathbb{G}_m]$ of the locus where x is non-zero. Then

$$\mathrm{ev}(f)_1\colon \mathbb{A}^1 = \underline{\mathrm{Map}}(\Theta, \mathbb{A}^1) \to \underline{\mathrm{Map}}(\Theta, [X/\mathbb{G}_m]) = \mathbb{A}^1 \sqcup \Big(\bigsqcup_{n<0} \mathbb{A}^1 \setminus 0 \Big)$$

which is the inclusion onto the first factor. Again, f is Θ-reductive but not Θ-surjective.

Example 3.32. Consider $f\colon [X/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ induced from the projection of the the non-separated affine line X on $\mathbb{A}^1$. We leave it for the reader to check that f is Θ-surjective but not Θ-injective nor Θ-reductive.

Example 3.33. Let $C \subset \mathbb{P}^2$ be the nodal cubic with a $\mathbb{G}_m$ action and consider the étale presentation $f\colon [W/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ where $W = \mathrm{Spec}(k[x, y]/xy)$ and $\mathbb{G}_m$ acts with weights 1 and -1 on x and y, respectively. Then f clearly maps closed points to closed points but we claim it is *not* Θ-surjective. Indeed, there is no lift in the diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[d] & {[\mathrm{Spec}(k[x,y]/xy)/\mathbb{G}_m]} \arrow[d, "f"] \\
\Theta \arrow[r] \arrow[ur, dashed] & {[C/\mathbb{G}_m]}
\end{tikzcd}$$

where $\mathrm{Spec}(k) \to [\mathrm{Spec}(k[x, y]/xy)/\mathbb{G}_m]$ is defined by $y = 0$ and $x \neq 0$, and $\Theta \to [C/\mathbb{G}_m]$ is the composition of the morphism $\Theta \to [\mathrm{Spec}(k[x, y]/xy)/\mathbb{G}_m]$ defined by $x = 0$ and the morphism f.

3.7. **Modifications and elementary modifications.** As in [Hei17, Section 2.B] the following stacks, which depends on a choice of discrete valuation ring R, plays an important role in our analysis of criteria for separatedness of coarse moduli spaces.

$$\overline{\mathrm{ST}}_R := [\mathrm{Spec}\big(R[s, t]/(st - \pi)\big)/\mathbb{G}_m], \tag{6}$$

where s and t have $\mathbb{G}_m$-weights 1 and -1 respectively, and π is a choice of uniformizer for R. A different choice of π results in an isomorphic stack.

Observe that $\overline{ST}_R \setminus 0 \cong \mathrm{Spec}(R) \cup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)$, where K is the fraction field of R, since the locus where $s \neq 0$ in $\overline{\mathrm{ST}}_R$ is isomorphic to $[\mathrm{Spec}\big(R[s, t]_s/(t - \pi/s)\big)/\mathbb{G}_m] \cong [\mathrm{Spec}(R[s]_s)/\mathbb{G}_m] \cong \mathrm{Spec}(R)$ and the locus where $t \neq 0$ has a similar description. A morphism $\overline{ST}_R \setminus 0 \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi'\colon \mathrm{Spec}(R) \to \mathcal{X}$ together with an isomorphism $\xi_K \simeq \xi'_K$.

Definition 3.34. Let $\mathcal{X}$ be an algebraic stack and let $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ be a morphism where R is a DVR.

(1) A *modification* of ξ is the data of a finite extension $R \to R'$ of DVRs and a morphism $\xi'\colon \operatorname{Spec}(R') \to \mathcal{X}$ such that the restrictions of ξ and ξ' to the fraction field K' of R' are isomorphic.

(2) An *elementary modification* of ξ is the data of a finite extension $R \to R'$ of DVRs and a morphism $h\colon \overline{\mathrm{ST}}_{R'} \to \mathcal{X}$ such that the restrictions $h|_{s\neq 0}$ and $\xi|_{R'}$ are isomorphic.

Remark 3.35. A modification of $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ is the data of a finite extension $R \to R'$ of DVRs and a morphism $h\colon \overline{\mathrm{ST}}_{R'} \setminus 0 \to \mathcal{X}$ such that $h|_{s\neq 0} \simeq \xi|_{R'}$; in this case, the other map $\xi'\colon \operatorname{Spec}(R') \to \mathcal{X}$ is given by $h|_{t\neq 0}$. Clearly, an elementary modification is also a modification.

Warning 3.36. The terminology here is inspired by the terminology of [Lan75], but does not exactly coincide. The "elementary modifications" of families of vector bundles over a DVR studied there are examples of the notion of elementary modification above which flip two-step filtrations, but our notion of elementary modification allows one to flip multi-step filtrations.

3.8. **S-complete morphisms.**

Definition 3.37. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks is *S-complete* if for any DVR R and any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{7}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram.

Remark 3.38. The motivation for the terminology "S-complete" comes from Seshadri's work on the S-equivalence of semistable vector bundles. Namely, if $\mathcal{X}$ is the moduli stack of semistable vector bundles over a smooth projective curve C over k, then $\mathcal{X}$ is S-complete (see e.g., Lemma 7.3). If R is a DVR with fraction field K and residue field k, and $\mathcal{E}, \mathcal{F}$ are two families of semistable vector bundles on C_R which are isomorphic over C_K, then S-completeness implies that the special fibers $\mathcal{E}_0$ and $\mathcal{F}_0$ on C are S-equivalent.

Remark 3.39. S-complete morphisms are stable under composition and base change. A morphism of algebraic spaces is S-complete if and only if it is separated (Proposition 3.45). While affine morphisms are always S-complete (Proposition 3.44(1)), it is not true that separated, representable morphisms are S-complete. For instance, the open immersion $\overline{\mathrm{ST}}_R \setminus 0 \to \overline{\mathrm{ST}}_R$ is not S-complete. This example also shows that S-complete morphisms do not satisfy smooth descent.

It suffices to check S-completeness on complete DVRs.

Proposition 3.40. *A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks is S-complete if and only if for every complete DVR R and any commutative diagram (7) of solid arrows, there exists a unique dotted arrow filling in the diagram.*

Proof. One can argue as in [Hei17, Rmk. 2.5]. □

The following proposition gives a criterion for when a quotient stack $[X/G]$ is S-complete. We use the notation introduced in §2.3.

Proposition 3.41. *Let $\mathcal{X} = [X/G]$ be a quotient stack, where X is a separated algebraic space locally of finite type over an algebraically closed field k and G is a geometrically reductive algebraic group over k. Then $\mathcal{X}$ is S-complete if and only if for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the morphism $\widetilde{X}_\lambda \to X \times X \times \mathbb{A}^1$ is proper.*

Proof. Properness of $\Gamma_\lambda\colon \widetilde{X}_\lambda \to X \times X \times \mathbb{A}^1$ can be checked using the valuative criterion for maps where $\operatorname{Spec}(K)$ (where K is the fraction field of a DVR R) lands in the dense open $\widetilde{X}_\lambda \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0)$. If $\operatorname{Spec}(R)$ also lands in this open, there is a unique lift since $\widetilde{X}_\lambda \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) \cong X \times (\mathbb{A}^1 \setminus 0)$ and the map $\Gamma_\lambda|_{\mathbb{A}^1 \setminus 0}$ corresponds to the diagonal $X \times (\mathbb{A}^1 \setminus 0) \to X \times X \times (\mathbb{A}^1 \setminus 0)$, which is a closed immersion.

Suppose $\mathcal{X}$ is S-complete. Let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup and $\operatorname{Spec}(R) \to \mathbb{A}^1$ be a morphism from a DVR R where the closed point maps to 0 and the generic point maps to $\mathbb{A}^1 \setminus 0$. A commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & \operatorname{Spec}(R) \arrow[d] \\ \widetilde{X}_\lambda \arrow[r, "\Gamma_\lambda"] & X \times X \times \mathbb{A}^1 \end{tikzcd} \tag{8}$$

corresponds to two morphisms $f_x, f_y\colon \operatorname{Spec}(R) \to X$ such that $f_x|_K = f_y|_K$. Note that $\operatorname{Spec}\big(R[x,y]/(xy-\pi)\big) \setminus 0$ is covered by $\operatorname{Spec}(R[x]_x)$ (where $x \neq 0$) and $\operatorname{Spec}(R[y]_y)$ (where $y \neq 0$). The morphisms $\lambda \cdot f_x\colon \operatorname{Spec}(R[x]_x) \to X$ and $\lambda^{-1} \cdot f_y\colon \operatorname{Spec}(R[y]_y) \to X$ glue to form a $\mathbb{G}_m$-equivariant morphism $F\colon \operatorname{Spec}\big(R[x,y]/(xy-\pi)\big) \setminus 0 \to X$ which descends to a morphism $\overline{\mathrm{ST}} \setminus 0 \to [X/G]$ fitting into a Cartesian square

$$\begin{tikzcd} \operatorname{Spec}\big(R[x,y]/(xy-\pi)\big) \setminus 0 \arrow[r, "F"] \arrow[d] & X \arrow[d] \\ \overline{\mathrm{ST}}_R \setminus 0 \arrow[r, "f"] & {[X/\mathbb{G}_m].} \end{tikzcd} \tag{9}$$

Since $[X/G]$ is S-complete, so is $[X/\mathbb{G}_m]$ (Proposition 3.44). Therefore, f extends to a unique morphism $\overline{\mathrm{ST}}_R \to [X/\mathbb{G}_m]$ which induces a $\mathbb{G}_m$-equivariant map $\operatorname{Spec}\big(R[x,y]/(xy-\pi)\big) \to X$; this is an R-point of $\widetilde{X}_\lambda$ filling in (8).

Conversely, a morphism $f\colon \overline{\mathrm{ST}}_R \setminus 0 \to [X/G]$ induces a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. One can choose morphisms $f_x, f_y\colon \operatorname{Spec}(R) \to X$ inducing Diagram 9 (see [Hei17, proof of Prop. 2.9]). The valuative criterion of Γ_λ shows that there is a unique $\mathbb{G}_m$-equivariant morphism $\operatorname{Spec}\big(R[x,y]/(xy-\pi)\big) \to X$ extending F. Taking $\mathbb{G}_m$-quotients, we obtain a unique extension $\overline{\mathrm{ST}}_R \to [X/G]$ of f. □

In the following examples, we work over an algebraically closed field k.

Example 3.42. If $X = \operatorname{Spec}(A)$ with an action of an algebraic group G over a field k, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces a $\mathbb{Z}$-grading $A = \bigoplus_{d \in \mathbb{Z}} A_d$. In this case, $\widetilde{X}_\lambda$ is the closed subscheme of $\mathbb{A}^1 \times X \times X$ defined by the ideal generated by $1 \otimes a_i \otimes 1 - t^i \otimes 1 \otimes a_i$ for $a_i \in A_i$ with $i \geq 0$ and $t^{-i} \otimes a_i \otimes 1 - 1 \otimes 1 \otimes a_i$ for $a_i \in A_i$ with $i \leq 0$, where t is the coordinate of $\mathbb{A}^1$. Thus, if G is geometrically reductive, Proposition 3.41 implies that $[\operatorname{Spec}(A)/G]$ is S-complete. We will give a more intrinsic proof of this fact in Proposition 3.44(2).

Example 3.43. Consider $X = \mathbb{A}^2$ where the coordinates x and y have $\mathbb{G}_m$-weights 0 and 1. Let $U = \mathbb{A}^2 \setminus 0$. It is not hard to show that $[U/\mathbb{G}_m]$ is S-complete.

Thus, for a DVR R, the algebraic stack $\Theta_R \setminus 0$ is S-complete but not Θ-reductive. This is dual to the fact that $\overline{\mathrm{ST}}_R \setminus 0$ is Θ-reductive (as it is a scheme) but not S-complete.

Let us summarize simple properties S-completeness that are analogous to those of Θ-reductivity from Proposition 3.20.

Proposition 3.44.

(1) An affine morphism of locally Noetherian algebraic stacks is S-complete.
(2) Let S be a locally Noetherian scheme. Let $G \to S$ be a geometrically reductive and étale-locally embeddable group scheme acting on a locally Noetherian scheme X affine over S. Then the morphism $[X/G] \to S$ is S-complete
(3) A good moduli space $\mathcal{X} \to X$, where $\mathcal{X}$ is a locally Noetherian algebraic stack with affine diagonal, is S-complete.

Proof. We may use the same argument as in the proof of Proposition 3.20. $\square$

Proposition 3.45. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of locally Noetherian algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ both have quasi-finite inertia. Then f is S-complete if and only if f is separated.*

Proof. The same argument in the proof of Proposition 3.22 show that any morphism $\overline{\mathrm{ST}}_R \to \mathcal{Y}$ factors uniquely through $\overline{\mathrm{ST}}_R \to \operatorname{Spec}(R)$. As $\overline{\mathrm{ST}}_R \setminus 0 = \operatorname{Spec}(R) \bigcup_{\operatorname{Spec}(K)} \operatorname{Spec}(R)$, we see that the valuative criterion of Diagram 7 is equivalent to the valuative criterion for separatedness. $\square$

Proposition 3.46. *If G is an algebraic group over a field k, then G is geometrically reductive if and only if BG is S-complete. In particular, an S-complete Noetherian algebraic stack with affine diagonal is locally geometrically reductive (Definition 2.1).*

Proof. From Proposition 3.44(2), we know that if G is geometrically reductive, then BG is S-complete. For the converse, we may assume that k is algebraically closed. Suppose that G is not geometrically reductive. Then by considering the unipotent radical G^{rad} of the reduced group scheme G^{red}, the induced morphism $BG^{\mathrm{rad}} \to BG$ is affine. Similarly, by taking a normal subgroup $\mathbb{G}_a \subset \mathbb{G}^{rad}$, there is an affine morphism $B\mathbb{G}_a \to BG^{\mathrm{rad}}$. The composition $B\mathbb{G}_a \to BG^{\mathrm{rad}} \to BG$ is affine. Since BG is S-complete, by Proposition 3.44(1) so is $B\mathbb{G}_a$, a contradiction. $\square$

Expanding on Proposition 3.44(3), we have the following criterion for when a good moduli space is separated.

Proposition 3.47. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack with affine diagonal and $\mathcal{X} \to X$ be a good moduli space. Then*

(1) the morphism $\mathcal{X} \to X$ is S-complete;
(2) the algebraic space X is separated if and only if $\mathcal{X}$ is S-complete; and
(3) the algebraic space X is proper if and only if $\mathcal{X}$ is universally closed and S-complete.

Proof. Part (1) is Proposition 3.44(3). The implication '$\Rightarrow$' in Part (2) follows from Part (1) and the fact that separated algebraic spaces are S-complete. Conversely, suppose $\mathcal{X}$ is S-complete. Suppose $f, g\colon \operatorname{Spec} R \to X$ are two maps such that $f|_K = g|_K$. After possibly a finite extension of R, we may choose a lift $\operatorname{Spec}(K) \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed, after possibly further extensions of R, we may choose lifts $\widetilde{f}, \widetilde{g}\colon \operatorname{Spec} R \to \mathcal{X}$ of f, g such that $\widetilde{f}|_K \cong \widetilde{g}|_K$. By applying the S-completeness of X, we can extend $\widetilde{f}, \widetilde{g}$ to a morphism $\overline{\mathrm{ST}}_R \to \mathcal{X}$. As $\overline{\mathrm{ST}}_R \to \operatorname{Spec} R$ is a good moduli space and hence universal for maps to algebraic spaces [Alp13, Thm. 6.6], the morphism $\overline{\mathrm{ST}}_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec} R \to X$ which necessarily must be equal to both f and g. We conclude that X is separated by the valuative criterion of separatedness. Part (3) follows from Part (1) using the fact that X is universally closed if and only if $\mathcal{X}$ is. □

Remark 3.48. Assume instead that $\mathcal{X} \to X$ is an adequate moduli space (rather than good moduli space) while keeping the other hypotheses on $\mathcal{X}$. The same argument as above shows that if $\mathcal{X}$ is S-complete (resp. universally closed and S-complete), then X is separated (resp. proper). We suspect that the conclusion of all parts of Proposition 3.47 hold but at the moment we cannot show this as we do not have a slice theorem to reduce to the case of $[\operatorname{Spec}(A)/G]$ with G geometrically reductive.

Corollary 3.49. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack with affine diagonal and $\mathcal{X} \to X$ be a good moduli space. Let R be any DVR and consider two morphisms $\xi_0, \xi_1 : \operatorname{Spec}(R) \to \mathcal{X}$ with $(\xi_0)|_K \cong (\xi_1)|_K$ Then following are equivalent:*

(1) ξ_0 and ξ_1 differ by an elementary modification,
(2) ξ_0 and ξ_1 differ by a finite sequence of elementary modifications,
(3) the compositions $\xi_i : \operatorname{Spec}(R) \to \mathcal{X} \to M$ agree for $i = 0, 1$.

Proof. Clearly (1) $\Rightarrow$ (2). The projection $\overline{ST}_R \to \operatorname{Spec}(R)$ is a good moduli space and hence universal for maps to algebraic spaces [Alp13, Thm. 6.6]. It follows that any two maps which differ by an elementary modification induce the same $\operatorname{Spec}(R)$-point of M, and thus (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (1) follows Proposition 3.47. □

Remark 3.50. The above conditions are not equivalent to saying that ξ_0 and ξ_1 are modifications such that the closures of $\xi_0(0)$ and $\xi_1(0)$ intersect. For instance, let X be the non-locally separated algebraic space obtained by taking the free $\mathbb{Z}/2$-quotient of the non-separated affine line, where the action of $\mathbb{Z}/2$ is via $x \mapsto -x$ and swaps the origins. Then there are two distinct maps $\xi_0, \xi_1\colon \operatorname{Spec} R \to X$ with $(\xi_0)|_K = (\xi_1)|_K$ and $\xi_0(0) = \xi_1(0)$.

3.9. **Hartogs's principle.** In the section, we unify the properties of Θ-reductivity and S-completeness by defining the following stronger valuative criterion.

Definition 3.51. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian algebraic stacks satisfies *Hartogs's principle* if for any regular local ring S of dimension 2 with closed point $0 \in \operatorname{Spec}(S)$ and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(S) \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec}(S) \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{10}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram.

This notion is clearly stable under composition and base change. Moreover, as in Proposition 3.19 and Proposition 3.40, Hartog's principle can be checked on complete DVRs. The same argument for Proposition 3.20 and Proposition 3.44 yields

Proposition 3.52.

(1) *An affine morphism of locally Noetherian algebraic stacks satisfies Hartog's principle.*
(2) *Let S be a locally Noetherian scheme. Let $G \to S$ be a geometrically reductive and étale-locally embeddable group scheme acting on a locally Noetherian scheme X affine over S. Then the morphism $[X/G] \to S$ satisfies Hartog's principle.*
(3) *A good moduli space $\mathcal{X} \to X$, where $\mathcal{X}$ is a locally Noetherian algebraic stack with affine diagonal, satisfies Hartog's principle.*

□

Proposition 3.53. *Any morphism of locally Noetherian algebraic stacks satisfying Hartogs's principle is both Θ-reductive and S-complete.*

Proof. This follows from a standard descent argument. □

3.10. **Unpunctured inertia.** We now give the last of the properties that will turn out to be necessary for the existence of good moduli spaces.

Definition 3.54. We say that a Noetherian algebraic stack has *unpuctured inertia* if for any closed point $x \in |\mathcal{X}|$ and versal deformation $p\colon (U, u) \to (\mathcal{X}, x)$, where U is the spectrum of a local ring with closed point u, each connected component of the inertia group scheme $\underline{\mathrm{Aut}}_U(p) \to U$ has non-empty intersection with the fiber over u.

Let us note a few situations in which this condition is easy to check. To cover more general situations, we will give some valuative criteria in Section 6.

Proposition 3.55. *If $\mathcal{X}$ is a Noetherian algebraic stack with quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.*

Proof. If $\mathcal{X}$ has finite inertia, then $\underline{\mathrm{Aut}}_U(p) \to U$ is finite so clearly the image of each connected component contains the unique closed point $u \in U$. For the converse, we may assume that U is the spectrum of a Henselian local ring in which case $\underline{\mathrm{Aut}}_U(p) = G \sqcup H$ where $G \to U$ finite and the fiber of $H \to U$ over u is empty. If $\underline{\mathrm{Aut}}_U(p)$ is not finite, then H is non-empty and any connected component of H will have non-empty intersection with the fiber over u. □

Proposition 3.56. *Let $\mathcal{X}$ be a Noetherian algebraic stack. If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.*

Proof. This is clear, by definition all fibers of $\underline{\mathrm{Aut}}_U(p) \to U$ are connected, so any connected component of Aut_U intersects the component containing the identity section. □

The following example shows that unpuncturedness need not be preserved when passing to open substacks.

Example 3.57. Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2$ via $t \cdot (a, b) = (ta, t^{-1}b)$ and $-1 \cdot (a, b) = (b, a)$. Note that every point $(a, b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The algebraic stack $[(X \setminus 0)/G]$ does not have unpunctured inertia by Proposition 3.55. However, it will follow from Theorem 6.20 that $[X/G]$ has unpunctured inertia.

4. Existence of good moduli spaces

The goal of this section is to prove the following theorem providing necessary and sufficient conditions for an algebraic stack to admit a good moduli space.

Theorem 4.1. *Let $\mathcal{X}$ be an algebraic stack, of finite type over a Noetherian algebraic space S, with affine diagonal. Then $\mathcal{X}$ admits a good moduli space if and only if*

(1) $\mathcal{X}$ is locally linearly reductive (Definition 2.1);
(2) $\mathcal{X}$ is Θ-reductive (Definition 3.9); and
(3) $\mathcal{X}$ has unpunctured inertia (Definition 3.54).

The idea of the proof is simple. We use the slice theorem to reduce to quotient stacks and glue the resulting moduli spaces. As this only works étale locally we need to apply the slice theorem carefully in a way that preserves unpuncturedness and such that the étale covering of the stack induces one on the level of coarse moduli spaces.

4.1. Preliminaries.

In this section, we recall the hypotheses on an étale morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks which ensure that an induced morphism on good moduli spaces is also étale (Proposition 4.2). We also show that for any representable, separated and quasi-finite morphism $f : \mathcal{X} \to \mathcal{Y}$, the locus of points in $\mathcal{X}$ where f is Θ-surjective is constructible (Proposition 3.29).

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks and $x \in |\mathcal{X}|$, we say that f is *stabilizer preserving at x* if there exists a representative $\widetilde{x} : \operatorname{Spec} l \to \mathcal{X}$ of x (equivalently, for all representatives of x), the natural map $\underline{\operatorname{Aut}}_{\mathcal{X}(l)}(\widetilde{x}) \to \underline{\operatorname{Aut}}_{\mathcal{Y}(l)}(f \circ \widetilde{x})$ is an isomorphism.

Proposition 4.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Noetherian algebraic stacks with affine diagonal. Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{Y}}"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{11}$$

where f is representable, étale and separated, and both $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are good moduli spaces. If f is Θ-surjective and f is stabilizer preserving at every closed point in $\mathcal{X}$, then g is étale and Diagram 11 is Cartesian.

Proof. If $x \in |\mathcal{X}|$ is a closed point, then after replacing Y with an open subspace, we may assume that $\pi_{\mathcal{Y}}(f(x)) \in |Y|$ is closed. Since f is Θ-surjective, $f(x) \in |\mathcal{Y}|$ is a closed point by Lemma 3.26. [Alp10, Thm. 6.10] implies that there is an open subspace $U \subset X$ containing $\pi_{\mathcal{X}}(x)$ such that $g|_U$ is étale and $\pi_{\mathcal{X}}^{-1}(U) = U \times_Y \mathcal{Y}$. The result follows. $\square$

4.2. **The existence result.** We first provide conditions on an algebraic stack insuring that there local quotient presentations which are θ-surjective and stabilizer preserving. This is the key ingredient in the proof of Theorem 4.1.

Proposition 4.3. *Let $\mathcal{Y}$ be an algebraic stack, of finite type over a Noetherian algebraic space S, with affine diagonal. Suppose that $\mathcal{Y}$ is locally linearly reductive and $y \in \mathcal{Y}$ is a closed point.*

(1) If $\mathcal{Y}$ is Θ-reductive, then there exists a local quotient presentation $f\colon \mathcal{X} \to \mathcal{Y}$ around y which is Θ-surjective.

(2) If $\mathcal{Y}$ has unpunctured inertia, then there exists a local quotient presentation $f\colon \mathcal{X} \to \mathcal{Y}$ around y which induces an isomorphism $I_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

Moreover, if $\mathcal{Y}$ is Θ-reductive and has unpunctured inertia, there exists a local quotient presentation $f\colon \mathcal{X} \to \mathcal{Y}$ around y which is Θ-surjective and induces an isomorphism $I_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

Proof. By Theorem 2.3, there exists a local quotient presentation $f\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$. For (1), consider Diagram 1 so that p_1 denotes the projection $\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \to \mathcal{X}$. By Lemma 3.6(6), the morphism $\mathrm{ev}(f)_1$ is an open immersion. As $\mathcal{X}$ is isomorphic to a quotient stack $[\operatorname{Spec} A/\operatorname{GL}_N]$, $\mathcal{X}$ is Θ-reductive by Proposition 3.20(2). Since $\mathcal{Y}$ is also Θ-reductive, it follows that $\mathrm{ev}(f)_1$ satisfies the valuative criterion of properness and therefore is a closed immersion.

Let $\mathcal{Z} \subset \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ be the open and closed complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$. By Equation (5), the image $p_1(\mathcal{Z}) \subset \mathcal{X}$ consists of the points where f is not Θ-surjective. By Proposition 3.29, the image $p_1(\mathcal{Z}) \subset \mathcal{X}$ is constructible. On the other hand, since $\mathcal{Y}$ is Θ-reductive, the image $p_1(\mathcal{Z})$ is closed under specializations.[1] Therefore, $p_1(\mathcal{Z})$ is closed.

We claim that $x \notin p_1(\mathcal{Z})$. Suppose

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d, "i"'] & \mathcal{X} \arrow[d, "f"] \\ \Theta \arrow[r, "\lambda"'] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

is a commutative diagram of solid arrows. But $y = f(x) \in \mathcal{Y}$ is a closed point so λ factors through the residual gerbe $\mathcal{G}_y$ of y. As the induced map $\mathcal{G}_x \to \mathcal{G}_y$ on residual gerbes is an isomorphism, λ lifts to a morphism $\Theta \to \mathcal{G}_x \to \mathcal{X}$.

Let $\phi\colon \mathcal{X} = [\operatorname{Spec} A/\operatorname{GL}_N] \to X = \operatorname{Spec} A^{\operatorname{GL}_N}$ be the adequate moduli space. Set $\mathcal{U} \subset \mathcal{X}$ to be the preimage of an affine open in $Y \setminus \phi(\mathcal{Z})$. Then $\iota\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is a Θ-isomorphism and the composition $\mathcal{U} \hookrightarrow \mathcal{X} \to \mathcal{Y}$ is a local quotient presentation around x. Consider the diagram

$$\begin{tikzcd} \underline{\mathrm{Map}}_S(\Theta, \mathcal{U}) \arrow[r, "\mathrm{ev}(\iota)_1"] \arrow[dr, "\mathrm{ev}(f\circ\iota)_1"'] & \mathcal{U} \times_{\mathcal{X}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \arrow[d, "\mathrm{ev}(f)_1"] & \\ & \mathcal{U} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[r] \arrow[d] & \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \arrow[d] \\ & \mathcal{U} \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

[1] It is here where the Θ-reductivity hypothesis on $\mathcal{Y}$ is used in an essential way.

where all squares are Cartesian. The substack $\mathcal{U}$ was chosen precisely such that $\mathcal{U}\times_{\mathcal{X}}\underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) \to \mathcal{U}\times_{\mathcal{Y}}\underline{\mathrm{Map}}_S(\Theta,\mathcal{Y})$ is an isomorphism. It follows that $\mathrm{ev}(f\circ\iota)_1$ is an isomorphism.

For (2), it suffices to find an open neighborhood $\mathcal{U}\subset\mathcal{X}$ of x such that $f|_{\mathcal{U}}\colon\mathcal{U}\to\mathcal{Y}$ induces an isomorphism $I_{\mathcal{U}}\to\mathcal{U}\times_{\mathcal{Y}} I_{\mathcal{Y}}$. We have a Cartesian diagram

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}. \end{tikzcd}$$

Since f is étale and affine, the morphism $I_{\mathcal{X}}\to\mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}}$ is an open and closed immersion; let $\mathcal{Z}\subset\mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{X}}$ be the open and closed complement. Denote $p_1\colon\mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}}\to\mathcal{X}$. We know that $x\notin p_1(\mathcal{Z})$ as f is stabilizer preserving at x. Moreover, if we choose a versal deformation $(U,u)\to(\mathcal{Y},y)$ where U is the spectrum of a complete local ring, then using that $\mathcal{Y}$ has unpunctured inertia, we know that the preimage of $\mathcal{Z}$ in $\mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}}\times_{\mathcal{Y}} U$ is empty (since each connected component of this preimage must intersect the fiber over u non-trivially which would imply that $x\in p_1(\mathcal{Z})$). This in turn implies that $x\notin\overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U}=\mathcal{X}\setminus\overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}}\to I_{\mathcal{Y}}\times_{\mathcal{Y}}\mathcal{Y}$ is an isomorphism. ☐

Using Proposition 4.3, we can now establish Theorem 4.1.

Proof of Theorem 4.1. For the sufficiency of these three conditions, we follow the proof of [AFS17, Thm. 2.1]. By taking a disjoint union of the local quotient presentations produced in Proposition 4.3, there exists an étale, affine and surjective morphism $f\colon\mathcal{X}_1=[\operatorname{Spec}A/\operatorname{GL}_N]\to\mathcal{X}$ such that (1) f is Θ-surjective and (2) f induces an isomorphism $I_{\mathcal{X}_1}\to\mathcal{X}_1\times_{\mathcal{X}} I_{\mathcal{X}}$. The second property implies that every closed point of $[\operatorname{Spec}A/\operatorname{GL}_n]$ has linearly reductive stabilizer. It follows from [AHR] that $[\operatorname{Spec}A/\operatorname{GL}_N]$ is cohomologically affine. We let $\phi_1\colon\mathcal{X}_1\to X_1:=\operatorname{Spec}(A^{\operatorname{GL}_N})$ be the induced good moduli space.

Set $\mathcal{X}_2=\mathcal{X}_1\times_{\mathcal{X}}\mathcal{X}_1$. The projections $p_1,p_2\colon\mathcal{X}_2\to\mathcal{X}_1$ are also étale, affine, surjective, and Θ-surjective morphisms that induce isomorphisms $I_{\mathcal{X}_2}\to\mathcal{X}_2\times_{\mathcal{X}_1} I_{\mathcal{X}_1}$. Since f is affine, $\mathcal{X}_2$ is cohomologically affine and admits a good moduil space $\phi_2\colon\mathcal{X}_2\to X_2$. By Proposition 4.2, both commutative squares in the diagram

$$\begin{tikzcd} \mathcal{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\phi_2"] & \mathcal{X}_1 \arrow[r, "f"] \arrow[d, "\phi_1"] & \mathcal{X} \\ X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 & \end{tikzcd}$$

are Cartesian. Moreover, by the universality of good moduli spaces, the étale groupoid structure on $\mathcal{X}_2\rightrightarrows\mathcal{X}_1$ induces a étale groupoid structure on $X_2\rightrightarrows X_1$. The fact that f induces isomorphisms of stabilizer groups implies that $\Delta\colon X_2\to X_1\times X_1$ is a monomorphism (see the argument of [AFS17, Prop. 3.1]). Thus, $X_2\rightrightarrows X_1$ is an étale equivalence relation and there exists an algebraic space quotient X. It follows from descent that there is an induced morphism $\phi\colon\mathcal{X}\to X$ which is a good moduli space.

Conversely suppose that $\mathcal{X}$ admits a good moduli space. Then the closed points of $\mathcal{X}$ have linearly reductive stabilizer, that is, $\mathcal{X}$ is locally linearly reductive. Moreover, Proposition 3.20(3) implies that $\mathcal{X}$ is Θ-reductive. The proof that $\mathcal{X}$ is unpunctured will take more effort. We will prove this in Theorem 6.20.

□

We note another consequence of Proposition 4.3, which will be used in §6 below.

Proposition 4.4. *Let $\mathcal{X}$ be an algebraic stack which is of finite type over a field k and has affine diagonal. Suppose that $\mathcal{X}$ is Θ-reductive and there exists a single closed point $x \in |\mathcal{X}|$ which has a linearly reductive stabilizer G_x. Then*

(1) $\mathcal{X}$ has a good moduli space;
(2) $\mathcal{X} = [\operatorname{Spec}(A)/G_x]$; and
(3) $\mathcal{X}$ is coherently complete along x.

Moreover, if $\mathcal{X}$ is reduced, then $\mathcal{X} \to \operatorname{Spec}(k)$ is the good moduli space.

Proof. Choose a local quotient presentation $f\colon (\mathcal{X}_1, x_1) \to (\mathcal{X}, x)$ with $\mathcal{W} = [\operatorname{Spec} B/G_x]$ such that $x_1 \in |\mathcal{X}_1|$ is the unique point mapping to x. Since $\mathcal{X}$ is Θ-reductive, by Proposition 4.3(1), we can assume that f is Θ-surjective. This implies that f sends closed points to closed points and both projections $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \rightrightarrows \mathcal{X}_1$ sends closed points to closed points. Since both $\mathcal{X}$ and $\mathcal{X}_1$ have a unique closed point and f induces an isomorphism of stabilizers $G_w \to G_x$, it follows that $\mathcal{X}_2$ has a unique closed point and that both projections $\mathcal{R} \rightrightarrows \mathcal{W}$ induce isomorphism of stabilizers at this point. Moreover, there are good moduli spaces $\mathcal{X}_1 \to X_1$ and $\mathcal{X}_2 \to X_2$. As in the proof of Theorem 4.1, Proposition 4.2 implies that the induced groupoid $X_2 \rightrightarrows X_1$ is an étale equivalence relation, and the quotient X_1/X_2 is a good moduli space for $\mathcal{X}$, which establishes (1).

Part (2) follows from [AHR15, Thm. 2.9]. For (3), since $\mathcal{X} = [\operatorname{Spec} A/G_x]$ is of finite type over k, the invariant ring A^{G_x} is a finitely generated local k-algebra. Therefore, A^{G_x} is Artianian and, in particular, complete. It follows from [AHR15, Thm. 1.3] that $\mathcal{X}$ is coherently complete along x. For the final statement, if $\mathcal{X}$ is reduced, so is $A^{G_x} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ by [Alp13, Thm. 4.16(viii)]. □

5. Semistable reduction and Θ-stability

In this section we explain how completeness properties of stacks induce similar properties of the substack of semistable objects, if these are defined using the theory of Θ-stability. Our key result is Theorem 5.3 that is inspired by Langton's algorithm for semistable reduction for families of vector bundles. Recall that this algorithm starts with a family of bundles over some projective variety parametrized by a discrete valuation ring R such that the generic fiber is semistable, but unstable special fiber. Using destabilizing subsheaves of the special fiber Langton modifies the family to obtain one in which the special fiber is less unstable, essentially by flipping the bundle in a way that transforms the the canonical destabilizing subbundle of the special fiber into a quotient bundle for the modified family. Surprisingly, it turns out that his construction admits an analog that relies only on the geometry of the algebraic stack representing the moduli problem, not on the particular type of objects classified by the moduli problem. The structure we will need is that of a Θ-stratification from [Hal14, Definition 2.1].

Definition 5.1. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a Noetherian algebraic space S.

(1) A Θ-*stratum in* $\mathcal{X}$ consists of a union of connected components $\mathcal{S} \subset \operatorname{Map}(\Theta, \mathcal{X})$ such that $\operatorname{ev}_1 : \mathcal{S} \to \mathcal{X}$ is a closed immersion.
(2) A Θ-*stratification of* $\mathcal{X}$ indexed by a totally ordered set Γ is a cover of $\mathcal{X}$ by open substacks $\mathcal{X}_{\leq c}$ for $c \in \Gamma$ such that $\mathcal{X}_{\leq c} \subset \mathcal{X}_{\leq c'}$ for $c < c'$,

along with a Θ-stratum $\mathcal{S}_c \subset \mathrm{Filt}(\mathcal{X}_{\leq c})$ in each $\mathcal{X}_{\leq c}$ whose complement is $\bigcup_{c'<c} \mathcal{X}_{\leq c'} \subset \mathcal{X}_{\leq c}$. We also require that $\forall x \in |\mathcal{X}|$ the subset $\{c \in \Gamma | x \in \mathcal{X}_{\leq c}\}$ has a minimal element.

(3) We say that a Θ-stratification is *well-ordered* if for any point $x \in |\mathcal{X}|$, the totally ordered set $\{c \in \Gamma | \operatorname{ev}_1(\mathcal{S}_c) \cap \overline{\{x\}} \neq \emptyset\}$ is well ordered.

Remark 5.2. It will be convenient for us to identify a theta stratum $\mathcal{S}$ with the closed substack it defines on $\mathcal{X}$, i.e., we will sometimes say that a closed substack $\mathcal{S} \subset \mathcal{X}$ is a Θ-stratum, if there exist a union of connected components $\mathcal{S}' \subset \mathrm{Map}(\Theta, \mathcal{X})$ such that $\operatorname{ev}_1 \colon \mathcal{S}' \to \mathcal{S} \subset \mathcal{X}$ identifies $\mathcal{S}$ and $\mathcal{S}'$.

Restricting a map $f : \Theta = \mathbb{A}^1/\mathbb{G}_m \to \mathcal{X}$ $\{0\}/\mathbb{G}_m \hookrightarrow \Theta$ defines a map $\operatorname{ev}_0 : \mathrm{Map}(\Theta, \mathcal{X}) \to \mathrm{Map}(B\mathbb{G}_m, \mathcal{X})$ which corresponds to "passing to the associated graded object" of the filtration f. Composition with the projection $\Theta \to B\mathbb{G}_m = [pt/\mathbb{G}_m]$ defines a section $\sigma : \mathrm{Map}(B\mathbb{G}_m, \mathcal{X}) \to \mathrm{Map}(\Theta, \mathcal{X})$ of the map ev_0 which corresponds to the "canonical filtration of a graded object." These maps define a canonical $\mathbb{A}^1$ deformation retract of $\mathrm{Map}(\Theta, \mathcal{X})$ onto $\mathrm{Map}(B\mathbb{G}_m, \mathcal{X})$, and in particular induce bijections on connected components [Hal14, Lemma 1.24]. We refer to the union of connected components $\mathcal{Z} \subset \mathrm{Map}(B\mathbb{G}_m, \mathcal{X})$ corresponding to $\mathcal{S}$ as the *center* of the Θ-stratum $\mathcal{S}$. The result is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, shift left, "\sigma"] & \mathcal{S} \arrow[l, shift left, "\operatorname{ev}_0"] \arrow[r, hook, "\operatorname{ev}_1"] & \mathcal{X}. \end{tikzcd}$$

5.1. The semistable reduction theorem.

Theorem 5.3 (Langton's algorithm)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a Noetherian algebraic space S, and let $\mathcal{S} \hookrightarrow \mathcal{X}$ be a Θ-stratum. Let R be a discrete valuation ring with fraction field K and residue field k. Let $\xi \colon \mathrm{Spec}(R) \to \mathcal{X}$ be an R-point such that the generic point ξ_K is not mapped to $\mathcal{S}$, but the special point ξ_k is mapped to $\mathcal{S}$:*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, hook] \arrow[d, "\xi_K"] & \operatorname{Spec} R \arrow[d, "\xi_R"] & \operatorname{Spec} k \arrow[l, hook'] \arrow[d, "\xi_k"] \\ \mathcal{X} - \mathcal{S} \arrow[r, hook, "j"] & \mathcal{X} & \mathcal{S}. \arrow[l, hook', "\iota"'] \end{tikzcd}$$

Then there exists an elementary modification of ξ such that $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ lands in $\mathcal{X} - \mathcal{S}$.

Remark 5.4. In the proof of the above result we will apply the slice theorem 2.6 for algebraic stacks. As the proof of this result has not appeared, we give an alternative argument using [AHR15, Theorem 1.2], which requires the additional hypothesis that S is the spectrum of a field and that for any $x \in \mathcal{X}(k)$, the automorphism group G_x is smooth – this suffices, in particular, for stacks over a field of characteristic 0.

This theorem is stated for a single stratum, but it immediately implies a version for a stack with a Θ-stratification:

Theorem 5.5 (Semistable reduction)**.** *Let $\mathcal{X}$ be a stack satisfying the hypotheses of Theorem 5.3 with a well-ordered Θ-stratification. Then any map $\mathrm{Spec}(R) \to \mathcal{X}$ admits a modification $\mathrm{Spec}(R') \to \mathcal{X}$, obtained by a finite sequence of elementary modifications, whose image lies in a single stratum of $\mathcal{X}$.*

Proof. Beginning with a map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ such that $\xi(K) \in \mathcal{S}_c$ and $\xi(k) \in \mathcal{S}_{c_0}$ for $c_0 > c$, we may apply Theorem 5.3 iteratively to obtain a sequence of modifications of ξ with special point in $\mathcal{S}_{c_i}$ for $c_0 > c_1 > \cdots$. Each $\mathcal{S}_{c_i}$ meets $\overline{\xi(K)}$, so the well-orderness condition guarantees that this procedure terminates, and it can only terminate when $c_i = c$. $\square$

Remark 5.6. In the relative situation, for a stack $\mathcal{X} \to B$, one can base-change the structure of a Θ-stratification along a smooth map $B' \to B$, so both Theorem 5.3 and Theorem 5.5 extend immediately to the case of a base stack B which locally admits a smooth surjection from a noetherian scheme S.

5.1.1. *Langton's algorithm in the basic situation.* The main idea of the proof is to reduce to the situation where $\mathcal{X} = [\mathrm{Spec}\, A/\mathbb{G}_m]$ is the quotient of an affine scheme by an action of $\mathbb{G}_m$, $\mathcal{Z} = [(\mathrm{Spec}\, A)^{\mathbb{G}_m}/\mathbb{G}_m]$ is the substack defined by the fixed point locus of the action and $\mathcal{S} = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m]$ is the attracting substack, where

$$I_+ := (\oplus_{n>0} A_n)$$

is the graded ideal generated by elements of positive weight. In this basic situation the theorem will then follow from an elementary calculation. We will first explain the proof of this special case and then show how to reduce to the basic situation.

Lemma 5.7. *In the setting of Theorem 5.3 suppose in addition that $\mathcal{X} = [\mathrm{Spec}\, A/\mathbb{G}_m]$ for a graded ring $A = \oplus_{n\in\mathbb{Z}} A_n$ and that $\mathcal{S} = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m]$. Then the conclusion of Theorem 5.3 holds.*

Proof. Let us denote $X := \mathrm{Spec}(A)$ and $S := \mathrm{Spec}(A/I_+)$. As $X \to \mathcal{X}$ is a $\mathbb{G}_m$-torsor, we can lift ξ to a map $\xi' \colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$, obtaining a diagram

$$\begin{tikzcd}
\mathrm{Spec}\, K \arrow[r, hook] \arrow[d, "\xi'_K"] & \mathrm{Spec}\, R \arrow[d, "\xi'"] & \mathrm{Spec}\, k \arrow[l, hook'] \arrow[d, "\xi'_k"] \\
X - S \arrow[r, hook, "j"] & \mathrm{Spec}\, A & \mathrm{Spec}(A/I_+). \arrow[l, hook', "\iota"']
\end{tikzcd}$$

As $\xi'_k \in S = \mathrm{Spec}(A/I_+)$ and A/I_+ is generated by elements of non-positive weight, the $\mathbb{G}_m$-orbit of ξ'_k, corresponding to a map of graded algebras $A/I_+ \to k[t^\pm]$ where t has weight -1, extends to an equivariant morphism $\mathbb{A}^1_k \to S$. Thus the $\mathbb{G}_m$-orbits of the points ξ'_K, ξ', ξ'_k define a diagram:

$$\begin{tikzcd}
\mathbb{G}_{m,K} \arrow[r, hook] \arrow[dd, "f_K"] & \mathbb{G}_{m,R} \arrow[dd, "f_R"] & \mathbb{G}_{m,k} \arrow[l, hook'] \arrow[d, hook] \\
 & & \mathbb{A}^1_k \arrow[d, "\overline{f}_k"] \\
X - S \arrow[r, hook, "j"] & \mathrm{Spec}\, A & \mathrm{Spec}\, A/I_+. \arrow[l, hook', "\iota"']
\end{tikzcd}$$

We know that $f_R^\#(I_+) \in \pi(R[t, t^{-1}])$ since $f_k^\#$ factors through A/I_+, and $K[t^{\pm 1}] f_R^\#(I_+) = K[t^{\pm 1}]$ since the image of f_K does not intersect S.

Let $a_i \in I_{d_i}$ be homogeneous generators of I_+. Then for all k we have $f_R^\#(a_i) = \epsilon_i \pi^{n_i} t^{-d_i}$ for some $n_i > 0$ and $\epsilon_i \in R^\times \cup \{0\}$. As $f_R^\#(I_+)$ is not 0 we can define

$$\frac{m}{d} := \min_i \{ \frac{n_i}{d_i} | f_R^\#(a_i) \neq 0 \}$$

and let $R' := R[\pi^{1/d}]$. Since $n_i - \frac{d_i m}{d} \geq 0$ for each i, we can write

$$f_R^{\#}(a_i) = \epsilon_i(\pi^{n_i - \frac{d_i m}{d}})(\pi^{\frac{m}{d}} x^{-1})^{d_i} = \epsilon_i(\pi^{n_i - \frac{d_i m}{d}}) s^{d_i}.$$

Since $f_R^{\#}$ maps elements of negative weight to $R[t]$, we have a homomorphism of graded rings

$$f'^{\#}_{R'} \colon A \to R'[s,t]/(st - \pi^{\frac{m}{d}}) = R'[t, \pi^{\frac{m}{d}} t^{-1}] \subset R'[t, t^{-1}]$$

Furthermore, composing with the map setting $s = 1$ at least one $f'^{\#}_{R'}(a_i)$ is not mapped to 0 mod $\pi^{1/d}$, i.e. $f'_{R'}|_{\{s=1\}} \colon \operatorname{Spec}(R') \mapsto \operatorname{Spec} A_{a_i} \subset X - S$. The graded homomorphism $f'^{\#}_{R'}$ defines a morphism

$$[\operatorname{Spec}\left(R'[s,t]/(st - \pi^{\frac{m}{d}})\right)/\mathbb{G}_m] \to \mathcal{X} = [X/\mathbb{G}_m].$$

As $\pi^{\frac{m}{d}}$ is not a uniformizer for R', this is not quite an elementary modification. However, we can embed $R'[s,t]/(st - \pi^{m/d}) \subset R'[s^{1/m}, t^{1/m}]/(s^{1/m}t^{1/m} - \pi^{1/d})$. If we regard $s^{1/m}$ and $t^{1/m}$ as having weight 1 and -1 respectively, the map $\operatorname{Spec}(R'[s^{1/m}, t^{1/m}]/(s^{1/m}t^{1/m} - \pi^{1/d})) \to \operatorname{Spec}(R'[s,t]/(st - \pi^{m/d}))$ is equivariant with respect to the group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given in coordinates by $z \mapsto z^m$. The resulting composition

$$\overline{\mathrm{ST}}_{R'} \to [\operatorname{Spec}\left(R'[s,t]/(st - \pi^{\frac{m}{d}})\right)/\mathbb{G}_m] \to \mathcal{X}$$

is the desired modification of ξ.

□

5.1.2. *Reduction to quasi compact stacks.* We first show that by replacing $\mathcal{X}$ by a suitable open substack we may assume that $\mathcal{X}$ is quasi-compact.

Let us recall

For convenience we will say ([Hal14, Definition 2.13]) that we say $\mathcal{S}$ induces a Θ-stratum in $\mathcal{Y}$ if $\mathrm{ev}_1 : \mathcal{S}' \to \mathcal{Y}$ is a Θ-stratum which is surjective onto $\pi^{-1}(\mathrm{ev}_1(\mathcal{S}))$. For example, it follows from [Hal14, Proposition 1.18] that a Θ-stratum $\mathcal{S} \subset \mathrm{Filt}(\mathcal{X})$ induces a Θ-stratum in an open substack $\mathcal{U} \subset \mathcal{X}$ if and only if for any $f \in \mathcal{S}$ such that $f(1) \in \mathcal{U}$, we have $f(0) \in \mathcal{U}$ as well.

Lemma 5.8. *Let $\mathcal{S} \subset \mathcal{X}$ be a Θ-stratum in $\mathcal{X}$ with center $\sigma : \mathcal{Z} \to \mathcal{S}$. Then for any point $x \in |\mathcal{Z}|$ and any open substack $\mathcal{U} \subset \mathcal{X}$ containing $\sigma(x)$, there is another open substack with $\sigma(x) \in \mathcal{V} \subset \mathcal{U}$ such that $\mathcal{S} \cap \mathcal{V}$ is a Θ-stratum in $\mathcal{V}$.*

Proof. We only need to find a substack $\mathcal{V} \subset \mathcal{X}$ such that for any $f \colon \Theta \to \mathcal{X}$ with $f \in \mathcal{S}$ and $f(1) \in \mathcal{V}$, we have $f(0) \in \mathcal{V}$ as well. Let $\mathcal{U}' = (\mathrm{ev}_1 \circ \sigma)^{-1}(\mathcal{U}) \subset \mathcal{Z}$, and let $\mathcal{Z}' = \mathcal{Z} \setminus \mathcal{U}'$ be its complement. Then the open substack

$$\mathcal{V} := \mathcal{U} \setminus (\mathcal{U} \cap \mathrm{ev}_1(\mathrm{ev}_0^{-1}(\mathcal{Z}'))) \subset \mathcal{X}$$

satisfies the condition. □

5.1.3. *Reminder on the normal cone to a Θstratum.* The problem in finding a presentation of the form $[\operatorname{Spec}(A/I_+)/\mathbb{G}_m] \subset [\operatorname{Spec}(A)/\mathbb{G}_m]$ is that for an arbitrary morphism $[\operatorname{Spec}(A)\mathbb{G}_m] \to \mathcal{X}$ the preimage of the theta stratum need not be defined by the ideal generated by the elements of positive weight. To find presentations for which this happens, we need to recall that the weights of the $\mathbb{G}_m$-action of the restriction of the conormal bundle of a Θ-stratum to its center $\mathcal{Z}$ are automatically positive, is classical. This property was already important in the work of Atiyah-Bott [AB83] and it appears in the language of spectral stacks in [Hal14, Section 1.2]. For completeness we provide a classical argument:

Lemma 5.9. *Let* $\mathrm{ev}_1 \colon \mathcal{S} \hookrightarrow \mathcal{X}$ *be a* Θ*-stratum,* $\mathcal{Z} \to \mathcal{S}$ *the center of* $\mathcal{S}$ *and* $x \in \mathcal{Z}(k)$ *a* k*-point. By abuse of notation we will also denote* $\sigma(x) \in \mathcal{X}(k)$ *by* x*.*

(1) *Let* $T_{\mathcal{X},x} = \oplus_{n\in\mathbb{Z}} T_{\mathcal{X},x,n}$ *be the decomposition of the tangent space at* x *into weight spaces with respect to the* $\mathbb{G}_m$ *action induced form the canonical cocharacter* $\lambda_x \colon \mathbb{G}_m \to \mathrm{Aut}_{\mathcal{X}}(x)$*. Then we have* $T_{\mathcal{S},x} = \oplus_{n\geq 0} T_{\mathcal{X},x,n}$*.*
(2) $\mathbb{G}_m$ *acts with non-negative weights on* $\mathrm{Lie}(Aut_{\mathcal{X}}(x))$*.*

Proof. Let us first show that $\oplus_{n\geq 0} T_{\mathcal{X},x,n} \subseteq T_{\mathcal{S},x}$. Let $t \in \mathcal{X}(k[\epsilon]/\epsilon^2)$ be a tangent vector in $T_{\mathcal{X},x,n}$ for some $n \leq 0$, i.e. t comes equipped with an isomorphism $t \mod \epsilon \cong x$.

This means that we have a 2-commutative diagram

$$\begin{tikzcd}
{[(\mathrm{Spec}\, k[\epsilon]/\epsilon^2)/\mathbb{G}_m]} \arrow[r, "t"] & \mathcal{X}, \\
{[\mathrm{Spec}\, k/\mathbb{G}_m]} \arrow[u, hook] \arrow[ur, "{(x,\lambda_x)}"'] &
\end{tikzcd}$$

where $\mathbb{G}_m$ acts on $\mathrm{Spec}\, k[\epsilon]/\epsilon^2$ via $(\lambda, \epsilon) \mapsto \lambda^n \epsilon$. In other words, we have a 2-commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}\, k[\epsilon]/\epsilon^2 \arrow[r, "{(\lambda,\epsilon\mapsto\lambda^n\epsilon)}"] \arrow[dr] & \mathrm{Spec}\, k[\epsilon]/\epsilon^2 \arrow[d, "t"] \\
& \mathcal{X}.
\end{tikzcd}$$

If $n \geq 0$ then the horizontal map extends to $\mathbb{A}^1$, i.e., we get an extension

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathrm{Spec}\, k[\epsilon]/\epsilon^2 \arrow[r, "{(\lambda,\epsilon\mapsto\lambda^n\epsilon)}"] \arrow[dr] & \mathrm{Spec}\, k[\epsilon]/\epsilon^2 \arrow[d, "t"] \\
& \mathcal{X}
\end{tikzcd}$$

and this defines an extension of t to a $k[\epsilon]/\epsilon^2$-valued point of $\underline{\mathrm{Map}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$.

Conversely, an extension of the constant map $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathrm{Spec}\, k/\mathbb{G}_m] \to \mathcal{X}$ to $[\mathbb{A}^1 \times \mathrm{Spec}(k[\epsilon]/(\epsilon^2))/\mathbb{G}_m] \to \mathcal{X}$ automatically factors through the first infinitesmal neighborhood of $x \in \mathcal{X}$. On a versal first order deformation this corresponds to a homomorphism of graded algebras $k[\epsilon_1, \ldots, \epsilon_d]/(\epsilon_i)^2_{i=1,\ldots d} \to k[\lambda, \epsilon]/(\epsilon^2)$, where we can choose ϵ_i to be homogeneous for the $\mathbb{G}_m$ action defined by λ_x. This has to vanish on those tangent directions ϵ_i on which λ_x acts with negative weights. This shows (1).

Similarly for (2), when we regard x as a k point of $\mathcal{Z} \hookrightarrow \mathcal{S} \subset \mathrm{Map}(\Theta, \mathcal{X})$, it corresponds to a map which factors as $\Theta_k \to BG_x \hookrightarrow \mathcal{X}$, where we abbreviated $G_x = \mathrm{Aut}_{\mathcal{X}}(x)$. We know that $\mathrm{Aut}_{\mathcal{S}}(x) \to \mathrm{Aut}_{\mathcal{X}}(x)$ is an equivalence, so by the classification of G_x-bundles on $[\mathbb{A}^1/\mathbb{G}_m]$ ([Hei17, Lemma 1.7] or [Hal14, Proposition A.1] this implies that for the canonical cocharacter $\lambda_x \colon \mathbb{G}_m \to G_x$ we have $G_x = P(\lambda_x)$ as an algebraic group. In particular this means that $\mathbb{G}_m$ acts with positive weights on the Lie algebra of $G_x = P(\lambda)$.

□

5.1.4. *Reduction to the basic situation - Case of smooth stabilizers over a field.*

Lemma 5.10. *Let $\mathcal{X}$ be a quasi-compact algebraic stack locally of finite type over an algebraically closed field k. Let $\mathcal{S} \subset \mathcal{X}$ be a Θ-stratum with center $\sigma : \mathcal{Z} \to \mathcal{S}$, and let $x_0 \in \mathcal{Z}(k)$ be a point such that $x := \sigma(x_0)$ has a smooth automorphism group. Then there is a smooth representable map $p : [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ whose image contains x and such that*

$$\mathcal{S}' = p^{-1}(\mathcal{S}) = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\mathrm{Spec}(A)/\mathbb{G}_m].$$

Proof. The point x_0 has a canonical non-constant homomorphism $(\mathbb{G}_m)_k \to \mathrm{Aut}_{\mathcal{Z}}(x_0)$, which induces a canonical homomorphism $\lambda : (\mathbb{G}_m)_k \to G_x := \mathrm{Aut}_{\mathcal{X}}(x)$. We may replace $(\mathbb{G}_m)_k$ with its image in G_x and thus assume that λ is injective. The quotient $G_x/\lambda(\mathbb{G}_m)$ is smooth, so we may apply [AHR15, Theorem 1.2] to obtain a smooth morphism

$$p\colon [\mathrm{Spec}\, A/\mathbb{G}_m] \to \mathcal{X}$$

together with a point $w \in \mathrm{Spec}(A)(k)$ in $p^{-1}(x)$ which is fixed by $\mathbb{G}_m$ and such that $p^{-1}(BG_x) \cong B(\mathbb{G}_m)_k$. The isomorphism $p^{-1}(BG_x) \cong B(\mathbb{G}_m)_k$ implies that the relative tangent space to $\tilde{p}\colon \mathrm{Spec}(A) \to \mathcal{X}$ at w is naturally identified with $\mathrm{Lie}(G_x)/\mathrm{Lie}(\mathbb{G}_m)$ on which $\mathbb{G}_m$ acts with non-negative weights by part (2) of lemma 5.9.

Note that connected components of $\mathrm{Spec}(A)^{\mathbb{G}_m}$ can be separated by invariant functions, so we may replace $\mathrm{Spec}(A)$ with a $\mathbb{G}_m$-equivariant affine open neighborhood of w so that $\mathrm{Spec}(A)^{\mathbb{G}_m}$ is connected. It follows that $\mathrm{Spec}(A/I_+)$ is connected as well.

This implies that $\mathcal{S}_A := [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subseteq [\mathrm{Spec}(A)/\mathbb{G}_m]$ is isomorphic to a connected component of $\mathrm{Map}(\Theta, [\mathrm{Spec}(A)/\mathbb{G}_m])$ and $\mathcal{Z}_A := [\mathrm{Spec}(A^{\mathbb{G}_m})/\mathbb{G}_m] \subset \mathcal{S}_A$ is the center of $\mathcal{S}_A$. As $p(x) \in \mathcal{Z}$ connectedness now implies that $p(\mathcal{Z}_A) \subset \mathcal{Z}_0$ and therefore we also have $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subset p^{-1}(\mathrm{ev}_1(\mathcal{S}))$.

To conclude that $\mathcal{S}_A \cong p^{-1}(\mathcal{S})$ after possibly shrinking A, it suffices to check that the inclusion $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subseteq p^{-1}(\mathrm{ev}_1(\mathcal{S}))$ of closed substacks of $[\mathrm{Spec}(A)/\mathbb{G}_m]$ is an isomorphism locally at w. Consider the pull-back:

$$\begin{tikzcd}
p^{-1}(\mathrm{ev}_1(\mathcal{S})) = \mathrm{Spec}(B) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "p"] \\
\mathcal{S} \arrow[r, hook] & \mathcal{X}
\end{tikzcd}$$

Then B is a graded ring and we still have an exact sequence

$$T_{p,w} \to T_{\mathrm{Spec}(B),w} \to \mathcal{T}_{\mathcal{S},x}.$$

As $\mathbb{G}_m$ acts with non-negative weight on the relative tangent bundle at w and also on $\mathcal{T}_{\mathcal{S},x}$ by Lemma lemma 5.9, this shows that $\mathbb{G}_m$ acts with non-negative weights on $T_{\mathrm{Spec}(B),w}$. In particular the maximal ideal $\mathfrak{m}_w \subset B$ of w is generated by elements of non-positive weight locally at w.

Therefore, after possibly shrinking A we may assume that $B = \oplus_{n\leq 0} B_n$ is non-positively graded. As $\mathrm{Spec}(A/I_+) \subset \mathrm{Spec}(A)$ was the contracting subscheme for $\mathbb{G}_m$ we find that locally around w we thus have $p^{-1}(\mathcal{S}) \subset \mathrm{Spec}(A/I_+)$ locally around w. This proves our claim.

□

5.1.5. *Reduction to the basic situation - general case.*

Lemma 5.11. *Let $\mathcal{S}$ be a Θ-stratum in a quasi-compact stack $\mathcal{X}$. Then there is a smooth representable map $p : [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ such that $p^{-1}(\mathcal{S})$ is the Θ-stratum*

$$\mathcal{S}' = p^{-1}(\mathcal{S}) = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\mathrm{Spec}(A)/\mathbb{G}_m],$$

and $\mathcal{S}$ is contained in the image of p.

Proof. Because $\mathcal{X}$ is quasi-compact, we may apply [Hal14, Lemma 4.26] to obtain a smooth surjective representable map $p : [\mathrm{Spec}(A)/\mathbb{G}_m^n] \to \mathcal{X}$ such that $\mathrm{Filt}([\mathrm{Spec}(A)/\mathbb{G}_m^n]) \to \mathrm{Filt}(\mathcal{X})$ is also smooth surjective and representable. From [Hal14, Theorem 1.36] , we know that $\mathrm{Filt}([\mathrm{Spec}(A)/\mathbb{G}_m^n])$ is the disjoint union indexed by cocharacters $\mathbb{G}_m \to \mathbb{G}_m^n$ of stacks of the form $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m^n]$, where I_+ is the ideal generated by positive weight elements with respect to a given cocharacter. Choosing different connected components if necessary and forgetting all but the relevant cocharacter in each component, we can construct a non-positively graded algebra $C = \bigoplus_{n\leq 0} C_n$ along with a smooth surjective representable map $[\mathrm{Spec}(C)/\mathbb{G}_m] \to \mathcal{S}$.

We now discard the previously constructed $\mathrm{Spec}(A)$ and apply the non-local version of the slice theorem of [AHR15] discussed in [Hal14, Theorem B.2] to the smooth surjective map $[\mathrm{Spec}(C)/\mathbb{G}_m] \to \mathcal{S}$, where we regard $\mathcal{S}$ as a closed substack of $\mathcal{X}$. The non-local version of the slice theorem provides a map $p : [\mathrm{Spec}(A')/\mathbb{G}_m] \to \mathcal{X}$ such that if $I_S \subset A'$ is the ideal corresponding to $p^{-1}(\mathcal{S})$, then we have a commutative diagram

$$\begin{tikzcd}
{[\mathrm{Spec}(C)/\mathbb{G}_m]} \arrow[drr, "\substack{\text{surj.}\\ \text{smooth}}"'] & & {[\mathrm{Spec}(A'/I_S)/\mathbb{G}_m]} \arrow[ll, "\substack{\text{surj.}\\ \text{étale}}"'] \arrow[r, hook] \arrow[d, "p"] & {[\mathrm{Spec}(A')/\mathbb{G}_m]} \arrow[d, "p"] \\
& & \mathcal{S} \arrow[r, hook] & \mathcal{X}
\end{tikzcd}.$$

Let us denote $C' = A'/I_S$, and $I_+^{C'}$ and I_+^C the corresponding ideals generated by positive degree elements. Because the map $\mathrm{Spec}(C') \to \mathrm{Spec}(C)$ is étale and surjective, the map $\mathrm{Spec}(C'/I_+^{C'}) \to \mathrm{Spec}(C/I_+^C)$ is also étale and surjective, which follows from the computation of the deformation theory of the space of filtrations in [Hal14, Section 1.2], or Lemma 5.9 below. Observing that $I_+^C = 0$ and $I_+^{C'} = (I_S + I_+)/I_S$, we have that

$$\mathrm{Spec}(A'/(I_S + I_+)) \to \mathrm{Spec}(C)$$

is étale and surjective. It follows that $\mathrm{Spec}(A'/(I_S + I_+)) \to \mathrm{Spec}(A'/I_S)$ is étale and hence a union of connected components, and only these components are necessary to surject onto $\mathrm{Spec}(C)$. In particular, we may invert a weight 0 element $a \in A'$ such that the latter map becomes an isomorphism. In other words we may replace A' with A'_a and assume that $I_S + I_+ = I_S$, i.e. $I_+ \subset I_S$.

Because p is smooth, the relative cotangent complex of $\mathrm{Spec}(C') \hookrightarrow \mathrm{Spec}(A')$ is $p^*(\mathbb{L}_{\mathcal{S}/\mathcal{X}})$. In particular, the fiber of the conormal bundle of $\mathrm{Spec}(C') \hookrightarrow \mathrm{Spec}(A')$ has positive weights at every point of $\mathrm{Spec}(C')^{\mathbb{G}_m}$ by [Hal14, Section 1.2], or Lemma 5.9 below. One may therefore find a collection of positive weight elements of I_S which generate the fiber of I_S at every closed point of $\mathrm{Spec}(C')^{\mathbb{G}_m}$. On the other hand the inclusion $I_+ \subset I_S$ implies that the algebra $C' = A'/I_S$ is non-positively graded, and hence the orbit closure of every point in $\mathrm{Spec}(C')$

meets the fixed locus $\mathrm{Spec}(C')^{\mathbb{G}_m}$. So by Nakayama's lemma we can actually find a collection of homogeneous elements of I_+ which generate the fiber of I_S at every point of $\mathrm{Spec}(C')$ and hence in a $\mathbb{G}_m$-equivariant open neighborhood of $\mathrm{Spec}(C')$. Again we may invert a weight 0 element of A' so that these elements of I_+ generate I_S and $C' = A'/I_S$ is unaffected.

In particular we have shown that after inverting a weight 0 element of A', we have a smooth map $p\colon [\mathrm{Spec}(A')/\mathbb{G}_m] \to \mathcal{X}$ such that $\mathrm{Spec}(A'/I_+) = p^{-1}(\mathcal{S})$ and the map $\mathrm{Spec}(A'/I_+) \to \mathcal{S}$ is surjective. □

In the previous proof, we have used a discussion of the cotangent complex of $\mathrm{Filt}(\mathcal{X})$ from [Hal14, Section 1.2] which makes use of derived algebraic geometry.

The proof of Lemma 5.11 referred to [Hal14, Theorem B.2], which is a strengthened form of [AHR15, Theorem 1.2] whose proof will appear in a forthcoming paper. In order to avoid citing results which have been announced but for which a proof is not yet publicly available, we prove a result analogous to Lemma 5.11 which only uses [AHR15, Theorem 1.2], but which makes the additional hypothesis of smooth automorphism groups in $\mathcal{X}$.

We can now prove the semistable reduction theorem:

Proof of Theorem 5.3. Consider a map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ as in the statement of the theorem. Observe that for any smooth map $p : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{S}$ induces a Θ-stratum $p^{-1}(\mathcal{S})$ in $\mathcal{Y}$ and the image of p contains the image of ξ, if we know the conclusion of the theorem holds for $\mathcal{Y}$ then the conclusion holds for $\mathcal{X}$ as well: indeed after an extension of R we may lift ξ to a map $\xi' : \mathrm{Spec}(R') \to \mathcal{Y}$, construct an elementary modification in $\mathcal{Y}$ such that the new map $\xi'' : \mathrm{Spec}(R'') \to \mathcal{Y}$ lies in $\mathcal{Y} \setminus p^{-1}(\mathcal{S})$, and observe that the composition of this elementary modification with p gives an elementary modification of ξ such that the new map $p \circ \xi'' : \mathrm{Spec}(R'') \to \mathcal{X}$ lies in $\mathcal{X} \setminus \mathcal{S}$.

Using this observation and the fact that ξ_k lies in $\mathcal{S}$, we may use Lemma 5.8 to replace $\mathcal{X}$ with a quasi-compact open substack, then use Lemma 5.11 to construct a smooth map $p : [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ whose image contains the image of ξ and for which $\mathcal{S}$ induces a Θ-stratum. Then we are finished by Lemma 5.7. □

5.2. **Comparison between a stack and its semistable locus.** As an immediate consequence of the semistable reduction theorem, we have the following:

Corollary 5.12. *Let $\mathcal{X}$ be a locally finite type algebraic stack with affine diagonal over a noetherian algebraic space S. Let $\mathcal{X} = \bigcup_{c\in\Gamma} \mathcal{X}_{\leq c}$ be a well-ordered Θ-stratification of $\mathcal{X}$. If $\mathcal{X} \to B$ satisfies the existence part of the valuative criterion for properness, then so does $\mathcal{X}_{\leq c} \to B$ for every $c \in \Gamma$.*

Proof. Consider a discrete valuation ring and a map $\mathrm{Spec}(R) \to B$ along with a lift $\mathrm{Spec}(K) \to \mathcal{X}^{\mathrm{ss}}$. If $\mathcal{X} \to B$ satisfies the existence part of the valuative criterion, then after a finite extension of R one can extend this lift to a lift $\mathrm{Spec}(R') \to \mathcal{X}$. By hypothesis the generic point lies in $\mathcal{X}^{\mathrm{ss}}$, so by Theorem 5.5 there is a sequence of elementary modifications resulting in a modification $\mathrm{Spec}(R'') \to \mathcal{X}^{\mathrm{ss}}$. Note that because $\mathrm{Spec}(R)$ is the good moduli space of $\overline{\mathrm{ST}}_R$, and good moduli spaces are universal for maps to an algebraic space [Alp13, Theorem 6.6], any elementary modification of a map $\mathrm{Spec}(R) \to B$ is trivial. It follows that our modified map $\mathrm{Spec}(R'') \to \mathcal{X}^{\mathrm{ss}}$ is a lift of the original map $\mathrm{Spec}(R) \to B$. □

Next let us briefly recall the notion of Θ-stability from [Hal14, Definition 4.1 & 4.4]. Given a cohomology class $\ell \in H^2(\mathcal{X}; \mathbb{R})$, we say that a point $p \in |\mathcal{X}|$ is *unstable* with respect to ℓ if there is a filtration $f : \Theta_k \to \mathcal{X}$ with $f(1) = p \in |\mathcal{X}|$ and such that $f^*(\ell) \in H^2(\mathcal{X};\mathbb{R}) \simeq \mathbb{R}$ is positive. The Θ-semistable locus $\mathcal{X}^{\mathrm{ss}}$ is the set of points which are not unstable. This is simply an intrinsic formulation of the Hilbert-Mumford criterion for semistability in geometric invariant theory.

We are somewhat flexible with what type of cohomology theory we use: if $\mathcal{X}$ is locally finite type over $\mathbb{C}$ we may use the Betti cohomology of the analytification of $\mathcal{X}$, if $\mathcal{X}$ is locally finite type over another field k, we can use Chow cohomology, and in general one may use the Neron-Severi group $NS(\mathcal{X})_{\mathbb{R}}$ for $H^2(\mathcal{X};\mathbb{R})$. In [Hal14, Section 3.7] we axiomatized the properties of the cohomology theory needed for the theory of Θ-stability.

Proposition 5.13. *Let $\mathcal{X}$ be a locally finite type algebraic stack with quasi-affine diagonal over a noetherian algebraic space S. Assume that the stack of Θ-semistable points with respect to a class $\ell \in H^2(\mathcal{X};\mathbb{R})$ is part of a Θ-stratification of $\mathcal{X}$, i.e. $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}_{\leq 0}$, such that for each HN filtration $f : \Theta_k \to \mathcal{X}$ of an unstable point one has $f^*(\ell) > 0$ in $H^2(\Theta_k;\mathbb{R})$.*

(1) If $\mathcal{X} \to S$ is S-complete, then so is $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}_{\leq 0}$.
(2) If $\mathcal{X} \to S$ is Θ-reductive, then so is $\mathcal{X}^{\mathrm{ss}} \to S$.

In the proof, we will need the following:

Lemma 5.14. *Under the hypotheses of Proposition 5.13, given a filtration $f : \Theta_k \to \mathcal{X}$ such that $f(1)$ is semistable with respect to ℓ, then $f^*\ell = 0$ if and only if $f(0)$ is semistable as well.*

Proof. The proof is a geometric reformulation of the corresponding argument for semistability for vector bundles. It is also a special case of [Hal14, Proposition 4.26]. Let us denote by $x_0 := f(0)$. One direction is easy: for any semistable point $x \in \mathcal{X}(k)$ and any cocharacter $\lambda : \mathbb{G}_m \to G_x$, the restriction of ℓ to $H^2([\mathrm{Spec}(k)/\mathbb{G}_m];\mathbb{R}) \simeq \mathbb{R}$ along the resulting map $f_\lambda\colon \Theta \to [\mathrm{Spec}(k)/\mathbb{G}_m] \to \mathcal{X}$ must vanish, because the invariants for λ and λ^{-1} differ by sign and are both non-positive.

For the converse suppose that $x_0 = f(0)$ is unstable, i.e., it lies in some proper Θ-stratum. Let $g\colon \Theta \times BG_{x_0} \to \mathcal{X}$ be the corresponding filtration of x_0.

Denote by $R = k[\![\pi]\!]$ the completion of the local ring of the affine line with coordinate π at 0. Then the map $f_R\colon \operatorname{Spec} R \to [\operatorname{Spec} k[t]/\mathbb{G}_m] = \theta_k \xrightarrow{f} \mathcal{X}_\kappa$ and $g_k\colon \Theta_k \to \Theta \times BG_{x_0} \to \mathcal{X}_\kappa$ define the datum needed to apply the gluing lemma A.1, i.e. for $n \gg 0$ there is a unique extension

$$F_R\colon [\mathrm{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m] \to \mathcal{X}$$

such that $F|_{t\neq 0} \cong f_R$ and $F|_{s=0} \cong g_k$ and $\mathbb{G}_m$ acts with weight n on s and weight -1 on t.

As f_R was the restriction of $f_{\mathbb{A}^1}\colon \mathbb{A}^1 \to \theta \to \mathcal{X}$ we find that this morphism extends canonically to

$$F\colon [\operatorname{Spec} k[\pi,s,t]/(st^n - \pi)/\mathbb{G}_m] \to \mathcal{X}.$$

By uniqueness of the extension F_R this morphism comes equipped with a descent datum for the standard $\mathbb{G}_m$ action on $\mathbb{A}^1 = \operatorname{Spec} k[\pi]$. We therefore obtain

$$\overline{F}\colon [\operatorname{Spec} k[\pi,s,t]/(st^n - \pi)/\mathbb{G}_m^2] \to \mathcal{X}.$$

where the action of the second copy of $\mathbb{G}_m$ is with weight -1 on π and trivial on t.

We restrict $\overline{F}$ to the quotient by the subgroup $\mathbb{G}_m \subset \mathbb{G}_m^2$ acting with weight -1 on t, weight $-n$ on π and trivial on s to get $F' \colon [\operatorname{Spec} k[\pi, s, t]/(st^n - \pi)/\mathbb{G}_m]$.

Then $F'|_{s=0} = g_k$, so $F'^{,*}\ell|_{(s,t)=(0,0)} > 0$ and $F|_{s=1}$ defines a filtration on $f(1)$ specializing to the point $F(0,0)$, for which the pull back of ℓ is thus also positive, which contradicts the assumption that $f(1)$ was semistable. □

Proof of Proposition 5.13. Consider a discrete valuation ring R and a diagram

$$\begin{tikzcd} \operatorname{Spec}(R) \cup_{\operatorname{Spec}(K)} \operatorname{Spec}(R) \arrow[r] \arrow[d] & \mathcal{X}^{\mathrm{ss}} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} .$$

By hypothesis we can fill the dotted arrow uniquely to a map $\overline{\mathrm{ST}}_R \to \mathcal{X}$. We claim that in fact the map $\overline{\mathrm{ST}}_R \to \mathcal{X}$ factors through $\mathcal{X}^{\mathrm{ss}}$. Because $\mathcal{X}^{\mathrm{ss}}$ is open, it suffices to check that the unique closed point maps to $\mathcal{X}^{\mathrm{ss}}$. By hypothesis the point $(\pi, s, t) = (0, 1, 0)$ and the point $(\pi, s, t) = (0, 0, 1)$ map to $\mathcal{X}^{\mathrm{ss}}$. Restricting the map $\overline{\mathrm{ST}}_R \to \mathcal{X}$ to the locus $\Theta_k \simeq \{s = 0\}$ and $\Theta_k \simeq \{t = 0\}$ give two filtrations in $\mathcal{X}$ of points in $\mathcal{X}^{\mathrm{ss}}$, and if one has $f^*(\ell) < 0$ then the other has $f^*(\ell) > 0$, which would contradict the fact that $f(1) \in \mathcal{X}^{\mathrm{ss}}$. Therefore $f^*(\ell) = 0$ for both filtrations, and it follows from Lemma 5.14 that $f(0) \in \mathcal{X}^{\mathrm{ss}}$ as well.

For the corresponding claim for Θ-reductivity is proved similarly. For the analogous filling diagram, we start with a map $f : \Theta_R \setminus \{(0,0)\} \to \mathcal{X}^{\mathrm{ss}}$ and fill it to a map $\tilde{f} : \Theta_R \to \mathcal{X}$. We claim that $(0,0)$ maps to $\mathcal{X}^{\mathrm{ss}}$ as well, and hence because $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is open it follows that $\tilde{f}$ lands in $\mathcal{X}^{\mathrm{ss}}$. Because the restriction f_K of f to $\Theta_K \subset \Theta_R \setminus \{(0,0)\}$ maps to $\mathcal{X}^{\mathrm{ss}}$, we know from Lemma 5.14 that $f_K^*(\ell) = 0$. The function $f^*(\ell) \in \mathbb{R}$, regarded as a function on $\operatorname{Filt}(\mathcal{X})$, is locally constant. It therefore follows that the restriction of $\tilde{f}$, $\tilde{f}_k : \Theta_k \to \mathcal{X}$, also has $\tilde{f}_k^*(\ell) = 0$. It follows that $\tilde{f}_k(0) \in \mathcal{X}^{\mathrm{ss}}$. □

Remark 5.15. The conclusion of Lemma 5.14 and hence the conclusion Proposition 5.13 remain true without the hypothesis that $\mathcal{X}^{\mathrm{ss}}$ is part of a Θ-stratification as long as $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is open and $\mathcal{X} \to B$ is Θ-reductive.

Proof. In the proof of Lemma 5.14, we only used the existence of HN filtrations to find a filtration g of $f(0)$ which is invariant under the action of $\mathbb{G}_m$ on $f(0)$ coming from the data of the filtration $f : \Theta_k \to \mathcal{X}$. If $\mathcal{X} \to B$ is Θ-reductive, then the representable map satisfies the valuative criterion for properness $\operatorname{Filt}_B(\mathcal{X}) \to \mathcal{X}$, so the fiber of this map over $f(0) \in \mathcal{X}(k)$, which is denoted $\operatorname{Flag}(f(0))$ is an algebraic space of finite type over k which satisfies the valuative criterion for properness. The action of $\mathbb{G}_m$ by automorphisms of $f(0)$ gives a $\mathbb{G}_m$-action on $\operatorname{Flag}(f(0))$. Given some point $g \in \operatorname{Flag}(f(0))(k)$ for which $g^*(\ell) > 0$, we can consider the orbit $\mathbb{G}_m \to \operatorname{Flag}(f(0))$ of g. Because $\operatorname{Flag}(f(0))$ satisfies the valuative criterion for properness, this map extends to an equivariant map $\mathbb{A}_k^1 \to \operatorname{Flag}(f(0))$. This map sends $0 \in \mathbb{A}_k^1$ to a fixed point for the action of $\mathbb{G}_m$ on $\operatorname{Flag}(f(0))$, which corresponds to a filtration g' of $\operatorname{ev}_0(f) \in \operatorname{Grad}(\mathcal{X})(k)$, and g' is on the same connected component of $\operatorname{Flag}(f(0))$ as g, so $(g')^*(\ell) = g^*(\ell) > 0$. □

The following depends on the results of the next section, but for the purposes of exposition we include it here:

Corollary 5.16. *Let $\mathcal{X}$ be a locally finite type algebraic stack with affine diagonal over an algebraic space S of finite type over a field of characteristic 0. Assume that $\mathcal{X} \to S$ is S-complete, Θ-reductive, and satisfies the valuative criterion A for unpunctured inertia (Definition 6.1). If*

$$\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_{c \in \Gamma} \mathcal{S}_c$$

is a Θ-stratification, where $\mathcal{X}^{\mathrm{ss}}$ is the Θ-semistable locus with respect to some class $\ell \in H^2(\mathcal{X};\mathbb{R})$ and $\mathcal{X}^{\mathrm{ss}} \to B$ is quasi-compact, then $\mathcal{X}^{\mathrm{ss}}$ admits a good moduli space which is separated over B. Furthermore if $\mathcal{X} \to B$ satisfies the existence part of the valuative criterion for properness, then the good moduli space for $\mathcal{X}$ is proper over B.

Proof. The map $\mathcal{X}^{\mathrm{ss}} \to B$ is S-complete and Θ-reductive by Proposition 5.13, and it has unpunctured inertia by Proposition 6.12 below. It follows from Theorem 4.1 that there is a good moduli space $\mathcal{X}^{\mathrm{ss}} \to M$, and by Proposition 3.47 M is separated over B. If $\mathcal{X} \to B$ satisfies the existence part of the valuative criterion for properness, then so does $\mathcal{X}^{\mathrm{ss}} \to B$, by Corollary 5.12, and hence $M \to B$ is proper by Proposition 3.47. □

5.3. **Application: Properness of the Hitchin fibration.** Let us illustrate how the semistable reduction theorem 5.5 can be used to simplify and extend classical semistable reduction theorems for principal bundles and Higgs bundles on curves.

The setup for these results is the following (see e.g.,[Ngô06, Section 2]). Let C be a smooth projective, geometrically connected curve over a field k and G a reductive algebraic group. As the notions are slightly easier to formulate over algebraically closed fields and the valuative criteria allow for extensions of the ground field, we will assume that k is algebraically closed in this section.

We denote by Bun_G the stack of principal G-bundles on C, i.e., for a k scheme S we have that $\mathrm{Bun}_G(S)$ is the groupoid of principal G-bundles on $C \times S$. Fix a line bundle $\mathcal{L}$ on C. A G-Higgs bundle with coefficients $\mathcal{L}$ on C is a pair $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is a G-bundle on C and $\phi \in H^0(C, (\mathcal{P} \times^G \mathrm{Lie}(G)) \otimes \mathcal{L})$. We denote by Higgs_G the stack of G-Higgs bundles with coefficients in $\mathcal{L}$.

The stack Higgs_G comes equipped with the forgetful morphism $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ and the Hitchin morphism $h\colon \mathrm{Higgs}_G \to \mathcal{A}_G$. Here $\mathcal{A}_G \cong \oplus_{i=1}^r H^0(C, \mathcal{L}^{d_i})$, where $d_1, \ldots, d_r$ are the degrees of the invariant polynomials of G and h is defined by mapping $(\mathcal{P}, \phi)$ to the characteristic polynomial of ϕ.

On both Bun_G and Higgs_G there is a classical notion of stability, which is defined in terms of reductions to parabolic subgroups.

Let us recall how this notion is related to Θ-stability. For vector bundles there is an equivalence (Proposition 2.8,[Hei17, Lemma 1.10])

$$\mathrm{Map}(\Theta, \mathrm{Bun}_{\mathrm{GL}_n}) \cong \langle (\mathcal{E}, \mathcal{E}^i)_{i \in \mathbb{Z}} | \begin{array}{l} \mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}_n}, \mathcal{E}^i \subseteq \mathcal{E}^{i+1} \subseteq \mathcal{E} \text{ subbundles} \\ \mathcal{E}^i = \mathcal{E} \text{ for } i \gg 0, \mathcal{E}^i = 0 \text{ for } i \ll 0 \end{array} \rangle$$

which is given by assigning to a weighted filtration of a vector bundle $\mathcal{E}$ the canonical $\mathbb{G}_m$-equivariant degeneration of $\mathcal{E}$ to into the associated graded bundle.

This construction has an analog for principal bundles. To state this we fix (as in 2.8) a complete set of conjugacy classes of cocharacters $\Lambda \subset \mathrm{Hom}(\mathbb{G}_m, G)$. As in Section 2.3 we denote by $P_\lambda^+ \subseteq G$ the parabolic subgroup defined by λ and by $L_\lambda \subset P_\lambda^+$ the Levi subgroup defined by λ which is isomorphic to the quotient of P_λ^+ by its unipotent radical $U_\lambda^+ \subset P_\lambda^+$. Then there is an equivalence (see e.g., [Hei17, Lemma 1.13])

$$\mathrm{Map}(\Theta, \mathrm{Bun}_G) \cong \coprod_{\lambda \in \Lambda} \mathrm{Bun}_{P_\lambda^+}.$$

For Higgs bundles note that the forgetful map $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ is representable and therefore $\mathrm{Map}(\Theta, \mathrm{Higgs}_G) \subset \mathrm{Map}(\Theta, \mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathrm{Higgs}_G$, i.e., a filtration on a Higgs bundle is the same as a filtration on the underlying principal bundle that preserves the Higgs field ϕ.

Recall that a G-bundle $\mathcal{E}$ is called semistable, if for all λ and all $\mathcal{E}_\lambda \in \mathrm{Bun}_{P_\lambda^+}$ with $\mathcal{E}_\lambda \times^{P_\lambda^+} G \cong \mathcal{E}$ we have $\deg(\mathcal{P}_\lambda \times^{P_\lambda^+} \mathrm{Lie}(P_\lambda^+)) \leq 0$. Similarly a Higgs bundle is called semistable if the same condition holds for all reductions that respect the Higgs field ϕ. This stability notion can be viewed as θ-stability induced from the so called determinant line bundle $\mathcal{L}_{\det}$ on the stack Bun_G whose fiber at a point $\mathcal{P}$ (resp. a point $(\mathcal{P}, \phi)$) is given by the one dimensional vector space $\det(H^1(C, \mathcal{P} \times^G \mathrm{Lie}(G))) \otimes \det(H^0(C, \mathcal{P} \times^G \mathrm{Lie}(G)))^{-1}$ (see [Hei17, Section 1.F], [Hal14]).

As usual we denote buy $\mathrm{Bun}_G^{ss} \subseteq \mathrm{Bun}_G$ the open substack of semistable bundles and for by $\mathrm{Bun}_{P_\lambda^+}^{ss} \subseteq \mathrm{Bun}_{P_\lambda^+}$ the open substack of bundles such that the associated L_λ-bundle is semistable.

Finally let us recall how the notion of Harder-Narasimhan reduction can be used to equip the stacks Bun_G and Higgs_G with a (well-ordered) Θ-stratification if the characteristic of k is not too small, i.e., such that Behrends conjecture hold for G (see [Hei08a, Theorem 1] for explicit bounds depending on G. Note that char 2 has to be excluded for groups of type B_n, D_n as well).

For any unstable G-bundle $\mathcal{P}$ there exists a canonical Harder-Naramsimhan reduction $\mathcal{P}^{HN}$ to a parabolic subgroup P_λ^+, where λ is uniquely determined up to a positive integral multiple. We denote by

$$\begin{aligned} \underline{d} := \underline{\deg}(\mathcal{P}^{HN})\colon\ \mathrm{Hom}(P_\lambda, \mathbb{G}_m) &\to \mathbb{Z} \\ \chi &\mapsto \deg(\mathcal{P}_\lambda^{HN} \times^\chi \mathbb{G}_m) \end{aligned}$$

the degree of $\mathcal{P}^{HN}$ and by $\mathrm{Bun}_{P_\lambda^+}^{\underline{d},ss} \subset \mathrm{Bun}_{P_\lambda^+}^{ss}$ the connected component defined by $\underline{d}$. The instability degree of $\mathcal{P}_\lambda$ is defined as

$$\mathrm{ideg}(\mathcal{P}) := \deg(\mathcal{P}^{HN} \times^{P_\lambda^+} \mathrm{Lie}(P)).$$

Behrend showed that the morphism $\mathrm{Bun}_{P_\lambda^+}^{\underline{d},ss} \to \mathrm{Bun}_G$ defined by the inclusion $P_\lambda^+ \subset G$ is radicial if the degree $\underline{d}$ is the degree of a HN-reduction ([Beh]) and the map is an embedding if Behrends conjectre holds for G ([Hei08b, Lemma 2.3]) this condition is satisfied if the characteristic of k is not too small with respect to G (e.g., > 31).

Moreover the instability degree ideg is upper semicontinuous in families and if this invariant is constant on a family, then the family admits a global Harder-Narasimhan reduction ([Beh, Proposition 7.1.3], [Hei08b, Proposition 2.2.]). Thus the HN-reduction of bundles defines a Θ-stratification on Bun_G if the characteristic of k is not too small. The same arguments apply for Higgs bundles and this shows the following lemma.

Lemma 5.17. *If the characteristic of k is large enough so that Behrend's conjecture holds for G, then the HN-stratifications of* Bun_G *and* Higgs_G *form a well-ordered Θ-stratification.*

To apply the semistable reduction theorem to $\mathrm{Higgs}_G \to \mathcal{A}_G$ we need to show that this morphism satisfies the existence part of the valuative criterion for properness. The existence result is probably well known (see e.g. [CL10, Section 8.4] for an argument over the regular locus) but we could not find a general reference.

Lemma 5.18. *Suppose that the characteristic of k is not a torsion prime for G and very good for G. Let R be a discrete valuation ring with fraction field K and $(\mathcal{E}_K, \phi_K) \in \mathrm{Higgs}_G(K)$ a Higgs bundle such that $h(\mathcal{E}_K, \phi_K) \in \mathcal{A}_G(R) \subset \mathcal{R}_G(K)$. Then there exists a finite extension R'/R and a point $(\mathcal{E}'_R, \phi'_R) \in \mathrm{Higgs}_G(R')$ extending $(\mathcal{E}_K, \phi_K) \in \mathrm{Higgs}_G(K)$.*

Proof. First let us assume that the derived group of G is simply connected.

Let R be a discrete valuation ring with fraction field K and $(\mathcal{E}_K, \phi_K) \in \mathrm{Higgs}_G(K)$ a Higgs bundle such that $h(\mathcal{E}_K, \phi_K) \in \mathcal{A}_G(R) \subset \mathcal{R}_G(K)$. The generic point of C will be denoted by η, $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathrm{car} := \mathfrak{g}//G$ is the space of characteristic polynomials of elements of $\mathfrak{g}$.

We argue as in [CL10, Section 8.4]. After a finite extension of K we may assume that $\mathcal{E}_K$ is trivial at the generic point $K(\eta)$ of C_K. Choosing a trivialization identifies ϕ_K with an element in $X_K \in \mathfrak{g}(K(\eta))$. To conclude the argument as in loc.cit., it is sufficient to show that after passing to a finite extension of K we can conjugate X_K to an element of $\mathfrak{g}(R(\eta))$.

We denote by $X_K = X_K^s + X_K^n$ the Jordan decomposition of X_K into the semisimple and nilpotent part of g_K.

As $h(\mathcal{E}, \phi)$ extends to R we know that the image of X_K in $\mathrm{car} = \mathfrak{g}//G$ defines an $R(\eta)$-valued point. We can use the Kostant section $\mathrm{car} \to \mathfrak{g}$ to obtain $Y_R \in \mathfrak{g}(R(\eta))$ with $h(Y_R) = h(X_K)$.

We claim that we can modify Y_R such that its generic fiber Y_K is semisimple. To see this let us consider the Jordan decomposition $Y_K = Y_K^s + Y_K^u$. By our assumptions on the characteristic of K the main result of [McN05] shows that there exists a parabolic subgroup $P(\lambda) \subset G$ defined by a cocharacter $\lambda\colon \mathbb{G}_m \to G_K$ such that Y_K^u is contained in the Lie algebra of the unipotent radical of $P(\lambda)$ and as Y_K^s is in the centralizer of Y_K^u the element Y_K also lies in $\mathrm{Lie}(P(\lambda))$. As parabolic subgroups extend over valuation rings, we find that Y_R is contained in a parabolic subgroup $P_R \subset G_R$ and we can choose λ to be a cocharacter defined over R as well.

As $P(\lambda)$ is defined to be the set of points such that $\lim_{t\to 0} \lambda)(t)p$ exists, the limit $\lim_{t\to 0} \lambda(t).Y_R$ will be an R-valued point Y'_R such that $Y'_K = Y_K^s$ is semisimple.

As the semi-simple part of X_K is the unique closed orbit in the conjugacy class of X_K we know that X_K^s and Y_K^s lie in the same closed orbit. As we assumed that the derived group of G is simply connected and the p is not a torsion prime for G the centralizer $Z_G(X_s)$ is a connected reductive group ([Ste75, Theorem 0.1]). By Steinberg's theorem, any $Z_G(X_s)$ torsor over $K(\eta)$ splits after a finite extension of K, so after possibly extending K the elements X_K^s and Y_K^s are conjugate. Thus after conjugating X_K we may assume that $X_K^s = Y_K^s$, i.e. we may assume that the semisimple part of X_K extends to R.

Now we can apply the previous argument to X_K, namely the element X_K is contained in a parabolic subalgebra defined by a cocharacter λ, such that X_K^u is contained in its unipotent radical, so that for some $a \in K$ the element $\lambda(a).X_K^u$ will extend to R as well.

Finally for any group G we can consider a z-extension

$$0 \to Z \to \widetilde{G} \to G \to 1$$

where Z is a central torus and the derived group G' of $\widetilde{G}$ is simply connected. Then the map $\mathrm{Bun}_{\tilde{G}} \to \mathrm{Bun}_G$ is a smooth surjection. Moreover the covering $G' \to G$ is separable, because we assumed that the fundamental group of G has no p-torsion. Therefore $\mathrm{Lie}(\tilde{G}) \cong \mathrm{Lie}(Z) \oplus \mathrm{Lie}(G)$ and therefore the map $\mathrm{Higgs}_{\tilde{G}} \to \mathrm{Higgs}_G$ also admits local sections. Thus it suffices to prove the result for $\tilde{G}$.

□

The semistable reduction theorem 5.5 now allows us to deduce:

Corollary 5.19. *Suppose that the characteristic p of k is large enough such that Behrend's conjecture holds for G, such that p is not a torsion prime for G and such that p is very good for G, then the Hitchin morphism*

$$h\colon \mathrm{Higgs}_G^{ss} \to \mathcal{A}_G$$

satisfies semistable reduction, i.e. if R is a discrete valuation ring with fraction field K and $x_K\colon \operatorname{Spec} K \to \mathrm{Higgs}_G^{ss}$ is a map such that $h(x_K)\colon \operatorname{Spec} K \to \mathcal{A}_G$ extends to R, then there exists a finite extension R'/R and $x_{R'} \in \mathrm{Higgs}_G(R')$ extending x_K.

Note that in characteristic 0 this result is due to Faltings [Fal93] and for a large part of the Hitchin fibration this is due to Chaudouard-Laumon [CL10].

Proof. By Lemma 5.18 we can find an extension of x_K to x_R. As Higgs_G admits a well ordered Θ-stratification by Harder-Narasimhan reductions we can therefore apply the semistable reduction Theorem 5.5 to conclude. □

6. Criteria for unpunctured inertia

In this section we discuss conditions which imply that a stack has unpunctured inertia (Definition 3.54). The results of this section are summarized in Theorem 6.6.

6.1. **Four valuative criteria.** We introduce four closely related properties of a stack which we call "valuative criteria," because they involve maps from a complete discrete valuation ring to $\mathcal{X}$

Definition 6.1 (Valuative criterion A)**.** Let $\mathcal{X}$ be an algebraic stack. We say that $\mathcal{X}$ satisfies the *valuative criterion A* if for any map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$, where R is a complete DVR with fraction field K, and any $g \in \mathrm{Aut}(\xi_K)$ of finite order prime to $\mathrm{char}(K)$, there is a modification of ξ to a map $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ such that the restriction $g|_{K'} \in \mathrm{Aut}(\xi'_{K'})$ extends to an automorphism of ξ'. We say that $\mathcal{X}$ satisfies the *strong valuative criterion A* if furthermore the modification can be chosen such $\xi'(0)$ is a specialization of $\xi(0)$.

Example 6.2. To illustrate the subtlety of this condition, let us exhibit in the context of Example 3.57 a map from a DVR to $[\mathbb{A}^2/G]$ (where $G = \mathbb{G}_m \rtimes (\mathbb{Z}/2)$) where performing an elementary modification allows a generic automorphism to extend. Let $R = k[[z]]$ and $K = k((z))$. Consider $\xi\colon \operatorname{Spec} R \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. Then $g = (z, -1) \in G(K)$ stabilizes ξ_K but does not extend to $G(R)$. Consider the degree 2 ramified extension $R \to R'$ with $R' = k[[\sqrt{z}]]$

and $K' = k((\sqrt{z}))$, and define $\xi' \colon \operatorname{Spec} R' \to X$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, ξ' is isomorphic as a point in $[\mathbb{A}^2/G]$ to the restriction $\xi|_{K'}$, because $(\sqrt{z}, -1) \cdot \xi'_{K'} = \xi|_{K'}$. Under this isomorphism our generic automorphism g becomes $g' = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1)$ which clearly extends to $\operatorname{Aut}(\xi')$.

The second valuative criterion is the following:

Definition 6.3 (Valuative criterion B)**.** Let $\mathcal{X}$ be an algebraic stack. We say that $\mathcal{X}$ satisfies the *valuative criterion B* if for any map $\xi : \operatorname{Spec}(R) \to \mathcal{X}$, where R is a complete DVR with fraction field K, and any geometrically connected component of $\underline{\operatorname{Aut}}_K(\xi_K)$, there is a modification of ξ to a map $\xi' : \operatorname{Spec}(R') \to \mathcal{X}$ such that the closure of this component in $\underline{\operatorname{Aut}}_{R'}(\xi')$ has non-empty intersection with the special fiber. We say that $\mathcal{X}$ satisfies the *strong valuative criterion B* if furthermore the modification can be chosen such $\xi'(0)$ is a specialization of $\xi(0)$.

Lemma 6.4. *The (strong) valuative criterion B is equivalent to the condition that for any map $\xi : \operatorname{Spec}(R) \to \mathcal{X}$, where R is a complete DVR with fraction field K, and for any geometrically connected component $H \subset \underline{\operatorname{Aut}}_K(\xi_K)$, there exist a modification ξ' of ξ (for which $\xi'(0)$ is a specialization of $\xi(0)$) and* some *$g \in \operatorname{Aut}(\xi')$ such that $g|_{K'}$ lies in H.*

The proof of this lemma follows immediately from the following fact:

Lemma 6.5. *Let $X \to \operatorname{Spec}(R)$ be a scheme of finite type over a DVR R, and let $H \subset X_K$ be a geometrically connected component. Then $\bar{H}$ meets the fiber over $0 \in \operatorname{Spec}(R)$ if and only if there is a finite extension of DVR's $R \subset R'$ such that $X_{R'} \to \operatorname{Spec}(R')$ admits a section whose generic point lies is H.*

Proof. The sufficiency of this condition is clear. For necessity, one may replace X by the reduced closure of H, which is faithfully flat over $\operatorname{Spec}(R)$. A faithfully flat map is universally submersive so after a finite extension of DVR's one has a section. $\square$

6.2. **Summary theorem.** We now summarize the results of this section:

Theorem 6.6 (Summary theorem)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type with affine diagonal over a field. Then we have a diagram of implications*

$$\begin{tikzcd}
 & \boxed{\textit{good moduli space}} \arrow[d, Rightarrow] & \\
\boxed{\textit{valuative criterion A}} \arrow[r, dashed, Rightarrow, "(a)\&(b)"'] \arrow[d, Rightarrow] & \boxed{\textit{strong valuative criterion A}} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\
\boxed{\textit{valuative criterion B}} \arrow[r, dashed, Rightarrow, "(a)\&(b)"'] & \boxed{\textit{strong valuative criterion B}} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \boxed{\textit{unpunctured inertia}} \arrow[ul, dashed, Rightarrow, "(a)"'] \arrow[uul, dashed, Rightarrow, "(a)\&(b)"']
\end{tikzcd}$$

where the boxes describe properties which $\mathcal{X}$ can have, and the solid arrows denote implications. The dotted arrows denote implications under the additional hypotheses (a): locally linearly reductive, and (b): Θ-reductive.

In particular, if $\mathcal{X}$ is Θ-reductive and locally linearly reductive over a field, then all of the four valuative criteria are equivalent to $\mathcal{X}$ having unpunctured inertia, which is also equivalent to $\mathcal{X}$ having a good moduli space.

This theorem will follow from Proposition 6.8, Proposition 6.11 and Theorem 6.20

Remark 6.7. The fact that 'unpunctured inertia' with conditions (a) and (b) implies 'good moduli space' was shown in the sufficiency of the conditions in Theorem 4.1. Conversely, the implication that a good moduli space has unpunctured inertia will follow from Theorem 6.20; recall that Theorem 6.20 was quoted in the proof of necessity of the conditions in Theorem 4.1.

6.3. Strong valuative criteria imply unpunctured inertia.

Proposition 6.8. *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine automorphism groups. Then the valuative criterion A implies the valuative criterion B, and we have implications*

$$\boxed{\begin{array}{c}\textit{strong valuative}\\ \textit{criterion A}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\textit{strong valuative}\\ \textit{criterion B}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\textit{unpunctured}\\ \textit{inertia}\end{array}},$$

and all three conditions are equivalent if $\mathcal{X}$ is locally linearly reductive and finite type over a field.

Proof. (Strong) valuative criterion A implies (strong) valuative criterion B:

It suffices to show that every connected component of $\underline{\mathrm{Aut}}_K(\xi_K)$ contains a finite type point of finite order. Let $g \in \underline{\mathrm{Aut}}_K(\xi_K)$ be a finite type point. After a finite field extension we can decompose $g = g_s g_u$ under the Jordan decomposition, where g_s is semisimple and g_u is unipotent. Then scaling g_u to 1 exhibits a family of group elements containing g and specializing to g_s. Now consider the reduced Zariski closed K-subgroup $H \subset \underline{\mathrm{Aut}}_K(\xi_K)$ generated by g_s. Because g_s is semisimple, H is a diagonalizable K-group and hence isomorphic to the product of a torus and copies of μ_{p^n} for p prime to $\mathrm{char}(K)$. Therefore every component of H contains an element of finite order prime to $\mathrm{char}(K)$.

Strong valuative criterion B implies unpunctured inertia:

Let $x \in \mathcal{X}$ be a closed point, and let $p : (U, u) \to (\mathcal{X}, x)$ be a versal deformation of x, and let $H \subset \underline{\mathrm{Aut}}_U(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains u. It follows that we can find a complete DVR R and a map $\operatorname{Spec} R \to U$ whose special point maps to u and whose generic point lies in the image of $H \to U$. After a finite extension of the DVR R, we may assume that the generic point $\mathrm{Spec}(K) \to U$ lifts to H, and that the connected component $H' \subset H|_{\mathrm{Spec}(K)}$ containing this lift is geometrically connected. By the strong valuative criterion B, after possibly further extending R, there exists a modification $\xi'\colon \operatorname{Spec} R \to \mathcal{X}$ of ξ such that the closure of H' in $\underline{\mathrm{Aut}}(\xi)$ meets the fiber over $0 \in \mathrm{Spec}(R)$ and $0 \in \mathrm{Spec}(R)$ still maps to u. By construction H' maps to H, which implies that $H \subset \underline{\mathrm{Aut}}_U(p)$ meets the fiber over u.

Unpunctured inertia implies strong valuative criterion A when $\mathcal{X}$ is locally linearly reductive and finite type over a field:

By Proposition 4.3 we can find an étale map $\mathcal{Y} = [X/G] \to \mathcal{X}$, with X affine and G linearly reductive, such that the induced map on inertia stacks $I_{\mathcal{Y}} \to \mathcal{Y} \times_{\mathcal{X}} I_{\mathcal{X}}$

is an isomorphism. The strong valuative criterion A is a property of the relative group scheme $I_{\mathcal{X}} \to \mathcal{X}$, and it can be checked étale locally, so it suffices to establish the strong valuative criterion for $\mathcal{Y} = [X/G]$. As $[X/G]$ has a good moduli space, this will follow from Theorem 6.20 below. $\square$

Remark 6.9. The valuative criterion B does not imply the valuative criterion A without additional hypotheses. Consider the group $\mathbb{G}_m \ltimes \mathbb{G}_a$ given coordinates (z, y) and the product rule $(z_1, y_1)\cdot(z_2, y_2) = (z_1 z_2, z_2 y_1 + y_2)$, and let $G \subset (\mathbb{G}_m \ltimes \mathbb{G}_a) \times \mathbb{A}^1_t$ be the hypersurface cut out by the equation $ty = 1 - z$. Then G is in fact a smooth subgroup scheme whose fiber over 0 is $\mathbb{G}_a$ and whose fiber everywhere else is $\mathbb{G}_m$.

Let $\mathcal{X} = BG$ and consider the map $\xi : \mathrm{Spec}(k[[t]]) \to \mathcal{X}$ which is just the completion of the canonical map $\mathbb{A}^1_t \to \mathcal{X}$ at the origin. Then for any modification of this map the special fiber still must map to $0 \in \mathbb{A}^1_t$ under the projection $\mathcal{X} \to \mathbb{A}^1_t$, so the automorphism group of ξ will be isomorphic to $G_{k[[t]]}$. There is a generic automorphism of ξ given by the formula $(\alpha, (1-\alpha)/t)$, where α is a non-identity n^{th} root of unity. This automorphism does not extend to 0, and the generic automorphism group is abelian and hence acts trivially on itself by conjugation. It follows that no modification of ξ will allow this generic automorphism to extend either.

6.4. **Relationship between weak and strong valuative criteria.** It is immediate from Definition 6.1 and Definition 6.3 that the strong valuative criteria imply the valuative criteria. In this section we prove a partial converse. We will use the following fact, which is of independent interest:

Lemma 6.10. *Let $\mathcal{X}$ be an algebraic stack of finite type over a field k which is Θ-reductive and locally linearly reductive. Then the closure of any k-point p contains a unique closed point x.*

Proof. Assume that x and x' are two closed points in the closure of p. After replacing k with a finite extension if necessary, we may assume that x and x' are k-rational. Hence the specializations $p \rightsquigarrow x$ and $p \rightsquigarrow x'$ are isotrivial. It follows from Lemma 3.25 that these specializations come from two filtrations $f, f' : \Theta_k \to \mathcal{X}$ with $f(1) \simeq f'(1) \simeq p$, $f(0) \simeq x$ and $f'(0) \simeq x'$. The maps f and f' glue to define a map $(\mathbb{A}^2_k - \{(0,0)\})/(\mathbb{G}^2_m)_k$, and this map extends uniquely to a map $\gamma : \mathbb{A}^2_k/(\mathbb{G}^2_m)_k \to \mathcal{X}$ by [Hal14, Proof of Proposition 4.16]. Then $\gamma(0,0)$ is a specialization of both $x \simeq \gamma(1,0)$ and $x' \simeq \gamma(0,1)$, which because x and x' are closed implies that $x \simeq \gamma(0,0) \simeq x'$. $\square$

Proposition 6.11. *Let $\mathcal{X}$ be an algebraic stack of finite type with affine diagonal over a field k. If $\mathcal{X}$ is Θ-reductive and locally linearly reductive, then the valuative criterion A (respectively B) is equivalent to the strong valuative criterion A (respectively B).*

Proof. Say we are given a map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ from a complete DVR R with fraction field K and residue field κ and a generic automorphism $g \in \mathrm{Aut}(\xi|_K)$ of finite order prime to $\mathrm{char}(K)$. The valuative criterion A produces a modification $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ of ξ along with an element $g' \in \mathrm{Aut}(\xi')$ extending $g|_{K'}$. Given this data, we must produce a further modification $\xi'' : \mathrm{Spec}(R'') \to \mathcal{X}$ with $g'' \in \mathrm{Aut}(\xi'')$ extending $g|_{K''}$ and such that $\xi''(0)$ is a specialization of $\xi(0)$. Let $\mathcal{Z} \subset \mathcal{X}_\kappa$ be the closure of the point $\xi(0)) \in \mathcal{X}_\kappa$. By Lemma 6.10 we know that $\mathcal{Z}$ has a unique closed point $x' \in \mathcal{Z}$, and in particular x' is a specialization of $\xi(0)$. If necessary we pass to a finite extension of κ and a corresponding finite extension of $R \simeq \kappa[[\pi]]$ so that we may assume that $x' \in \mathcal{Z}(\kappa)$ as well. In order to verify the strong valuative criterion A, we will construct our modification ξ'' of ξ' such that $\xi''(0) = x'$.

Proposition 4.4 implies that $\mathcal{Z} \simeq [\mathrm{Spec}(A)/G_{x'}]$ for some affine $G_{x'}$-scheme $\mathrm{Spec}(A)$. Let $p = \xi'(0) \in [\mathrm{Spec}(A)/G](\kappa)$. Kempf's theorem [Kem78] implies that after passing to a finite purely inseparable extension of κ there is a canonical filtration $f\colon \Theta_\kappa \to \mathrm{Spec}(A)/G_{x'}$ with an isomorphism $f(1) \simeq p$ such that $f(0) = x'$. The fact that f is canonical means that any automorphism of $p = f(1)$ extends to an automorphism of the map f. In particular the automorphism g' of ξ' restricts to an automorphism of $p = \xi'(0)$ which extends uniquely to an automorphism of f which we also denote g'.

We now apply the strange gluing lemma (Corollary A.2), which states that after composing f with a suitable ramified cover $(-)^n : \Theta_\kappa \to \Theta_\kappa$, the data of the map $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ and the filtration $f : \Theta_\kappa \to \mathcal{X}$, comes from a *unique* map $\gamma : \overline{\mathrm{ST}}_{R'} \to \mathcal{X}$, where f is the restriction of γ to the locus $\{s = 0\}$ and ξ' is the restriction of γ to the locus $\{t \neq 0\}$. The uniqueness of this extension guarantees that the automorphism g' of ξ' and f extends uniquely to an automorphism of γ, which we again denote g'. Finally we construct our modification as the composition

$$\xi'' : \mathrm{Spec}(R'[\sqrt{\pi}]) \to \overline{\mathrm{ST}}_{R'} \xrightarrow{\gamma} \mathcal{X},$$

where the first map is given in (s, t, π) coordinates by $(\sqrt{\pi}, \sqrt{\pi}, \pi)$, which maps the special point of $\mathrm{Spec}(R'[\sqrt{\pi}])$ to the point $\{s = t = \pi = 0\}$ of $\overline{\mathrm{ST}}_R$. By construction the automorphism g' restricts to an automorphism g'' of ξ'' extending $g|_{K''}$, and the special point $\xi''(0)$ maps to the closed point x' of $\mathcal{Z}$, which verifies the strong valuative criterion A.

The argument that the valuative criterion B implies the strong valuative criterion B is similar: if $H \subset \underline{\mathrm{Aut}}_K(\xi_K)$ is geometrically connected component, then by Lemma 6.4, we may assume that a modification $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ of our original map ξ has been constructed such that there is an automorphism $g' \in \mathrm{Aut}(\xi')$ whose restriction to K' lies in H. Now we may repeat the argument above verbatim to produce a modification $\xi'' : \mathrm{Spec}(R'') \to \mathcal{X}$ with $\xi''(0) = x'$ and an automorphism g'' of ξ'' (which may no longer be of finite order) such that $g''|_{K''}$ lies in H. $\square$

6.5. **Valuative criteria and Θ-stratifications.** In this subsection, we assume that $\mathcal{X}$ is an algebraic stack locally of finite type and with affine diagonal over a base stack B which locally admits a smooth surjection from a G-ring. We consider a well-ordered Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_\alpha \mathcal{S}_\alpha$ (We recalled the notion of a Θ-stratification in Section 5). We denote the center of the stratum $\mathcal{S}_\alpha$ by $\mathcal{Z}_\alpha^{\mathrm{ss}}$.

Proposition 6.12. *If $\mathcal{X}$ satisfies the valuative criterion A (respectively B), then so do $\mathcal{X}^{\mathrm{ss}}$, $\mathcal{S}_\alpha$, and $\mathcal{Z}_\alpha^{\mathrm{ss}}$ for all α. Conversely if $\mathcal{X}$ is defined over a field and $\mathcal{X}^{\mathrm{ss}}$ and $\mathcal{Z}_\alpha^{\mathrm{ss}}$ satisfy the valuative criterion A for all α, then $\mathcal{X}$ satisfies the valuative criterion A as well.*

Lemma 6.13. *Any Θ-stratification of $\mathcal{X}$ induces a Θ-stratification of $I_{\mathcal{X}}$.*

Proof. By definition $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X}^2} \mathcal{X}$. The formation of $\mathrm{Filt}(-) := \underline{\mathrm{Map}}(\Theta, -)$ commutes with fiber products, so we have a canonical isomorphism $I_{\mathrm{Filt}(\mathcal{X})} \simeq \mathrm{Filt}(I_{\mathcal{X}})$, and this isomorphism identifies the canonical map $\mathrm{ev}_1 : \mathrm{Filt}(I_{\mathcal{X}}) \to I_{\mathcal{X}}$ with the map one gets by applying $I_{(-)}$ to $\mathrm{ev}_1 : \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$. Now let $\mathcal{S} \subset \mathrm{Filt}(\mathcal{X})$ be a union of connected component such that $\mathrm{ev}_1 : \mathcal{S} \to \mathcal{X}$ is a closed immersion.

Then $\mathcal{S}' : I_{\mathcal{X}}|_{\mathcal{S}} \simeq I_{\mathcal{S}} \subset I_{\mathrm{Filt}(\mathcal{X})}$ is a union of connected components which is identified under the equivalence $\mathrm{Filt}(I_{\mathcal{X}}) \simeq I_{\mathrm{Filt}(\mathcal{X})}$ with the preimage of $\mathcal{S}$ under the projection $\mathrm{Filt}(I_{\mathcal{X}}) \to \mathrm{Filt}(\mathcal{X})$. It follows that $\mathrm{ev}_1 : \mathcal{S}' \to I_{\mathcal{X}}$ is a Θ-stratum, i.e. $\mathcal{S}$ induces a Θ-stratum in $I_{\mathcal{X}}$. The result for a Θ-stratification with more than one stratum follows as well. □

Proof of Proposition 6.12. Assume that $\mathcal{X}$ satisfies the valuative criterion A, and let $\xi : \mathrm{Spec}(R) \to \mathcal{X}^{\mathrm{ss}}$ be a map from a DVR along with a generic automorphism $g \in \mathrm{Aut}(\xi_K)$ of finite order prime to $\mathrm{char}(K)$. Then by hypothesis we can modify the composition $\mathrm{Spec}(R) \to \mathcal{X}^{\mathrm{ss}} \to \mathcal{X}$ to a map $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$ so that $g|_{K'}$ extends to an automorphism $g' \in \mathrm{Aut}(\xi')$. Regard g' as lift

$$\begin{tikzcd} & I_{\mathcal{X}} \arrow[d] \\ \mathrm{Spec}(R') \arrow[ur, dashed, "g'"] \arrow[r, "\xi'"] & \mathcal{X} \end{tikzcd} .$$

By Lemma 6.13, the Θ-stratification of $\mathcal{X}$ induces a Θ-stratification of $I_{\mathcal{X}}$, and by hypothesis $g'(\mathrm{Spec}(K')) \subset (I_{\mathcal{X}})^{\mathrm{ss}}$. Then Theorem 5.3 implies that we can find a modification $g'' : \mathrm{Spec}(R'') \to I_{\mathcal{X}}$ of g' which lands entirely in $(I_{\mathcal{X}})^{\mathrm{ss}}$. Then the composition of g'' with $(I_{\mathcal{X}})^{\mathrm{ss}} \to \mathcal{X}^{\mathrm{ss}}$ is a modification of the original ξ for which $g|_{K''}$ extends. Hence $\mathcal{X}^{\mathrm{ss}}$ satsifies the valuative criterion A. Furthermore, this implies that every $\mathcal{S}_\alpha$ satisfies the valuative criterion because the Θ-stratification induces a Θ-stratification of closed substack $\bigcup_{\beta \geq \alpha} \mathcal{S}_\beta$ (with its reduced structure) whose semistable locus is $\mathcal{S}_\alpha$ [Hal14, Lemma 2.14].

Finally the map $\sigma : \mathcal{Z}_\alpha^{\mathrm{ss}} \to \mathcal{S}_\alpha$ is a section for the retract $\mathrm{ev}_0 : \mathcal{S}_\alpha \to \mathcal{Z}_\alpha^{\mathrm{ss}}$, i.e. $\mathrm{ev}_0 \circ \sigma \simeq \mathrm{id}_{\mathcal{Z}_\alpha^{\mathrm{ss}}}$. Therefore given a family ξ: $\mathrm{Spec}(R) \to \mathcal{Z}_\alpha^{\mathrm{ss}}$ and a generic automorphism, if one can find a modification ξ' of $\sigma \circ \xi \in \mathcal{S}_\alpha(R)$ such that the generic automorphism extends to $\mathrm{Spec}(R)$, then $\mathrm{ev}_0 \circ \xi'$ is a modification of ξ such that the generic automorphism extends. Therefore if $\mathcal{S}_\alpha$ satisfies the valuative criterion A then so does $\mathcal{Z}_\alpha^{\mathrm{ss}}$.

The converse:

Given a map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ and a generic automorphism of ξ, we apply the semistable reduction theorem to find a modification of ξ whose image lies in a single stratum. Therefore if $\mathcal{X}^{\mathrm{ss}}$ and all unstable strata $\mathcal{S}_\alpha$ satisfy the valuative criterion A, then we can find a further modification of ξ such that g extends. It therefore suffices to show that if $\mathcal{Z}_\alpha^{ss}$ satisfies the valuative criterion A, then so does $\mathcal{S}_\alpha$.

A map $\xi : \mathrm{Spec}(R) \to \mathcal{S}_\alpha$ corresponds to a map $f : \Theta_R \to \mathcal{X}$. The generic automorphism g of $\xi|_K$ corresponds to a map $\gamma : \Theta_K \times (\mathrm{pt}/\Gamma) \to \mathcal{X}$, where Γ is the cyclic group whose order is the order of g, along with an isomorphism $\gamma|_{\Theta_K} \simeq f|_{\Theta_K}$. Restricting γ to $\mathrm{Spec}(K) \times \{0\}/(\mathbb{G}_m \times \Gamma)$ corresponds to a map $\mathrm{Spec}(K) \times (\mathrm{pt}/\Gamma) \to \mathcal{Z}_\alpha^{ss}$ which extends to $\mathrm{Spec}(R)$ non-Γ-equivariantly. If $\mathcal{Z}_\alpha^{ss}$ satisfies the valuative criterion A, then one can extend this to a map $\mathrm{Spec}(R) \times (\mathrm{pt}/\Gamma) \to \mathcal{Z}_\alpha^{ss}$. Equivalently, one can extend the previous map to a map $\mathrm{Spec}(R) \times \{0\}/(\mathbb{G}_m \times \Gamma)$.

Now by [AHR15], one can find a smooth representable map $\mathrm{Spec}(A)/(\mathbb{G}_m \times \Gamma) \to \mathcal{X}$ such that the map from the special point of $\mathrm{Spec}(R)$, $\{o\} \times \{0\}/(\mathbb{G}_m \times \Gamma) \to \mathcal{X}$ lifts to $\mathrm{Spec}(A)/(\mathbb{G}_m \times \Gamma)$. Using the fact that $\mathrm{Spec}(R) \times \{0\}/(\mathbb{G}_m \times \Gamma)$ is coherently complete at the special point, and using an argument analogous to [], one may lift the map $\mathrm{Spec}(R) \times \{0\}/(\mathbb{G}_m \times \Gamma) \to \mathcal{X}$ to $\mathrm{Spec}(A)/(\mathbb{G}_m \times \Gamma)$. In particular we have a lift

$$\begin{tikzcd} & \operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma) \arrow[d] \\ \operatorname{Spec}(K) \times \{0\}/(\mathbb{G}_m \times \Gamma) \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

Using the same argument one may lift the map $f : \Theta_K \times (\mathrm{pt}/\Gamma) \to \mathcal{X}$ to $\operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma)$. Thus we have managed to lift our original K-point of $\mathcal{S}_\alpha$ along with the automorphism g to $\operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma)$.

Up to isomorphism the map $f : \Theta_K \times (\mathrm{pt}/\Gamma) \to \operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma)$ corresponds to a $\mathbb{G}_m$-equivariant and Γ-invariant map $\mathbb{A}^1_K \to \operatorname{Spec}(A)$. Equivalently, it is given by a homomorphism of $\mathbb{Z} \times (\Gamma)^\vee$-graded algebras $A \to K[t]$, where the latter is trivially $\Gamma^\vee$-graded, and t has $\mathbb{Z}$-weight -1. By construction, the composition $A \to K[t] \to K = K[t]/(t)$ factors (uniquely) through $R \subset K$. As A is finitely generated, its image in $K[t]$ is generated as an algebra over the ground field by a finite list of elements of the form $u_i \pi^{a_i} t^{b_i}$, where $\pi \in R$ is a uniformizer, $u_i \in R$ is a unit, and $a_i, b_i \in \mathbb{Z}$ with $b_i \geq 0$. The condition that the composition $A \to K$ factors through R amounts to the condition that $a_i \geq 0$ if $b_i = 0$. This condition implies that if we compose the map $A \to K[t]$ with the map $K[t] \mapsto K[t]$ given by $t \mapsto \pi^n t$ for $t \gg 0$, the new map $A \to K[t]$ factors uniquely through $R[t]$. The result is a map $\Theta_R \times (\mathrm{pt}/\Gamma) \to \operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma)$ whose restriction to $\Theta_K \times (\mathrm{pt}/\Gamma)$ is isomorphic to f. The composition of this map with the projection $\operatorname{Spec}(A)/(\mathbb{G}_m \times \Gamma) \to \mathcal{X}$, followed by restriction to $\operatorname{Spec}(R) \times (\mathrm{pt}/\Gamma) = \{t \neq 0\} \subset \Theta_R \times (\mathrm{pt}/\Gamma)$, gives a map $\operatorname{Spec}(R) \times (\mathrm{pt}/\Gamma) \to \mathcal{S}_\alpha$ extending the original map $\operatorname{Spec}(K) \times (\mathrm{pt}/\Gamma) \to \mathcal{S}_\alpha$. Hence, $\mathcal{S}_\alpha$ satisfies the valuative criterion A. □

6.6. **Methods for checking the valuative criterion.** A key input to establishing the valuative criterion will be the following somewhat technical result, whose proof was communicated to us by Brian Conrad:

Lemma 6.14. *Let R be a complete DVR and let $1 \to H \to G \to F \to 1$ be a short exact sequence of R-group schemes with F finite and H smooth with connected reductive geometric fibers. Then if $g \in G(K)$ has finite order prime to $\operatorname{char}(K)$, there is a finite extension $R \subset R'$ so that g is conjugate in $G(K')$ to an element of $G(R')$.*

Proof. Step 1: reduce to the case where H is semisimple of adjoint type.

Let $Z \subset H$ be the scheme-theoretic center in H, which is a closed subgroup of multiplicative type by [Con14, Theorem 3.3.4] or [Mil13, XII, 4.11]. Assume that the conclusion of the lemma holds for the quotient exact sequence,

$$1 \to H/Z \to G/Z \to F \to 1,$$

then we claim that then the conclusion follows for G itself. Indeed, let g' denote the image of g in $(G/Z)(K)$ and let $h' \in (G/Z)(K)$ such that $h'g'(h')^{-1} \in (G/Z)(R)$. Then after passing to a suitable finite extension $R \subset R'$ with function field K' we may assume that h' lifts to $h \in G(K')$ and that $h'g'(h')^{-1}$ admits a lift to $g'' \in G(R')$, so that

$$hgh^{-1} = g'' \cdot z, \text{ for some } z \in Z(K).$$

We claim that after a finite extension of R, z lies in $Z(R)$, so that hgh^{-1} lies in $G(R)$ as we are trying to show. We know that $g^n = 1$, so

$$z^n = (g'')^{-n} \in G(R') \cap Z(K') = Z(R').$$

Let m be such that $z^{mn} \in Z^0(R')$, where Z^0 is the identity component. Passing to another finite extension $R' \subset R''$, we may find a $z_0 \in Z^0(R'')$ such that $z^{mn} = z_0^{mn}$, hence z/z_0 has finite order. Because Z is a group of multiplicative type, any finite order element of $Z(K'')$ extends to $Z(R'')$, and this shows that $z = z_0 \cdot (z/z_0)$ lies in $Z(R'') \subset Z(K'')$.

Step 2: reduce to the case where $H \simeq (\mathbb{G}_m^r)_R$ is a split torus.

By the previous step, it suffices to prove the claim of the lemma when H is semisimple of adjoint type. Because the order of g is prime to the characteristic of K, the automorphism of H_K induced by conjugation by g is "semisimple" in the sense of [Ste68, p.51]. By [Ste68, Theorem 7.5], every semisimple automorphism of a connected linear algebraic group over an algebraically closed field preserves a Borel subgroup and some maximal torus inside it. In particular after a finite extension of K we can find a split maximal K-torus T in H_K normalized by g.

After passing to a finite extension of R, we may arrange that H is R-split, so there is a fiberwise-maximal split R-torus $S \subset G$. Then S_K and T are $H(K)$-conjugate, so by conjugating by an element of $H(K)$ we can arrange that $T = S_K$. By [Con14, Proposition 2.1.2], the schematic normalizer $N_G(S)$ exists as a smooth closed R-subgroup scheme of H that contains g and meets H in an extension of S by the Weyl group of H. Thus we may replace G with $N_G(S)$ and H with a split R-torus in the formulation of the lemma.

Step 3: reduce to the case where $F = (\mathbb{Z}/n)_R$.

F is finite, so the projection of $g \in G(K)$ to $F(K)$ extends uniquely to $F(R)$. This defines a map $(\mathbb{Z}/n)_R \to F$, and we can pull back the short exact sequence $1 \to H \to G \to F \to 1$ along this map to obtain a sequence

$$1 \to H \to G' \to (\mathbb{Z}/n\mathbb{Z})_R \to 1,$$

where $G' := (\mathbb{Z}/n\mathbb{Z})_R \times_F G$ obtains a unique R-group-scheme structure such that $G' \to (\mathbb{Z}/n\mathbb{Z})_R$ and $G' \to G$ are homomorphisms (this can be checked on the functor of points), and H is the kernel of the surjective homomorphism $G' \to (\mathbb{Z}/n\mathbb{Z})_R$.

By construction, g canonically lifts to an element of $G'(K)$ of order n which projects to a generator of $(\mathbb{Z}/n\mathbb{Z})_R$. It suffices to show that this element is conjugate in $G'(K)$ to an element of $G'(R)$ after passing to a finite extension of R, so for the remainder of the proof we may assume $F = (\mathbb{Z}/n\mathbb{Z})_R$.

Step 4: prove the claim for the semidirect product of $(\mathbb{Z}/n\mathbb{Z})_R$ with a split torus S.

The extension on R points $1 \to S(R) \to G(R) \to (\mathbb{Z}/n\mathbb{Z})(R) \to 1$ is split after extending to K-points $1 \to S(K) \to G(K) \to (\mathbb{Z}/n\mathbb{Z})(K) \to 1$ by the element $g \in G(K)$. We must show both that the extension on R-points is split, and it is split by an element of order n in $G(R)$ which is conjugate to g in $G(K)$.

Let $X_*(S)$ denote the cocharacter lattice of S, and consider the short exact sequence of $\mathbb{Z}/n\mathbb{Z}$-modules

$$0 \to S(R) \to S(K) \to X_*(S) \to 0,$$

where the map $S(K) \to X_*(S)$ is determined by the valuation isomorphism $K^*/R^* \simeq \mathbb{Z}$. This induces a long exact sequence in group cohomology

$$\cdots \to H^1(\mathbb{Z}/n\mathbb{Z}, S(R)) \to H^1(\mathbb{Z}/n\mathbb{Z}, S(K)) \to H^1(\mathbb{Z}/n\mathbb{Z}, X_*(S)) \to \\ \to H^2(\mathbb{Z}/n\mathbb{Z}, S(R)) \to H^2(\mathbb{Z}/n\mathbb{Z}, S(K)) \to \cdots$$

The extension $1 \to S(R) \to G(R) \to (\mathbb{Z}/n\mathbb{Z})(R) \to 1$ is classified by a class in $H^2(\mathbb{Z}/n\mathbb{Z}, S(R))$, and the fact that it splits after restriction to K implies that it comes from a class $c \in H^1(\mathbb{Z}/n\mathbb{Z}, X_*(S))$. For a totally unramified extension K/K' of degree d, the induced pullback map

$$H^1(\mathbb{Z}/n\mathbb{Z}, S(K)/S(R)) \to H^1(\mathbb{Z}/n\mathbb{Z}, S(K')/S(R'))$$

is identified with multiplication by d on $H^1(\mathbb{Z}/n\mathbb{Z}, X_*(S))$ under the valuation isomorphisms $S(K)/S(R) \simeq X_*(S) \simeq S(K')/S(R')$. It follows that the class c vanishes after pullback to a totally unramified extension of degree n, and hence the extension $1 \to S(R') \to G(R') \to (\mathbb{Z}/n\mathbb{Z})(R') \to 1$ splits.

The element $g \in G(K')$ is K'-conjugate to an R' point of G if and only if there is a splitting of $G(R') \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})(R')$ which restricts to a splitting of $G(K') \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})(K')$ which differs by an inner automorphism of $G(K')$ from the splitting determined by g. The set of splittings of the extension $G(K') \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})(K')$ up to inner automorphism is a torsor for $H^1(\mathbb{Z}/n\mathbb{Z}, S(K'))$, and likewise the set of splittings for the extension of R'-points is a torsor for $H^1(\mathbb{Z}/n\mathbb{Z}, S(R'))$. Choose a splitting of the map $G(R') \to (\mathbb{Z}/n\mathbb{Z})(R')$ and restrict this splitting to a splitting on K' points. The difference between this and the splitting determined by g is classified by some $c' \in H^1(\mathbb{Z}/n\mathbb{Z}, X_*(S))$. The splitting determined by g extends after inner automorophism to a splitting of $G(R') \to (\mathbb{Z}/n\mathbb{Z})(R')$ if and only if $c' = 0$. By the previous calculation, c' vanishes after a degree n totally unramified extension of R', which completes the proof. $\square$

Corollary 6.15. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a map of algebraic stacks which is a gerbe whose geometric fibers are classifying stacks for connected reductive groups. If $\mathcal{Y}$ satisfies the valuative criterion A, then so does $\mathcal{X}$.*

Proof. Let $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ be a map, and let g be an automorphism of $\xi|_K$ whose order is prime to $\mathrm{char}(K)$. Let Γ denote the cyclic group whose order is the order of g. By hypothesis we may find a modification of $p \circ \xi$ such that the automorphism over K induced g extends to an automorphism over R. By passing to an étale cover of $\mathrm{Spec}(R')$ if necessary, we may lift this to a map $\xi' : \mathrm{Spec}(R') \to \mathcal{X}$. After a suitable extension of the DVR if necessary, $\xi'|_{K'} \simeq \xi|_{K'}$, hence ξ' is a modification of ξ. Furthermore, under this isomorphism g is identified with an automorphism g' of $\xi'|_{K'}$, and by construction the image of g' under $\mathrm{Aut}(\xi'|_{K'}) \to \mathrm{Aut}(p \circ \xi'|_{K'})$ extends to R.

We have a short exact sequence of group schemes $\{1\} \to H \to \mathrm{Aut}(\xi') \to \mathrm{Aut}(p \circ \xi') \to \{1\}$, where H is a smooth group scheme with connected reductive geometric fibers. We can pull this short exact sequence back along the map $\Gamma_R \to \mathrm{Aut}(p \circ \xi')$ determined by the extension of $p(g') \in \mathrm{Aut}(p \circ \xi'|_{K'})$ to $\mathrm{Spec}(R)$. The result is a short exact sequence of groups $\{1\} \to H \to G \to \Gamma_R \to \{1\}$ along with an element of finite order $g' \in G(K')$ prime to $\mathrm{char}(K')$, which is precisely the set up of Lemma 6.14. It follows that after conjugation by an element of $G(K')$, g' lies in $G(R')$. Geometrically, this corresponds to an automorphism of $\xi'|_{K'}$ after which the automorphism g' extends to R, which gives the desired modification of ξ' and hence ξ. $\square$

Lemma 6.16. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a proper representable map of Noetherian stacks. If $\mathcal{Y}$ satisfies the valuative criterion A (respectively valuative criterion B), then so does $\mathcal{X}$.*

Proof. p is representable and separated, so for any DVR R and map $\xi : \operatorname{Spec}(R) \to \mathcal{X}$, we have a closed immersion $\underline{\operatorname{Aut}}(\xi) \hookrightarrow \underline{\operatorname{Aut}}(p \circ \xi)$ of group schemes over $\operatorname{Spec}(R)$. Furthermore, because p is proper, any modification of $p \circ \xi$ lifts uniquely to a modification of ξ. Therefore, given a generic automorphism of ξ, we may modify $p \circ \xi$ so that this generic automorphism extends, and then this lifts uniquely to a modification of ξ such that the given generic automorphism extends. □

Recall that a G-projective map of G-schemes $X \to Y$ is a G-equivariant map which admits a G-equivariant relatively ample invertible sheaf.

Corollary 6.17. *Let $\mathcal{X}$ be an algebraic stack locally finite type with affine diagonal over a G-ring such that for any point $x \in \mathcal{X}$ there is a map $f : ([W/G], w) \to (\mathcal{X}, x)$ with the following properties: 1) G is a reductive group scheme over the base and W is proper over the base, 2) there is a distinguished point $w \in [W/G]$ mapping to x, and 3) f is étale and inertia preserving in a neighborhood of w. Then $\mathcal{X}$ satisfies the valuative criteria A and B.*

Proof. Under these hypotheses, any map $\xi : \operatorname{Spec}(R) \to \mathcal{X}$ from a complete DVR along with any generic automorphism $g \in \operatorname{Aut}(\xi_K)$ lifts to some stack of the form W/G, so it suffices to prove the claim for such a stack, and it suffices to show the valuative criterion A by Proposition 6.8, which follows immediately from Lemma 6.16 and Lemma 6.14. □

Example 6.18. Let $X \to S$ be a projective morphism of schemes, and let $\underline{Coh}_S(X)$ be the S-stack parameterizing flat families of coherent sheaves on X. Then there is a family of maps $Q/\operatorname{GL}_n \to \underline{Coh}_S(X)$ and open substacks $U/\operatorname{GL}_n \subset Q/\operatorname{GL}_n$, where Q is a Quot scheme of a locally free sheaf of the form $\mathcal{O}_X(-n)^{\oplus m}$, such that $U/\operatorname{GL}_n \to \underline{Coh}_S(X)$ is an open immersion, and these open substacks cover $\underline{Coh}_S(X)$. Because the Quot scheme is projective, the previous corollary implies that $\underline{Coh}_S(X)$ satisfies the valuative criteria for unpunctured inertia.

Furthermore, over a field we can strengthen this.

Corollary 6.19. *In the context of Corollary 6.17, if $\mathcal{X}$ is defined over a field, then we may assume that the local models $[W/G]$ are such that G is* linearly *reductive and W is proper over an affine G-scheme.*

Proof. In the proof of Corollary 6.17, we reduced the statement to proving the claim for the local model W/G. If W was projective over the base ring, then no further work was needed, so here we prove the claim when W is proper over its affinization and G is linearly reductive. By Lemma 6.16 it suffices to assume that W is affine. After choosing a closed G-equivariant embedding $W \hookrightarrow \mathbb{A}^n$, it suffices to prove the claim for $[\mathbb{A}^n/G]$ where G acts linearly on $\mathbb{A}^n$. Now consider the equivariant embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^{n+1}$. Choosing a Weyl-group invariant inner product on the cocharacter lattice of a maximal torus $T \subset G$, we can consider the Θ-stratification of $\mathbb{P}^{n+1}$ associated to this class and the line bundle $\mathcal{O}(a, 1)$ using geometric invariant theory.[2] For $a \gg 0$, the normalized Hilbert-Mumford

[2]For a reductive group over a field, geometric invariant theory only produces a weak Θ-stratification when $\operatorname{char}(k) > 0$. However, we are assuming G is linearly reductive, so in positive

numerical invariant of points in the divisor $D = \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ are larger than any unstable point in $\mathbb{A}^n = \mathbb{P}^{n+1} \setminus D$. It follows that D is a union of Θ-strata, so by Proposition 6.12 it suffices to prove that $[\mathbb{P}^{n+1}/G]$ satisfies the valuative criterion. The map $[\mathbb{P}^{n+1}/G] \to [\mathrm{pt}/G]$ is proper and representable, so Lemma 6.16 reduces us to the case of the stack $[\mathrm{pt}/G]$. This follows from Lemma 6.14. □

6.6.1. *Valuative criteria for stacks with a good moduli space.*

Theorem 6.20. *Let $\mathcal{X}$ be a stack of finite type with affine diagonal over a field k. If $\mathcal{X}$ admits a good moduli space, then $\mathcal{X}$ satisfies the strong valuative criterion A of Definition 6.1. In particular $\mathcal{X}$ has unpunctured inertia.*

Proof. Note that it suffices to verify the valuative criterion A, rather than the strong valuative criterion, by Proposition 6.11. Let $\mathcal{X} \to X$ be the good moduli space. By [AHR15, Thm. 2.9], one can find an étale cover $U \to X$ such that the base change $\mathcal{X}_U \simeq \mathrm{Spec}(A)/G$ for some linearly reductive group G and finitely generated k-algebra A. $\mathcal{X}_U$ satisfies the valuative criterion A by Corollary 6.19. Given a map $\xi : \mathrm{Spec}(R) \to \mathcal{X}$, one can find a lift $\xi' : \mathrm{Spec}(R') \to \mathcal{X}_U$ after passing to a finite extension of the DVR $R \subset R'$. Furthermore, the map $\mathcal{X}_U \to \mathcal{X}$ is inertia preserving in the sense that $I_{\mathcal{X}_U} \simeq I_{\mathcal{X}}|_{\mathcal{X}_U}$, so any automorphism of $\xi|_K$ lifts to an automorphism of $\xi'_{K'}$ as well. If one can modify ξ' so that this generic automorphism extends, then composing with the map $\mathcal{X}_U \to \mathcal{X}$ gives a modification of ξ for which the generic automorphism extends. □

Remark 6.21. In this case where $\mathcal{X} = [V/G]$ for a linear representation of G, this theorem has the following interpretation: for every DVR R with fraction field K and for every $v \in V(R)$ with an element $g \in G(K)$ of finite order prime to $\mathrm{char}(K)$ such that $g \cdot v = v$ in $V(K)$, there exists a finite extension $R \subset R'$ of DVR's and an element $h \in G(K')$ (where $K' = \mathrm{Frac}(R')$) such that $hgh^{-1} \in G(R')$ and $h \cdot v \in V(R')$. Despite the purely representation-theoretic nature of this claim, we have not been able to find a completely elementary proof.

7. Good moduli spaces for moduli of $\mathcal{G}$-torsors

To illustrate our general theorems we now construct coarse moduli spaces for semistable torsors under Bruhat-Tits group schemes. This generalizes the results obtained by Balaji and Seshadri who constructed such moduli spaces for generically split groups over the complex numbers.

As in this article we are interested in existence theorems for good moduli spaces (instead of adequate moduli) we will have to assume that we work over a base field k of characteristic 0 in this section.

Let us briefly introduce the setup from [Hei17]. Let C be a smooth geometrically connected, projective curve over a field k and $\mathcal{G}/C$ a smooth Bruhat-Tits group scheme over C, i.e., $\mathcal{G}$ is smooth a affine group scheme over C that has geometrically connected fibers, such that over some dense open subset $U \subset C$ the group scheme is reductive and over all local rings at points p in $\mathrm{Ram}(\mathcal{G}) := C \setminus U$ the group scheme $\mathcal{G}|_{\mathrm{Spec}\, \mathcal{O}_{C,p}}$ is a connected parahoric Bruhat-Tits group. The simplest examples are of course reductive groups $G \times C$.

The stack of $\mathcal{G}$-torsors is denoted by $\mathrm{Bun}_{\mathcal{G}}$ and this is a smooth algebraic stack. To define stability one usually chooses a line bundle on $\mathrm{Bun}_{\mathcal{G}}$. As explained in

characteristic G will be of multiplicative type. The weak Θ-stratification coming from geometric invariant theory will be a Θ-stratification in this case, because the adjoint representation of G is trivial [Hal14, Lemma 2.5].

[Hei17, Section 3.B] there are natural choices in our situation. First there is the determinant line bundle $\mathcal{L}_{\det}$ given by the adjoint representation, i.e., the fiber at a bundle $\mathcal{E} \in \mathrm{Bun}_{\mathcal{G}_p}$ is $\mathcal{L}_{\det,\mathcal{E}} = \det(H^*(C, \mathrm{ad}(\mathcal{E})))^\vee$, where $\mathrm{ad}(\mathcal{E}) = \mathcal{E} \times^{\mathcal{G}} \mathrm{Lie}(\mathcal{G}_p/C)$ is the adjoint bundle of $\mathcal{E}$.

Next any collection of characters $\underline{\chi} \in \prod_{p\in \mathrm{Ram}(\mathcal{G})} \mathrm{Hom}(\mathcal{G}_p, \mathbb{G}_m)$ defines line bundles on the classifying stacks $B\mathcal{G}_p$ and one obtains a line bundle $\mathcal{L}_{\underline{\chi}}$ on $\mathrm{Bun}_{\mathcal{G}}$, by pull back via the map $\mathrm{Bun}_{\mathcal{G}} \to B\mathcal{G}_p$ defined by restriction of $\mathcal{G}$ torsors on C to the point p. We will denote by $\mathcal{L}_{\det,\underline{\chi}} := \mathcal{L}_{\det} \otimes \mathcal{L}_{\underline{\chi}}$, call the corresponding notion of stability $\underline{\chi}$-stability and denote by $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-ss} \subset \mathrm{Bun}_{\mathcal{G}}$ the substack of $\underline{\chi}$-semistable torsors.

Under explicit numerical conditions on $\underline{\chi}$ this satisfies the positivity assumption of loc.cit. ([Hei17, Proposition 3.3]), i.e., the restriction of $\mathcal{L}_{\det}$ to the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},p}$ classifying $\mathcal{G}$ bundles together with a trivialization on $C \smallsetminus p$ is nef. The parameter $\underline{\chi}$ will be called positive if $\mathcal{L}_{\det,\underline{\chi}}$ is ample on $\mathrm{Gr}_{\mathcal{G},p}$ for all p.

Theorem 7.1 (Good moduli for semistable $\mathcal{G}$-torsors)**.** *Assume k is a field of characteristic 0, C is a smooth, projective, geometrically connected curve over k, $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme over k and $\underline{\chi}$ is a positive stability parameter. Then $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-ss}$ admits a proper good moduli space $M_{\mathcal{G}}$.*

As remarked before, in the case that $\mathcal{G}$ is a generically split group scheme, the space $M_{\mathcal{G}}$ was constructed by Balaji and Seshadri [BS15].

To prove the theorem we only need to check that $\mathrm{Bun}_{\mathcal{G}}$ satisfies the assumptions of Corollary 5.13. This will be done in a series of Lemmas.

Lemma 7.2. *The canonical reduction of $\mathcal{G}$-torsors defines a Θ-stratification on $\mathrm{Bun}_{\mathcal{G}}$ with semistable locus $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-ss}$. This stratification admits a well-ordering.*

Proof. By definition $\mathcal{L}_{\det,\underline{\chi}}$-stability is defined in terms of maps $\Theta \to \mathrm{Bun}_{\mathcal{G}}$.

Moreover, by [Hei17] any unstable bundle $\mathcal{E}$ admits a canonical filtration $\mathrm{HN}_{\mathcal{E}} \colon \Theta \to \mathrm{Bun}_{\mathcal{G}}$ with $\mathrm{HN}_{\mathcal{E}}(1) = \mathcal{E}$, which is defined by maximizing the invariant $\mu_{\max}(\mathcal{E})$ ([Hei17, Section 3.F]). By [Hei17, Lemma 3.17] this invariant is semicontinuous under specialization in the sense that for any family $\mathcal{E}_R \in \mathrm{Bun}_{\mathcal{G}}(R)$ defined over a discrete valuation ring R with fraction field K and residue field κ we have $\mu_{\max}(\mathcal{E}_K) \leq \mu_{\max}(\mathcal{E}_\kappa)$ and equality holds only if the canonical filtration over K extends to the family.

Now in [Hei17, Proposition 3.18] it is shown that the stratification of $\mathrm{Bun}_{\mathcal{G}}$ defined by $\mu_{\max}$ is constructible. Since the invariant $\mu_{\max}$ is semicontinuous this implies that for any constant c the substacks $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}\leq c}$ defined by the condition $\mu_{\max}(\mathcal{E}) \leq c$ are open.

To show that this defines a Θ-stratification we are therefore left to show that the closed substacks $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}\leq c} \setminus \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}< c}$ are unions of connected components of $\mathrm{Map}(\Theta, \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}\leq c})$. To prove this, we may pass to the algebraic closure of our base field and thus assume that k is algebraically closed. In this situation, by [Hei17, Lemma 3.9] the points of $\mathrm{Map}(\Theta, \mathrm{Bun}_{\mathcal{G}})$ can be described as reductions of bundles to subgroups $\mathcal{P}_\lambda \subset \mathcal{G}$. The proof of this Lemma also shows that this description holds in families, i.e., the components of the mapping stack are isomorphic to moduli stacks $\mathrm{Bun}_{\mathcal{P}_\lambda}$ of torsors. Let us fix an unstable bundle $\mathcal{E}$ with $\mu_{\max}(\mathcal{E}) = c$ and canonical reduction given as a reduction to $\mathcal{P}_\lambda$. Let us denote by $p_\lambda \colon \mathrm{Bun}_{P_\lambda}^{\mathrm{HN}} \to \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}\leq c}$ the restriction of the canonical map $\mathrm{Bun}_{\mathcal{P}_\lambda} \to \mathrm{Bun}_{\mathcal{G}}$. As the canonical filtration is unique and the weight is semi-continuous, the map

p_λ is an isomorphism on closed points. As in the case of Behrend's conjecture it is not hard to see that the map is also injective on tangent spaces as follows. First note that any automorphism of a $\mathcal{G}$-torsor preserves the canonical reduction and since we are in characteristic 0 this implies that for any point $\mathcal{E}_\lambda \in \mathrm{Bun}_{P_\lambda}^{HN}$ the map $\mathrm{Aut}_{\mathrm{Bun}_{\mathcal{P}_\lambda}}(\mathcal{E}_\lambda) \to \mathrm{Aut}_{\mathrm{Bun}_\mathcal{G}}(p_\lambda(\mathcal{E}_\lambda))$ is surjective. Now derivative of p_λ at $\mathcal{E}_\lambda$ is induced from the cohomology of the sequence of vector bundles on C:

$$0 \to \mathcal{E}_\lambda \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{P}_\lambda/C) \to \mathcal{E}_\lambda \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{G}/C) \to \mathcal{E}_\lambda \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{P}_\lambda/C) \to 0.$$

It is therefore sufficient to show that $H^0(C, \mathcal{E}_\lambda \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{P}_\lambda/C)) = 0$. This is true because the point $\mathcal{E}_\lambda$ also defines a filtration $f\colon \Theta_\mathcal{E} \to \mathrm{Bun}_{P_\lambda}$ and at the point $f(0) = \mathcal{E}_0 \in \mathrm{Bun}_{P_\lambda}$ the above filtration splits, as there the quotient bundle is the subbundle on which λ acts with negative weights. As the dimension of H^0 is semi-continuous it suffices to show that $H^0(C, \mathcal{E}_0 \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{P}_\lambda)) = 0$. However $H^0(C, \mathcal{E}_0 \times^{\mathcal{P}_\lambda} \mathrm{Lie}(\mathcal{G}))$ is also the tangent space to $\mathrm{Aut}_{\mathrm{Bun}_\mathcal{G}}(\mathcal{E}_0 \times^{\mathcal{P}_\lambda} \mathcal{G})$ and we already saw that the map from $\mathrm{Aut}_{\mathrm{Bun}_{\mathcal{P}_\lambda}}(\mathcal{E}_0)$ to this group scheme is surjective. Thus the map p_λ is a closed embedding.

Finally, stability defines a theta stratification that admits a well-ordering, because for any c and any connected component of $\mathrm{Bun}_\mathcal{G}$ the open substack $\mathrm{Bun}_\mathcal{G}^{\mu_{\max}\leq c}$ are of finite type, so $\mathrm{Bun}_\mathcal{G}^{\mu_{\max}\leq c} \setminus \mathrm{Bun}_\mathcal{G}^{\mu_{\max}< c}$ can only contributre finitely many strata on each component of $\mathrm{Bun}_\mathcal{G}$. $\square$

Lemma 7.3. *The stack $\mathrm{Bun}_\mathcal{G}^{\chi-ss}$ is S-complete and locally linearly reductive.*

Proof. S-completeness holds because of the existence of a blow up of ST_R to linking two specializations. $\mathcal{L}_{\det,\underline{\chi}}$ is positive on the exceptional lines, so if the blow-up was necessary, one of the bundles was unstable.

In particular every closed substack of $\mathrm{Bun}_\mathcal{G}^{\chi-ss}$ is again S-complete, so that by Proposition 3.46 the automorphism groups of closed points are geometrically reductive. As we assumed our base field to be of characteristic 0 in this section, these groups are linearly reductive. $\square$

Lemma 7.4. *The stacks $\mathrm{Bun}_\mathcal{G}^{\chi-ss}$ have unpunctured inertia.*

Proof. We will show the valuative vriterion (A) for the stack $\mathrm{Bun}_\mathcal{G}$. As semistability is defined by a Θ-stratification we can invoke Proposition 6.12 to deduce that $\mathrm{Bun}_\mathcal{G}^{\chi-ss}$ then also satisfies the valuative criterion and this will imply that $\mathrm{Bun}_\mathcal{G}^{\chi-ss}$ is unpunctured by Theorem 6.6.

Let us prove the valuative criterion, i.e., given a discrete valuation ring R with fraction field K, $\mathcal{E}_R \in \mathrm{Bun}_\mathcal{G}(R)$ and $g_K \in \mathrm{Aut}_\mathcal{G}(\mathcal{E}_K)$ an element of some finite order n we have to construct a modification $\mathcal{E}'_{R'}$ of $\mathcal{E}_R$ such that g_K extends to an automorphism of $\mathcal{E}'_{R'}$. Viewing g_K as a section of the affine group scheme $\mathrm{Aut}_{\mathcal{G}/C}(\mathcal{E}) \to C$, we see that it suffices to find a modification of $\mathcal{E}_R$ such that such that this section extends to the generic point of the special fiber.

Since we may pass to a finite extension of our ground field k, we may assume that the generic fiber of $\mathcal{G}$ is quasi-split and we will fix $\mathcal{T} \subset \mathcal{G}$ a maximal torus containing a maximal split torus. We will denote the generic fibers of $\mathcal{G}$ and $\mathcal{T}$ by G and T. We will denote the generic point of C by η.

By Steinbergs theorem for any connected reductive group G and any algebraically closed field K/k we have $H^1(\overline{K}(\eta), G) = 0$ (see e.g., [Ser94, Théorème 2, Chapter III]), i.e., after possibly passing to a finite extension of K we may assume that $\mathcal{E}_K$ is trivial over an open subset of C_K.

At the generic point $K(\eta)$ automorphism defines an element $g_K \in G(K(\eta))$ of finite order. In particular this is a semisimple element and we claim that it is conjugate to an element of $G(k(\eta)) \subset G(R(\eta)) \subset G(K(\eta))$ after passing to a finite extension K' of K.

Over the algebraic closure $\overline{K(\eta)}$ every semisimple element is conjugate to an element of our maximal torus T and the elements of order n of T define a finite subscheme of T, in particular the conjugacy classes of order n define a finite subscheme of the adjoint quotient $G//G = T//N(T)$ and so the conjugacy class of g_K is defined over some finite extension of the ground field k. Now the map $G \to G//G$ is an affine quotient and the semisimple elements are the unique closed orbits of the fiber. Thus passing to a finite extension of k we find a semisimple element $g_k \in G(k(\eta))$ that in that lies in the conjugacy class of g_K.

Let us temporarily assume that the derived group of G is simply connected. The obstruction to conjugate g_K to g_k over $K(\eta)$ is a torsor under the centralizer $Z_G(g_k)$, which is a connected reductive group because the derived group of $\mathcal{G}$ was assumed to be simply connected ([SS70, 3.9]). Applying Steinberg's theorem again we know that $H^1(\overline{K}(\eta), Z_G(g_k)) = 0$, so this obstruction vanishes after a finite extension of K, i.e., we can change the trivialization of $\mathcal{E}_K$ over the generic point such that g_K lies in $G(k(\eta))$. Now choose any extension of $\mathcal{E}_K$ to R that is trivial on a neighborhood of $\mathrm{Spec}(R(\eta)) \subset C_R$ to conclude (e.g. [Hei17, Proposition 3.3]).

For a general group $\mathcal{G}$ we can find an z-extension $0 \to T' \to G' \to G \to 0$, where T' is an induced torus that is central in G' and such that the derived group of G' is simply connected ([Kot82, Lemma 1.1]). In particular for any field K the sequence $1 \to T(K) \to G'(K) \to G(K) \to 1$ is exact, because $H^1(K,T) = 0$ if T is an induced torus by Hilbert's theorem 90. We claim that we can find a preimage g'_K of g_K such that its conjugacy class in $G'//G'$ is defined over k.

From our previous argument we know that the conjugacy class $[g_K] \in G//G$ is already defined over k and that we can find $g_k \in G(k(\eta))$ that defines the same conjugacy class. Choose preimages g'_K of g_K and g'_k of g_k. As T is a central torus the map $G'//G \to G//G$ is again a T-torsor so the conjugacy classes g'_K and g'_k differ by an element t_K of $T(K(\eta))$, so $t_K^{-1} g'_K$ will be a preimage of g_K whose conjugacy class is the one of g'_k. Now we can argue as before to see that g'_K is conjugate to g'_k after a finite extension of K and we can conclude as before.

□

Proof of Theorem 7.1. We just proved that $\mathrm{Bun}_{\mathcal{G}}^{\chi-ss}$ is S-complete, locally linearly reductive with unpunctured intertia. By [Hei17, Proposition 3.3] the stack $\mathrm{Bun}_{\mathcal{G}}$ satisfies the existence criterion for properness, i.e., if R is a discrete valuation ring with fraction field K and $\mathcal{E}_K \in \mathrm{Bun}_{\mathcal{G}}(K)$ is a $\mathcal{G}$-torsor over C_K then there exists a finite extension R' of R such that $\mathcal{E}_K$ extends to a torsor over C_R. Therefore we can apply Corollary 5.13 to deduce the existence of a proper good moduli space. □

Appendix A. Strange gluing lemma

Let R be a discrete valuation ring with residue field κ, and let $\pi \in \mathfrak{m} \subset R$ be a uniformizer for the maximal ideal. We refer to the closed point in $\mathrm{Spec}(R)$ as o. For $n > 0$ we will consider the following quotient stack

$$\overline{\mathrm{ST}}_R^{n,1} = \mathrm{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m$$

where the $\mathbb{G}_m$-action is encoded by giving s weight $n > 0$ and giving t weight -1. We have a closed immersion $i : \Theta_\kappa = \{s = 0\} \hookrightarrow \overline{\mathrm{ST}}_R^{n,1}$ and an open immersion $j : \mathrm{Spec}(R) = \{t \neq 0\} \subset \overline{\mathrm{ST}}_R^{n,1}$. We can restrict any map $m : \overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$ to get two maps $m \circ i : \Theta_\kappa \to \mathcal{X}$ and $m \circ j : \mathrm{Spec}(R) \to \mathcal{X}$ along with an isomorphism $m \circ j(o) \simeq m \circ i(1)$ in $\mathcal{X}(\kappa)$. We shall regard a triple

$$(f : \Theta_\kappa \to \mathcal{X}, \xi : \mathrm{Spec}(R) \to \mathcal{X}, \phi : \xi(o) \simeq f(1))$$

as *gluing data* for the map m. Note that such a triple can be regarded as potential gluing data for a map $\overline{ST}_R^{n,1}$ for any $n > 0$.

Proposition A.1. *Let $\mathcal{X}$ be an algebraic stack of finite type with quasi-affine diagonal over a locally noetherian quasi-separated algebraic space. Consider gluing data $f : \Theta_\kappa \to \mathcal{X}$, $\xi : \mathrm{Spec}(R) \to \mathcal{X}$ and an isomorphism $\phi : \xi(o) \simeq f(1)$. For all $n \gg 0$, there is a map $m : \overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$, unique up to unique isomorphism, which restricts to the gluing data (f, ξ, ϕ).*

This theorem is inspired by the perturbation theorem [Hal14, Proposition 3.53], which is an analogous result for constructing map $\mathbb{A}_k^2/(\mathbb{G}_m^2)_k \to \mathcal{X}$ from maps from the loci $\{s = 0\}$ and $\{t \neq 0\}$. We first give a direct proof, then we sketch a proof using deformation theory along the lines of [Hal14, Proposition 3.53], which gives effective bounds on n. In the body of the paper, we will use $\overline{\mathrm{ST}}_R = \overline{\mathrm{ST}}_R^{1,1}$ only, so we note the following:

Corollary A.2. *In the context of Proposition A.1, for $n \gg 0$ the gluing data $(f, \xi|_{\mathrm{Spec}(R[\pi^{1/n}])}, \phi)$ extends canonically to a map $m : \overline{\mathrm{ST}}_{R[\pi^{1/n}]}^{1,1} \to \mathcal{X}$.*

Proof. Compose the uniquely defined map $\overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$ of Proposition A.1 with the canonical map $\overline{\mathrm{ST}}_{R[\pi^{1/n}]}^{1,1} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. □

First proof of Proposition A.1. Let $C = R[t, \pi/t, \pi/t^2, \ldots] \subset R[t^\pm]$. Then we claim that the diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa) \arrow[r, "j"] \arrow[d, "i"] & \Theta_\kappa \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(C)/\mathbb{G}_m \end{tikzcd} \tag{12}$$

is a pushout in the full subcategory of noetherian tannakian stacks in the sense of [BHL17, Definition 3.1], which includes noetherian algebraic stacks with quasi-affine diagonal [BHL17, Theorem 1.4]. By [BHL17, Lemma 3.13] for a noetherian tannakian stack $\mathcal{Y}$ and any stack T the map

$$\mathrm{Map}(T, \mathcal{Y}) \to \mathrm{Fun}^c_\otimes(\mathrm{APerf}(\mathcal{Y})^{cn}, \mathrm{APerf}(T)^{cn})$$

is an equivalence, where the former denotes the ∞-groupoid of maps of stacks, and the latter denotes the ∞-category of symmetric monoidal ∞-functors between the symmetric monoidal ∞-categories of pseudo-coherent complexes which preserve finite colimits. To verify that this is a pushout diagram, it therefore suffices to show that

$$\mathrm{APerf}(\mathrm{Spec}(C)/\mathbb{G}_m)^{cn} \simeq \mathrm{APerf}(R)^{cn} \times_{\mathrm{APerf}(\kappa)^{cn}} \mathrm{APerf}(\Theta_\kappa)^{cn}. \tag{13}$$

The map j can be presented by the $\mathbb{G}_m$-equivariant open immersion of schemes $\mathrm{Spec}(\kappa[t^{\pm}]) \subset \mathrm{Spec}(\kappa[t])$, and i can be presented as the $\mathbb{G}_m$-equivariant closed immersion $\mathrm{Spec}(R[t^{\pm}]) \to \mathrm{Spec}(\kappa[t^{\pm}])$. Forgetting the $\mathbb{G}_m$-equivariant structure, the fact that restriction gives an equivalence

$$\mathrm{APerf}(C) \simeq \mathrm{APerf}(R[t^{\pm}])^{cn} \times_{\mathrm{APerf}(\kappa[t^{\pm}])^{cn}} \mathrm{APerf}(\kappa[t])^{cn}$$

follows from [Lur11, Proposition 7.7], which deduces this for many different categories of complexes from the version of this statement for $\mathrm{D}(-)^{cn}$, which is [Lur11, Theorem 7.2]. The equivalence (13) follows from smooth descent and the fact that limits of ∞-categories commutes: $\mathrm{APerf}(\mathrm{Spec}(C)/\mathbb{G}_m)$ is a limit of categories of the form $\mathrm{APerf}(\mathrm{Spec}(C) \times \mathbb{G}_m^p$ for $p \geq 0$, and each of these categories decomposes as a fiber product in the same manner as $\mathrm{APerf}(\mathrm{Spec}(C))$ above.

Now that we have shown that (12) is a pushout diagram in the ∞-category of noetherian tannakian stacks, it follows that any gluing data as in the statement of the proposition glues to a unique map $\mathrm{Spec}(C)/\mathbb{G}_m \to \mathcal{X}$, which is unique up to unique isomorphism. Write C as a union $C = \bigcup C_n$, where $C_n := R[t, \pi/t^n]$ and $C_n \subset C_m$ for $m > n$. For convenience we denote C by C_∞. Denoting π/t^n by s, we have an isomorphism of graded rings $C_n \simeq R[s,t]/(st^n - \pi)$. It follows that the inclusion $R \subset C_n$ is the $\mathbb{G}_m$-invariant subring for all n, and thus the same holds for C. Therefore $\mathrm{Spec}(R)$ is the good moduli space for all of the stacks $\mathrm{Spec}(C_n)/\mathbb{G}_m$, including $n = \infty$.

We wish to show that the map $\mathrm{Spec}(C)/\mathbb{G}_m \to \mathcal{X}$ factors uniquely through the map $\mathrm{Spec}(C)/\mathbb{G}_m \to \mathrm{Spec}(C_n)/\mathbb{G}_m$ for all $n \gg 0$. Uniqueness follows from the fact that $\mathrm{Spec}(C/\mathbb{G}_m) \to \mathrm{Spec}(C_n)/\mathbb{G}_m$ is a monomorphism , so it suffices to show existence of such a factorization. By hypothesis there is a finite type map $\mathcal{X} \to S$, where S is a locally noetherian quasi-separated algebraic space, and the composition $\mathrm{Spec}(C)/\mathbb{G}_m \to S$ factors uniguely through the good moduli space $\mathrm{Spec}(C)/\mathbb{G}_m \to \mathrm{Spec}(R)$ [Alp13, Theorem 6.6]. All of the maps $\mathrm{Spec}(C_n)/\mathbb{G}_m \to \mathrm{Spec}(C)/\mathbb{G}_m$ are maps over $\mathrm{Spec}(R)$, so it suffices to assume that $S = \mathrm{Spec}(R)$.

$\mathcal{X}$ is colimit preserving as a functor of R-algebras, so $\mathcal{X}(C) = \mathrm{hocolim}_n\, \mathcal{X}(C_n)$, and the same holds for $\mathcal{X}(C \otimes \mathcal{O}_{\mathbb{G}_m^p})$ for $p = 1, 2$. The description of $\mathrm{Map}(\mathrm{Spec}(C_n)/\mathbb{G}_m, \mathcal{X})$ in terms of descent for the groupoid scheme $\mathrm{Spec}(C) \times \mathbb{G}_m \rightrightarrows \mathrm{Spec}(C)$ for $n = 1, \cdots, \infty$ involves a finite homotopy limit of groupoids, so because finite homotopy limits commute with filtered homotopy colimits, we have

$$\mathrm{Map}(\mathrm{Spec}(C)/\mathbb{G}_m, \mathcal{X}) \simeq \mathrm{hocolim}_n\, \mathrm{Map}(\mathrm{Spec}(C_n)/\mathbb{G}_m, \mathcal{X}).$$

This shows that any map $\mathrm{Spec}(C)/\mathbb{G}_m \to \mathcal{X}$ factors through a map $\mathrm{Spec}(C_n)/\mathbb{G}_m \to \mathcal{X}$ for all $n \gg 0$. □

The second proof of our proposition will use the following

Lemma A.3. *The stack $\overline{ST}_R^{n,1}$ is coherently complete along $i : \Theta_\kappa \hookrightarrow \overline{\mathrm{ST}}_R^{n,1}$. This means that if $\Theta_\kappa^{(m)}$ denotes the m^{th} infinitesimal thickening of Θ_κ, then the canonical restriction map*

$$\mathrm{APerf}(\overline{\mathrm{ST}}_R^{n,1}) \to \varprojlim_m \mathrm{APerf}(\Theta_\kappa^{(m)})$$

is an equivalence of symmetric monoidal ∞-categories, which restricts to an equivalence on the full subcategory of connective complexes.

The proof of this lemma is straightforward, and is very similar to [Hal14, Lemma 3.54], so we shall omit it. It will also be subsumed by a much more general statement in [AHR].

Second proof of Proposition A.1. First note that again using the tannakian formalism, Lemma A.3 implies that the restiction map

$$\operatorname{Map}(\overline{\mathrm{ST}}^{n,1}_R, \mathcal{X}) \to \varprojlim_m \operatorname{Map}(\Theta^{(m)}_\kappa, \mathcal{X})$$

is an equivalence of groupoids. The analogous claim holds for the m^{th} infinitesimal neighborhoods in $\{t \neq 0\} \simeq \operatorname{Spec}(R) \subset \overline{\mathrm{ST}}^{n,1}_R$ of the closed point $\{t \neq 0\} \simeq \operatorname{Spec}(\kappa) \subset \Theta_\kappa$. It therefore suffices to show that the restriction map

$$\varprojlim_m \operatorname{Map}(\Theta^{(m)}_\kappa, \mathcal{X}) \to \operatorname{Map}(\Theta^{(0)}_\kappa, \mathcal{X}) \times_{\mathcal{X}(\kappa)} \varprojlim_m \operatorname{Map}(\Theta^{(m)}_\kappa - \{0\}, \mathcal{X}) \tag{14}$$

is an equivalence for $n \gg 0$.

Let $f_0 = f : \Theta_\kappa \to \mathcal{X}$ be a map. We must show that $g : \operatorname{Spec}(R) \to \mathcal{X}$ along with the isomorphism $g(o) \simeq f_0(1)$ uniquely defines an iterative sequence of extensions

$$\begin{tikzcd} \Theta^{(m)}_\kappa \arrow[r, "f_m"] \arrow[d] & \mathcal{Y} \\ \Theta^{(m+1)}_\kappa \arrow[ur, dashed, "f_{m+1}"'] & \end{tikzcd}$$

Note that $\Theta_\kappa \hookrightarrow \overline{ST}^{n,1}_R$ is a regular closed immersion cut out by the variable s with weight n, so we have a square-zero extension

$$i_*(\mathcal{O}_{\Theta_\kappa}\langle -nm\rangle) \to \mathcal{O}_{\Theta^{(m)}_y} \to \mathcal{O}_{\Theta^{(m-1)}_y},$$

where $\mathcal{O}_{\Theta_y}\langle -mn\rangle$ is the equivariant locally free sheaf generated in weight mn and $i : \Theta_\kappa \hookrightarrow \Theta^{(m)}_\kappa$ is the inclusion. Therefore given a map f_m, the obstruction to extending to a map f_{m+1} is an element in

$$\operatorname{Hom}_{\Theta^{(m)}_\kappa}(f_m^* \mathbb{L}_\mathcal{X}, i_*(\mathcal{O}_{\Theta_\kappa}\langle -mn\rangle[1])) \simeq \operatorname{Hom}_{\Theta_\kappa}(f_0^* \mathbb{L}_\mathcal{X}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle[1]).$$

If an extension exists, the set of extensions is a torsor for $\operatorname{Hom}_{\Theta_\kappa}(f_0^*\mathbb{L}_\mathcal{X}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle)$.

The same analysis reduces a pro-system of maps $\Theta^{(m)}_\kappa - \{0\} \to \mathcal{X}$ to an iterated extension problem, so it suffices to show that

$$\operatorname{RHom}_{\Theta_\kappa}(f_0^*\mathbb{L}_\mathcal{X}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle) \to \operatorname{RHom}_{\Theta_\kappa - \{0\}}(f_0^*\mathbb{L}_\mathcal{X}|_{\Theta_\kappa - \{0\}}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle)$$

is an isomorphism in H^0 and H^1. In other words, it suffices to show that

$$R\underline{\Gamma}_{\{0\}} \operatorname{RHom}_{\Theta_\kappa}(f_0^*\mathbb{L}_\mathcal{X}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle)$$

has vanishing hypercohomology in degrees $0, 1, 2$. Note that $R\underline{\Gamma}_{\{0\}}\mathcal{O}_{\Theta_\kappa} \simeq (k[t^\pm]/k[t])[-1]$, so the complex above has a filtration whose associated graded is

$$\begin{aligned} &\bigoplus_{a \geq 1} \mathcal{O}_{\{0\}}\langle -a\rangle \otimes \operatorname{RHom}_{\Theta_\kappa}(f_0^*\mathbb{L}_\mathcal{X}, \mathcal{O}_{\Theta_\kappa}\langle -mn\rangle)[-1] \\ &\qquad \simeq \bigoplus_{a \geq 1} \operatorname{RHom}_{\{0\}/\mathbb{G}_m}(f_0^*\mathbb{L}_\mathcal{X}|_{\{0\}}, \mathcal{O}_{\{0\}}\langle -mn - a\rangle)[-1]. \end{aligned} \tag{15}$$

From the previous analysis, it suffices to show that

$$\operatorname{RHom}_{\{0\}/\mathbb{G}_m}(f_0^*\mathbb{L}_\mathcal{X}|_{\{0\}}, \mathcal{O}_{\{0\}}\langle -mn - a\rangle)[-1]$$

vanishes in low cohomological degree for all $a, m > 0$. The $\mathbb{G}_m$-weights of $f_0^*\mathbb{L}_{\mathcal{X}}|_{\{0\}}$ in high cohomological degree are bounded above. As soon as n is larger than the highest weight of $f_0^*\mathbb{L}_{\mathcal{X}}|_{\{0\}}$ in high cohomological degree we have the desired vanishing for all $a, m > 0$. □

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