Moduli theory

Daniel Halpern-Leistner

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Contents

Lecture 0

Notation and background

I have tried to keep the background limited to the material from Chapters 2 and 3 of [H3], and some basic category theory. The notions from category theory that I will assume some familiarity with are:

1. categories, functors, and natural transformations

2. limits and colimits (of sets)

3. filtered category (often used as the indexing category for a colimit)

4. adjunction between functors, unit and counit

5. abelian categories, and symmetric monoidal categories

If $Y' \to Y$ and $X \times Y$ are morphisms of schemes, I will sometimes use the notation $X_{Y'}$ for the fiber product $X \times_Y Y'$. If $Y' = \operatorname{Spec}(A)$ is affine, I will also sometimes denote this X_A.

Lecture 1

Introduction to moduli problems

References: [V], [F2]
Date: 1/22/2020
Exercises: 8

1.1 About the course

The course will have two halves. The first will be a survey on the methods
and foundational results in the theory of algebraic stacks. In the second
half, we will discuss some recent developments extending the results of
geometric invariant theory to algebraic stacks. We will give special focus to
the moduli of vector bundles (and principal G-bundles) on a smooth curve,
as a motivating example.

1.1.1 Goals for Part I

By the end of this unit, you should have some familiarity with:

1. Definitions and basic notions for algebraic stacks and spaces

2. Quasi-coherent sheaves on algebraic stacks

3. Deformation theory

4. Artin's criteria

5. Tannaka duality

6. Local structure of algebraic stacks

1.1.2 Goals for Part II

There is an algebraic stack $\mathcal{M}_{n,d}(C)$ parametrizing rank n degree d vector bundles on a smooth projective curve C. This stack is an important object in geometric representation theory and mathematical physics. For our purposes, it is a great illustration of the pathologies which arise in practice: it is highly non-separated, and it is unbounded (meaning there is no finite type variety that can parameterize all vector bundles of rank n had degree d). To remedy this, $\mathcal{M}_{n,d}(C)$ has a special stratification by locally closed substacks (due to Harder, Narasimhan, and Shatz). The dense open stratum, which parameterizes "semistable bundles," has a projective good moduli space.[1]

A general method for producing this structure – a special stratification and existence of moduli spaces for semistable objects – has been developed recently in my work and the work of others. I've referred to it as Θ-*stability* and the theory of Θ-*stratifications*. It generalizes geometric invariant theory. The foundations of the theory are more-or-less in place, and there are lots of interesting examples to try to apply these methods to. So the goal of Part II is to try to give an accessible overview of these recent developments.

We will develop this general story in tandem with a detailed study of the stack $\mathcal{M}_{n,d}(C)$, which will serve as a concrete example demonstrating these methods. This will also give us a chance to discuss some striking results on the stack $\mathcal{M}_{n,d}(C)$, which will provide great source material for student presentations! Two influential classic results that I hope to get to are (restricting our attention to $\mathcal{M}_{2,d}$ for simplicity):

(i) The Atiyah-Bott formula [AB] for the Poincaré polynomial, $P_t(-) := \sum_{i \geqslant 0} t^i \dim H^i(-, \mathbb{Q})$, of the moduli stack of semistable vector bundles over a complex curve:

$$P_t(\mathcal{M}_{2,d}^{ss}) = P_t(\mathcal{M}_{2,d}) - \sum_{k>d/2} t^{\#_k} P_t(\mathcal{M}_{1,k}^{ss}) P_t(\mathcal{M}_{1,d-k}^{ss})$$

$$= \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \sum_{k>d/2} t^{\#_k} \left(\frac{(1+t)^{2g}}{1-t^2} \right)^2$$

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[1]A good moduli space is a formal mathematical notion, introduced in [A2]. For now, just take it to mean there's a projective scheme which parameterizes all semistable vector bundles up to a manageable equivalence relation, called S-equivalence.

where $\#_k = 2k - d + g - 1$. It is somewhat surprising that when d is odd, $P_t(\mathcal{M}_{2,d}^{ss})(1 - t^2)$ is a polynomial, but one can deduce this geometrically.

(ii) The Verlinde formula [B1, F1, T1]: There is a unique "positive" generator L of $\mathrm{Pic}(\mathcal{M}_{2,0}^{ss})$, and the Verlinde formula says that $H^i(\mathcal{M}_{2,0}^{ss}, L^{\otimes k}) = 0$ for $i > 0$ and

$$\dim H^0(\mathcal{M}_{2,0}^{ss}, L^{\otimes k}) = \left(\frac{k+2}{2} \right)^{g-1} \sum_{j=1}^{k+1} (\sin(\pi j/(k+2)))^{2-2g}.$$

Again, it is somewhat surprising that the right hand side of this formula is even an integer.

1.2 What is a moduli problem?

There is a guiding meta-problem in mathematics: the classification of mathematical objects. A famous example is simple Lie algebras over $\mathbb{C}$, which are classified by a Dynkin diagram of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2. However, many geometric objects do not admit a discrete classification.

Example 1.1. There is a space $\mathcal{M}_g$ parameterizing all smooth Riemann surfaces of genus $g > 1$. There is a quick construction from the perspective of differential geometry: If S is a smooth surface of genus $g > 1$, define the Teichmüller space $T(S)$ to be the quotient space $H(S)/\mathrm{Diff}_0$, where $H(S)$ is the space of Riemannian metrics of constant curvature -1 and Diff_0 is the group of diffeomorphisms isotopic to the identity map. This quotient space can be identified with $\mathbb{R}^{6(g-1)}$, and it has a canonical complex structure. Moreover, if we let $MCG = \mathrm{Diff}/\mathrm{Diff}_0$ be the mapping class group of S, then $T(S)/MCG$ can be identified with $\mathcal{M}_g$, where $\mathcal{M}_g$ is the set of Riemann surface structures on S, up to holomorphic isomorphism. One can show that $\mathcal{M}_g$ inherits a topology, and is homeomorphic to a quasi-projective variety over $\mathbb{C}$. This understanding of $\mathcal{M}_g$ is very useful because questions about metrics on S can be translated into questions about a quasi-projective variety.

This is historically the first moduli problem studied. The term "moduli problem" is somewhat informal, and generally refers to studying the classification of geometric objects of a certain kind, especially when these objects admit continuous deformations. In algebraic geometry, continuous deformations are studied by considering "families of objects" over a base scheme, where the meaning of "family" depends on context.

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For many algebro-geometric objects of interest, the isomorphisms classes are determined

1. first by some discrete data (e.g., genus), along with

2. a point on a finite dimensional space of continuous parameters (e.g., $\mathcal{M}_g$), and this space has the structure of a variety.

Our first goal for the course is to give a formal framework for studying moduli problems and thereby give a precise meaning to the statement above that the moduli space "has the structure of an algebraic variety."

1.3   Everything is a moduli space

One way to give a topological space X the structure of an algebraic variety over $\mathbb{C}$ is to specify, for any algebraic variety T, which continuous maps $T \to X$ are algebraic. The data $T \mapsto \mathrm{Map}_{\mathrm{alg}}(T, X) \subset \mathrm{Map}(T, X)$ can be organized into a functor $\mathfrak{h} : \{Varieties\} \to \mathrm{Fun}(\{Varieties\}^{op}, \mathrm{Set})$. The Yoneda lemma says that this is a fully faithful embedding, i.e., the structure of an algebraic variety on X is uniquely determined by its functor of points.

Lemma 1.2 (Yoneda). *For any category $\mathcal{C}$, let $\mathfrak{h} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ be the functor taking $X \in \mathcal{C}$ to $\mathrm{Map}_{\mathcal{C}}(-, X)$. Then for any $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$,*

$$F(X) \cong \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})}(\mathfrak{h}_X, F).$$

This perspective is so fundamental that we sometimes identify X with its functor of points, and write $X(T) = \mathrm{Map}(T, X)$ for the set of T-*points.* Given an algebraic map $\phi : T \to X$, if we can can regard the assignment $t \in T \mapsto \phi(t) \in X$ as an algebraically varying family of points in X. This tautology, that X parameterizes families of points in X, is the entry point into moduli theory.

The same discussion applies to arbitrary schemes, although the set of maps of schemes $T \to X$ is no longer a subset of the set of continuous maps. We will denote the category of schemes as Sch, and the category of rings as Ring. Given a category $\mathcal{C}$, the functor category $\mathscr{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is referred to as the category of presheaves on $\mathcal{C}$.

Replacing a scheme with its functor of points might seem like a lot of extra data, but in practice it is often easy to specify the functor of points. You are probably already familiar with these examples:

Example 1.3. $\mathrm{Map}(\mathrm{Spec}(R), \mathbb{G}_m) = R^{\times}$

Example 1.4. $\mathrm{Map}(\mathrm{Spec}(R), GL_n) = \{\text{Automorphisms of the free module } R^n\}$. This is actually a group scheme, i.e., a group object in the category of schemes.

Example 1.5. A finite type affine scheme $S = \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ represents the functor $R \mapsto \{r_1, \ldots, r_n \in R | 0 = f_i(r_1, \ldots, r_n), \forall i\}$.

Exercise 1.1. *Recall the functor of points for projective space $\mathbb{P}^n$.*

1.3.1 First encounter with descent

You may notice that I have only specified the functor of maps $\mathrm{Spec}(R) \to X$ for each of the schemes above. It turns out that this is enough, i.e. the fully faithful embedding $\mathrm{Spec} : \mathrm{Ring} \to \mathrm{Sch}^{op}$ defines a restriction functor $\mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$. It turns out that the composition

$$\mathrm{Sch} \xrightarrow{\natural} \mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$$

is fully faithful.

Exercise 1.2. *Show that the restriction $\mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ is not fully faithful.*

The fully-faithfulness of the functor $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ is a special instance, and our first encounter with, the theory of descent. It follows from the fact that for any map $T \to X$ of schemes, one can cover T by affine open subschemes, and one can identify maps $T \to X$ with compatible families of maps from these affine open subschemes. In other words, it follows from

Lemma 1.6. *For any schemes T and X, the assignment $U \subset T \mapsto \mathrm{Map}(U, X)$ is a sheaf of sets on T.*

Exercise 1.3. *Prove this lemma, and use it to show that the functor $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ is fully faithful.*

This is a general principal that applies for any moduli problem:

Principle 1.7. A family of objects over a scheme T should be determined by its restriction to a collection of open sets which cover T.

1.4 Our first stack: $B\,\mathrm{GL}_n$

We have seen that any scheme parameterizes families of points in X, but there are many functors parameterizing families of objects that are not representable by any scheme (i.e., not in the essential image of $\mathfrak{h}$.)

We will focus on the moduli of families of vector spaces of rank n. Whatever a family of vector spaces over a scheme T should be, it should at the very least assign a vector space $V_{k(t)}$ of rank n over the residue field $k(t)$ to any point $t \in T$. From this perspective, a natural definition of a family of vector spaces over T would be a vector space object $V \in \mathscr{P}(\mathrm{Ring}_{/T}) \cong \mathscr{P}(\mathrm{Ring})/\mathfrak{h}_T$, i.e., it has maps

$$+ : V \times V \to V, 0 : \mathrm{pt} \to V, \text{ and } (-) \cdot (-) : \mathbb{A}^1 \times V \to V$$

satisfying the usual axioms of a vector space, and for any point $t \in T$, $V(k(t))$ is an n-dimensional vector space over $k(t)$.

In general this notion contains rather wild objects, but you can narrow it down quite a bit by requiring that V is representable by a scheme and finitely presented over T. It turns out that if T is a reduced scheme over a field of characteristic 0, then any such object is an n-dimensional vector bundle, i.e.,

$$V \cong \mathbb{V}_T(\mathcal{E}) := \mathrm{Spec}_T(\mathrm{Sym}(\mathcal{E}))$$

for some locally free sheaf $\mathcal{E}$ of $\mathcal{O}_T$-modules. The same is true for arbitrary T in characteristic 0 under the hypothesis that $V \to T$ is flat. We will be able to prove this using the techniques we will develop later in the course.

Exercise 1.4. *Prove that for any category $\mathcal{C}$ and $T \in \mathcal{C}$, the canonical functor is an equivalence of categories $\mathcal{P}(\mathcal{C}_{/T}) \cong \mathcal{P}(\mathcal{C})_{/\mathfrak{h}_T}$.*[2]

Note that over an affine scheme, locally free sheaves correspond to projective modules, so the natural functor of points $\mathcal{F} \in \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ parameterizing families of n-dimensional vector spaces would be (showing both $\mathcal{F}$ and its restriction to $\mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$):

$\mathcal{F}(X) := \{$isomorphism classes of locally free sheaves of rank n on $X\}$

$\mathcal{F}(R) := \{$isomorphism classes of projective R-modules of rank $n\}$

Unfortunately, this violates Principle 1.7. By definition every locally free sheaf is locally isomorphic to the constant sheaf, but there are certainly non-constant locally free sheaves, such as $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathbb{P}^n$.

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[2]For any category $\mathcal{C}$ and object $T \in \mathcal{C}$, recall the definition of the *slice* category, $\mathcal{C}_{/T}$: objects are morphisms $X \to T$ in $\mathcal{C}$, and morphisms in $\mathcal{C}_{/T}$ are morphisms $X \to Y$ in $\mathcal{C}$ such that the composition $X \to Y \to T$ is the same as the given map $X \to T$.

Exercise 1.5. *Show that the functor $\mathcal{F}$ above is not representable by a scheme.*

A locally free sheaf $\mathcal{E}$ over T is determined, of course, by its restriction to an open cover $\{U_i\}$ of T, but you also have to to remember the isomorphisms over $U_{ij} := U_i \cap U_j$ by which the sheaf is glued together. So instead, we define (again showing $\mathcal{F}$ and its restriction to rings):

$$\mathcal{F}(X) := \{\text{locally free sheaves on } X\}^{\text{isomorphisms}} \quad \text{\{E:def\_BGLn\}} \tag{1.1}$$

$$\mathcal{F}(R) := \{\text{projective } R\text{-modules of rank } n\}^{\text{isomorphisms}} \tag{1.2}$$

where the superscript "isomorphisms" indicates the largest subcategory which is a groupoid, i.e., the largest subcategory in which all morphisms are isomorphisms.[3] Note that the category Set embeds in the category of groupoids, which we denote Gpd, by regarding a set as a category whose only morphisms are identity morphisms. In that sense we are enlarging our previous notion of presheaf.

1.4.1 The subtleties involved in this definition

The definition of $\mathcal{F}(X)$ leaves ambiguous the following important question: *how does one give $\mathcal{F}$ the structure of a functor $\text{Sch}^{op} \to \text{Gpd}$?* It is more concrete to think about the restriction of $\mathcal{F}$ to rings.

Given a homomorphism $\phi_0 : R \to R_0$ the induced map $(\phi_0)^* : \mathcal{F}(R) \to \mathcal{F}(R_0)$ should map a projective R-module M to $R_0 \otimes_R M$. But given a second ring map $\phi_1 : R_0 \to R_1$, the modules $(\phi_1\phi_0)_\sharp(M)$ and $(\phi_1)_\sharp(\phi_0)_\sharp(M)$ are not the same, but only isomorphic. This is related to the fact that Gpd is actually a 2-category, where two morphisms of groupoids can be identified via a natural isomorphism, so it is a bit awkward to demand that $(\phi_1\phi_0)_\sharp = (\phi_1)_\sharp(\phi_0)_\sharp$ on the nose.

You could try to give $\mathcal{F}$ the structure of a lax 2-functor, that is you fix isomorphisms:

1. $\epsilon_R : R \otimes_R (-) \cong \text{id}$ as functors $\mathcal{F}(R) \to \mathcal{F}(R)$, for any $R \in \text{Ring}$, and

2. $\alpha_{\phi_1,\phi_0} : (\phi_1)_\sharp(\phi_0)_\sharp(-) \cong (\phi_1\phi_0)_\sharp(-)$ as functors $\mathcal{F}(R) \to \mathcal{F}(R_1)$, for any pair of homomorphisms $R \xrightarrow{\phi_0} R_0 \xrightarrow{\phi_1} R_1$.

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[3]$\mathcal{F}(X)$ is not a small category (class of objects is not a set), so this raises potential issues, but this can be handled using the theory of Grothendieck universes. $\mathcal{F}(X)$ is also essentially small (equivalent to a small groupoid), so it behaves like a small category. In any event, we will safely ignore these set-theoretic issues.

The issue you run into is that for a sequence of many homomorphisms $R \to R_0 \to R_1 \to R_2 \to \cdots \to R_N$, there are many ways to realize the N-fold composition as a sequence of binary compositions. So there are some compatibility conditions required so α can really be regarded as a "canonical" equivalence. Formulating gluing here becomes a bit of a mess...

Exercise 1.6. *Formulate these compatibility conditions (see* [V, Def. 3.10]*).*

Another approach would be to try to "strictify" in some sense, that is, to eliminate the redundancy implicit in the definition of $\mathcal{F}(R)$ by *choosing* a single representative M_α of every isomorphism class, and choosing an isomorphism of each M with one of these representatives M_α. This approach also involves some care, but it can be made to work [V, Theorem 3.45], but the choices involved make it impractical...

The most elegant solution is to encode $\mathcal{F}$ in an entirely different structure which, rather than defining the pullback operation $\phi_\sharp = R' \otimes_R (-) : \mathcal{F}(R) \to \mathcal{F}(R')$ explicitly, characterizes $R' \otimes_R M$ implicitly by its universal property.

We let $\mathcal{C}$ denote the category of pairs (R, M) with $R \in \mathrm{Ring}$ and M is a projective R-module of rank n. A morphism $(R, M) \to (R', M')$ in $\mathcal{C}$ is a ring homomorphism $\phi : R \to R'$ and a homomorphism of R-modules $\psi : M \to M'$ satisfying the condition that for any R'-module N, composition with ψ induces an isomorphism

$$\mathrm{Hom}_{R'}(M', N) \cong \mathrm{Hom}_R(M, N). \tag{1.3}$$

Then the forgetful functor $\pi : \mathcal{C} \to \mathrm{Ring}$ mapping $(R, M) \mapsto R$ encodes the functor $\mathcal{F}$ in the following sense:

- The objects of $\mathcal{F}(R)$ are pairs whose underlying ring is (literally) R, and the morphisms are those which map to the identity arrow in Ring.

- For any ring map $\phi_\sharp : R \to R'$ and projective R-module M of rank n, the object $\phi_\sharp(M)$ is the object in $\mathcal{F}(R')$ satisfying the universal property (1.3).

Note that $\phi_\sharp(M)$ is only defined up to canonical isomorphism, but one can upgrade $\phi_\sharp$ to an actual functor $\mathcal{F}(R) \to \mathcal{F}(R')$ using the Yoneda embedding. The morphism $\pi : \mathcal{C} \to \mathrm{Ring}$ is an example of a *cocartesian fibration*, and more specifically a *category fibered in groupoids* over Ring (because each fiber $\mathcal{F}(R)$ is a groupoid).

Note that the additional structure required to give $\mathcal{F}$ the structure of a functor directly has been replaced by the condition (1.3), which is somewhat easier to handle. We will also see that there are elegant ways to formulate Principle 1.7 for the functor $\pi : \mathcal{C} \to \mathrm{Ring}$.

1.4.2   The bottom line

In practice, people will often specify a moduli problem with an expression such as Equation (1.1), but they implicitly mean a category fibered in groupoids over Ring. In order to work in this subject, it is important to get comfortable going back and forth between the two.

1.4.3   A concrete approach

There is another approach to thinking about $\mathcal{F}(R)$, which is really a different approach to thinking about what a locally free sheaf *really is*, which makes Principle 1.7 manifest. For any locally free sheaf E on a scheme X, there is a Zariski open cover of X by open subschemes U_α such that $E|_{U_\alpha}$ is trivializable for all α. Let $a : V_0 := \bigsqcup_\alpha U_\alpha \to X$, and choose a trivialization $s : \mathcal{O}^n_{U'} \cong a^*(E)$. Let $V_1 := \bigsqcup_{\alpha \neq \beta} U_\alpha \cap U_\beta$, along with its two natural projections $d_0, d_1 : V_1 \to V_0$ whose compositions with a agree. Then we have an automorphism

$$\varphi : \mathcal{O}^n_{V_1} \xrightarrow{d_0^*(s)} d_0^*(a^*(E)) \cong d_1^*(a^*(E)) \xrightarrow{d_1^*(s^{-1})} \mathcal{O}^n_{V_1},$$

which we interpret as a map of schemes $V_1 \to \mathrm{GL}_n$ (see Example 1.4). φ satisfies a cocycle condition on the triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$, which make use of the group structure on GL_n. Thus a locally free sheaf consists of an open cover of X and a cocycle on this open cover with values in the group scheme GL_n. We will see that in general moduli problem one can given a similarly concrete description of families over any scheme, but instead of cocycles taking values in the group scheme GL_n, they will take values in a more general object known as a *groupoid scheme*.

Lecture 2

Stacks I

Last time we discussed the example of the moduli functor of families of vector spaces of rank n. In the following two lectures, we will take a step back and discuss the general formalism of sheaves of groupoids, of which the previous construction was an example.

Remark 2.1. We will make mild use of the concept of a 2-category, by which we mean a *strict* 2-category. A 2-category is a category enriched over Cat, i.e., for any two objects $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a category, called the category of 1-morphisms, and the composition functors $\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$ are strictly associative. There are also functors $\{*\} \to \mathrm{Map}_{\mathcal{C}}(X, X)$ which play the role of identities. The set of 2-morphisms between two 1-morphisms is the set of morphisms in $\mathrm{Map}_{\mathcal{C}}(X, Y)$. The standard example of a 2-category is Cat, where 1-morphisms are functors, and 2-morphisms are natural transformations of functors. Another is Gpd $\subset$ Cat, in which all 2-morphisms are automatically invertible. For further discussion, see [M1, XII.2].

2.1   Topologies

First we have to discuss notions of coverings which are "finer" than that of a Zariski covering. This is forced on us by basic examples: The moduli functor $\mathcal{F}$ of locally free sheaves is associated to the group scheme GL_n, so it

is natural to ask for an analog for other group schemes, e.g. $O_n, SL_n, \ldots$. It turns out the right notion here is a principal G-bundle (do not worry if you have not seen this notion before, we will provide the formal definition when we need it later). The problem is that a principal G-bundle is not necessarily locally trivial:

Exercise 2.1. *When G is finite, a principal G-bundle over a complex variety X is just a covering space $Y \to X$ in the analytic topology with group of deck transformations G acting transitively on each fiber. When X is a complex curve, any surjection $\pi_1(X^{an}) \twoheadrightarrow G$ defines such a cover. Show that these coverings are locally trivial in the analytic topology, but not trivializable Zariski-locally.*

The fix is the following notion:

Definition 2.2. A *Grothendieck topology* on a category $\mathcal{C}$ that has fiber products is a collection of sets of arrows $\{U_i \to U\}_{i \in I}$, called a covering of U, such that

  1. an isomorphism $\{U' \to U\}$ is a covering,

  2. coverings are closed under base change,

  3. coverings are closed under composition.

A *site* is a category that has fiber products[1] together with a Grothendieck topology.

Example 2.3. In the Zariski topology, a covering of U is a collection of Zariski open subsets $U_i \subset U$ such that $\bigsqcup_{i \in I} U_i \to U$ is a surjective morphism of schemes.

There are many different topologies on the category of schemes [S4, Tag 020K]. The most important for us are the étale topology and smooth topology, so we will recall some important facts about smooth and étale morphisms of schemes.

Definition 2.4. A map of rings $A \to B$ is *smooth* if it is of finite presentation, flat, and has regular geometric fibers. $A \to B$ is *étale* if it is smooth of relative dimension 0. A map of schemes $\pi : X \to Y$ is smooth (resp. étale) if for any open affine $U \subset Y$, $\pi^{-1}(U)$ can be covered by affine opens for which the corresponding ring homomorphisms are smooth (resp. étale).

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[1]It is not necessary to assume this [????], but all of our sites will have fiber products, so we take this as part of the definition.

There are many equivalent definitions of smooth and étale morphisms, and I encourage you to spend some time looking over them, if you are not already familiar. The stacks project definition [S4, Tag 00T2] is that a ring map $R \to S$ is smooth (resp. étale) if it is of finite presentation, and for any presentation $0 \to I \to R[x_1, \ldots, x_n] \xrightarrow{\alpha} S \to 0$, the two term complex

$$\mathbb{NL}(\alpha) = [I/I^2 \xrightarrow{d} S \otimes_{R[x_1,\ldots,x_n]} \Omega^1_{R[x_1,\ldots,x_n]/R}] \quad \text{\{E:naive\_cotangent\}}(2.1)$$

has homology which is a projective S-module in degree 0 and 0 otherwise (resp. $\mathbb{NL}(\alpha)$ is acyclic). $\mathbb{NL}(\alpha)$ does not depend, up to quasi-isomorphism, on the choice of presentation [S4, Tag 00S1].

One advantage of this (equivalent) definition is that it allows easy proofs of local structure results for smooth morphisms: a ring morphism $R \to S$ is *standard smooth* if there is a presentation of the form $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $c \leq n$ such that

$$\det\left(\left[\frac{\partial f_i}{\partial x_j}\right]_{i,j=1,\ldots,c}\right) \in S$$

is a unit. A ring map $R \to S$ is smooth if and only if $\text{Spec}(S)$ admits an open cover by standard affines $D(g)$ such that $R \to S_g$ is standard smooth [S4, Tag 00TA]. Similarly $R \to S$ is étale if and only if $c = n$ in the local standard smooth presentations.

Lemma 2.5 (Weak implicit function theorem). [S4, Tag 054L] *Consider a smooth morphism of schemes $X \to Y$. Then Zariski-locally on X and Y, the morphism admits a factorization $U \to \mathbb{A}^d_V \to V$ in which the first map is étale.*

Proof. This can be reduced to the case of a standard smooth morphism by the above remarks, in which case this is just the observation that $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ factors as $R \to R[x_{c+1}, \ldots, x_n] \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$, and the second ring map is standard smooth with equal numbers of generators and relations, hence étale. $\square$

Corollary 2.6. *If $X \to Y$ is a smooth morphism of schemes, then there is a surjective étale morphism $U \to Y$ which factors through a lift $U \to X$, i.e., the base change $X \times_Y U \to U$ admits a section.*

Exercise 2.2. *Show that if $X \to Y$ is an étale morphism of smooth varieties over $\mathbb{C}$, then every point of X has a neighborhood in the analytic topology which is a homeomorphism onto its image.*

One can give a slightly stronger local structure statement for étale morphisms: a morphism of schemes $X \to S$ is étale if and only if Zariski-locally it is induced by a *standard étale* ring map $R \to R[x]_h/(g)$, i.e., g is monic and g' is a unit in $R[x]_h/(g)$ [S4, Tag 00UE].

Definition 2.7. An étale (resp. smooth) covering of a scheme U is a set of morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that every f_i is étale (resp. smooth) and $\bigsqcup U_i \to U$ is surjective.

We say that a collection of arrows $\{V_j \to U\}_{j \in J}$ is a refinement of a collection $\{U_i \to U\}_{i \in I}$ if each $V_j \to U$ factors through some $U_i \to U$.

Exercise 2.3. *Show that any smooth cover of a quasi-compact scheme U admits a refinement by a finite cover $\{V_i \to U\}_{i=1,\dots,N}$ where each V_i is étale and affine (or even standard étale) over an affine open subset of U.*

2.2 Presheaves of categories

Let $p : \mathcal{F} \to \mathcal{C}$ be a functor between two categories.

Definition 2.8. An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ is *p-cartesian* if for any object $\zeta \in \mathcal{F}$, the canonical map

$$\mathrm{Map}_{\mathcal{F}}(\zeta, \xi) \to \mathrm{Map}_{\mathcal{F}}(\zeta, \eta) \times_{\mathrm{Map}_{\mathcal{C}}(p(\zeta), p(\eta))} \mathrm{Map}_{\mathcal{C}}(p(\zeta), p(\xi)),$$

which maps $\theta \mapsto (\phi \circ \theta, p(\theta))$, is bijective. If ϕ is cartesian, we say that ξ is a *pullback* of η along the map $p(\xi) \to p(\eta)$. The functor p is a *cartesian fibration* (also known as a fibered category), if for any arrow $f : X \to p(\eta)$, there is a cartesian arrow $\phi : \xi \to \eta$ with $p(\phi) = f$, i.e., pullbacks along any morphism exist.

A *cleavage* is defined to be a class $K \subset \mathrm{Mor}(\mathcal{F})$ consisting of one cartesian arrow ϕ_f lifting each arrow $f : X \to p(\xi)$ for every $X \in \mathcal{C}$ and $\xi \in \mathcal{F}$. As remarked in Lecture 1, a fibered category with a cleavage defines a lax 2-functor from $\mathcal{C}^{\mathrm{op}}$ to the 2-category of categories, $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, which we denote $\mathcal{F}(-)$. For $X \in \mathcal{C}$, $\mathcal{F}(X)$ is the subcategory of $\mathcal{F}$ consisting of arrows which map to id_X. Given a morphism $f : X \to Y$, the induced functor $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ maps $\xi \in \mathcal{F}(X)$ to the source of the unique cartesian arrow $\phi_f \in K$ lifting f. Given another morphism $g : Y \to Z$, the two functors $f^*(g^*(-))$ and $(g \circ f)^*(-)$ are not equal, but only canonically isomorphic as functors. We will not spell out precisely what this means, as we will not need it. See [V, Sect. 3.1]

Example 2.9. Let $\mathcal{C} = \mathrm{Sch}_{/S}$ be the category of schemes over a fixed base scheme. The fibered category $p : \mathrm{QCoh}^{\mathrm{op}}_{/S} \to \mathrm{Sch}_{/S}$ has objects consisting of pairs $(X \in \mathrm{Sch}_{/S}, E \in \mathrm{QCoh}(X))$, where the fiber functor p maps (X, E) to X. A morphism $(X, E) \to (Y, F)$ is a map of schemes $f : X \to Y$ and a map $F \to f_*(E)$. Note that given a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the composition $g_*(f_*(E)) = (g \circ f)_*(E)$ on the nose, so we can define the composition of morphisms $(X, E) \to (Y, F) \to (Z, G)$ in $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ to be the composed map $g \circ f : X \to Z$ along with the composed homomorphism of quasi-coherent sheaves on Z, $G \to g_*(F) \to g_*(f_*(E)) = (g \circ f)_*(E)$.

Exercise 2.4. *Use the adjunction between f^* and f_* to show that $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ is a fibered category over Sch/S, and an arrow $(X, E) \to (Y, F)$ is cartesian if and only if the homomorphism $F \to f_*(E)$ induces an isomorphism $f^*(F) \cong E$. Show that the fiber of $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ over $X \in \mathrm{Sch}_{/S}$ is the opposite category $\mathrm{QCoh}(X)^{op}$.*

Given a fibered category $p : \mathcal{F} \to \mathcal{C}$, one can construct another fibered category as follows: objects of $\mathcal{F}^{\mathrm{rev}}$ are objects of $\mathcal{F}$, and morphisms are defined by

$$\mathrm{Map}_{\mathcal{F}^{\mathrm{rev}}}(\xi, \eta) = \left\{ \text{diagrams } \xi \leftarrow \xi' \to \eta \ \middle| \ \begin{matrix} \xi' \to \eta \text{ is cartesian, and} \\ p(\xi) = p(\xi'), p(\xi \to \xi') = \mathrm{id}_{p(\xi)} \end{matrix} \right\} / \sim,$$

where two diagrams are equivalent if they fit into a commutative diagram in $\mathcal{F}$

which is necessarily unique if it exists. Given two arrows $\alpha = (\eta \leftarrow \eta' \to \chi)$ and $\beta = (\xi \leftarrow \xi' \to \eta)$ in $\mathcal{F}^{\mathrm{rev}}$ one can choose a cartesian arrow $\xi'' \to \eta'$ over $p(\xi' \to \eta)$ and use the universal property of $\xi' \to \eta$ to deduce the existence and uniqueness of a dotted arrow over $\mathrm{id}_{p(\xi)}$ that makes the following diagram commute

We then define $\alpha \circ \beta$ to be the outer compositions of this diagram $\alpha \circ \beta = (\xi \leftarrow \xi'' \rightarrow \chi)$. $\mathcal{F}^{\mathrm{rev}}$ is a fibered category as well, in which a cartesian arrow is a diagram $(\xi \leftarrow \xi' \rightarrow \eta)$ for which $\xi' \rightarrow \xi$ is an isomorphism. The fiber $\mathcal{F}^{\mathrm{rev}}(X)$ is canonically isomorphic to $\mathcal{F}(X)^{\mathrm{op}}$. If we replace $\mathcal{F}$ with an equivalent fibered category which admits a splitting and hence corresponds to a strict functor $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Cat}$, then $\mathcal{F}^{\mathrm{rev}}$ corresponds to the functor $F^{\mathrm{rev}}(X) = (F(X))^{\mathrm{op}}$. Note also that if $\mathcal{F}^{\mathrm{cart}} \subset \mathcal{F}$ denotes the subcategory consisting of all cartesian arrows of $\mathcal{F}$, then the assignment

$$(\xi \xleftarrow{f} \xi' \xrightarrow{g} \eta) \mapsto (g \circ f^{-1} : \xi \rightarrow \eta)$$

defines an equivalence of categories fibered in groupoids $(\mathcal{F}^{\mathrm{rev}})^{\mathrm{cart}} \cong \mathcal{F}^{\mathrm{cart}}$.
E:fibered_cat_qcoh_op

Example 2.10 (Quasi-coherent sheaves). We apply this construction to the fibered category $\mathrm{QCoh}_{/S}^{\mathrm{op}}$ over $\mathrm{Sch}_{/S}$ to obtain a fibered category

$$\mathrm{QCoh}_{/S} := (\mathrm{QCoh}_{/S}^{\mathrm{op}})^{\mathrm{rev}}, \qquad \{\texttt{E:define\_qcoh}\} \quad (2.2)$$

which call the *fibered category of quasi-coherent sheaves.* The fiber over $X \in \mathrm{Sch}_{/S}$ is isomorphic to $\mathrm{QCoh}(X)$, and the pullback functors correspond to the usual pullback of quasi-coherent sheaves. An alternate construction of $\mathrm{QCoh}_{/S}$, which involves choosing a clevage for $\mathrm{QCoh}_{/S}^{\mathrm{op}}$, is discussed in [V, Sect. 3.2.1].

The class of fibered categories over $\mathcal{C}$ has the structure of a 2-category, $\mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cart}}$. A morphism of fibered categories $\mathcal{F} \rightarrow \mathcal{F}'$ is a functor of categories over $\mathcal{C}$ which preserves cartesian arrows. A 2-morphism between base-preserving functors $f, g : \mathcal{F} \rightarrow \mathcal{F}'$ is a *base-preserving natural transformation,* i.e., a natural transformation $\eta : f \Rightarrow g$ such that for any $\xi \in \mathcal{F}$, the homorphism $\eta_\xi : f(\xi) \rightarrow g(\xi)$ maps to the identity morphism of $p'(f(\xi)) = p'(g(\xi))$.

One can show that a morphism of fibered categories $\mathcal{F} \rightarrow \mathcal{F}'$ is fully faithful or an equivalence if and only if the same is true for the functor $\mathcal{F}(X) \rightarrow \mathcal{F}'(X)$ for any $X \in \mathcal{C}$ [V, Prop. 3.36]. Another useful fact is that a composition of cartesian fibrations $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{C}$ is again a cartesian fibration [V, Prop. 3.7]

Definition 2.11. A fibered category $p : \mathcal{F} \rightarrow \mathcal{C}$ is a *category fibered in groupoids* if every fiber $\mathcal{F}(X)$ is a groupoid, i.e., all arrows in $\mathcal{F}$ mapping to an identity arrow are invertible. This is equivalent to the condition that all arrows in $\mathcal{F}$ are p-cartesian by [V, Prop. 3.22]. We let $\mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cart}, \cong} \subset \mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cart}}$ denote the full 2-subcategory of categories fibered in groupoids over $\mathcal{C}$.

For categories fibered in groupoids, base-preserving functors are automatically morphisms of fibered categories, and all natural transformations are equivalences, i.e., the category $\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(\mathcal{F}, \mathcal{F}')$ is a groupoid.

E:representable_fibered_cat

Example 2.12. An important example is given an object $X \in \mathcal{C}$, the forgetful functor from the slice category $\mathcal{C}_{/X} \to \mathcal{C}$ is a category fibered in groupoids. In fact, the fibers are *sets*. There is an equivalence of categories between categories fibered in sets and functors $\mathcal{C}^{op} \to \mathrm{Set}$, under which $\mathcal{C}_{/X}$ corresponds to the representable functor $\mathfrak{h}_X(-) = \mathrm{Map}_{\mathcal{C}}(-, X)$.

Lemma 2.13 (2-Yoneda lemma). *Given $\mathcal{F} \in \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}$ and $X \in \mathcal{C}$, the functor*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(\mathcal{C}_{/X}, \mathcal{F}) \to \mathcal{F}(X),$$

which takes a morphism of fibered categories $F : \mathcal{C}_{/X} \to \mathcal{F}$ to the object $F(\mathrm{id}_X) \in \mathcal{F}(X)$, is an equivalence of categories. [V, Sect. 3.6.2]

Proof. To construct the inverse functor, we choose a cleavage K of $\mathcal{F}$, which allows one to define pullback functors $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ for any $f : X \to Y$ in $\mathcal{C}$. For any $f : X \to Y$ and $\xi \in \mathcal{F}(Y)$, we let $\phi_{f,\xi} : f^*(\xi) \to \xi$ denote the unique arrow in K lifting f.

For any $\xi \in \mathcal{F}(X)$, we let $F_\xi : \mathcal{C}_{/X} \to \mathcal{F}$ be the functor which takes $(\alpha : U \to X) \in \mathcal{C}_{/X}$ to $\alpha^*(\xi) \in \mathcal{F}(U)$. F_ξ maps a morphism in $\mathcal{C}_{/X}$, given by a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ \ f\ \ } & V \\
& {\alpha}\searrow \quad \swarrow{\beta} & \\
& X &
\end{array}
\ ,
$$

to the unique arrow $\alpha^*(\xi) \to \beta^*(\xi)$ corresponding to the element $(f, \phi_{\alpha,\xi})$ in $\mathrm{Map}(U, V) \times_{\mathrm{Map}(U,X)} \mathrm{Map}(\alpha^*(\xi), \xi)$. We refer to [V, Sect. 3.6.2] to show that F_ξ is a morphism of fibered categories, and $\xi \mapsto F_\xi$ and $F \mapsto F(\mathrm{id}_X)$ are mutually inverse functors. $\qquad\square$

2.2.1 Straightening and unstraightening.

Let $\mathcal{F} \to \mathcal{C}$ be a fibered category, and let $K \subset \mathrm{Mor}(\mathcal{F})$ be a cleavage of $\mathcal{F}$. If K contains the identity morphisms and is closed under composition, then we say K is a *splitting*. In this case the lax 2-functor corresponding to $\mathcal{F}$ is actually a strict functor $F : \mathcal{C}^{op} \to \mathrm{Cat}$, where $F(X) = \mathcal{F}(X)$ and any

morphism $f : X \to Y$ induces a pullback functor $f^* : F(Y) \to F(X)$ via the splitting.

On the other hand, given a strict functor $F : \mathcal{C}^{op} \to \mathrm{Cat}$, we can define a fibered category $\mathrm{Un}_{\mathcal{C}}(F) \to \mathcal{C}$, which we call the unstraightening of F. The objects of $\mathrm{Un}_{\mathcal{C}}(F)$ are pairs $(X \in \mathcal{C}, \xi \in F(X))$, and a morphism $(X, \xi) \to (Y, \eta)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ along with a morphism $\xi \to f^*(\eta)$ in $F(X)$. Note that $\mathrm{Un}_{\mathcal{C}}(F)$ admits a canonical splitting, consisting of arrows of the form $(f : X \to Y, \mathrm{id} : f^*(\eta) \to f^*(\eta))$.

We equip $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat})$ with the structure of a 2-category as follows:

- A 1-morphism $\phi : F \to G$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat})$ is an assignment of functors $\phi_X : F(X) \to G(X), \forall X \in \mathcal{C}$ such that for any $f : X \to Y$ the following diagram of functors *strictly* commutes

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{\phi_Y} & G(Y) \\
\downarrow{f^*} & & \downarrow{f^*} \\
F(X) & \xrightarrow{\phi_X} & G(X)
\end{array}
$$

- A 2-morphism $a : \phi_1 \Rightarrow \phi_2$ between two morphisms $\phi_1, \phi_2 : F \to G$ is an assignment for every $X \in \mathcal{C}$ a natural transformation $a_X : (\phi_1)_X \Rightarrow (\phi_2)_X$ of functors $F(X) \to G(X)$, satisfying the condition that for any morphism $f : X \to Y$, $a_X \circ f^* = f^* \circ a_Y$ as a natural transformations after identifying $(\phi_1)_X \circ f^* = f^* \circ (\phi_1)_Y$ and $(\phi_2)_X \circ f^* = f^* \circ (\phi_2)_Y$, i.e. the two diagrams are *equal*:

$$
\left[F(Y) \underset{(\phi_2)_X \circ f^*}{\overset{(\phi_1)_X \circ f^*}{\rightrightarrows}} {\Downarrow a_x \circ f^*} \; G(X) \right] = \left[F(Y) \underset{f^* \circ (\phi_2)_X}{\overset{f^* \circ (\phi_1)_X}{\rightrightarrows}} {\Downarrow f^* \circ a_X} \; G(X) \right]
$$

`L:fully_faithful`

Lemma 2.14. $\mathrm{Un}_{\mathcal{C}}$ *admits the structure of a functor of* 2*-categories, which is bijective on* 2*-morphisms and identifies* 1*-morphisms* $F \to G$ *with morphisms of fibered categories* $\mathrm{Un}_{\mathcal{C}}(F) \to \mathrm{Un}_{\mathcal{C}}(G)$ *which map arrows in the canonical splitting of* $\mathrm{Un}_{\mathcal{C}}(F)$ *to arrows in the canonical splitting of* $\mathrm{Un}_{\mathcal{C}}(G)$.

Proof. Given a 1-morphism $\phi : F \to G$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat})$, the corresponding fiber functor $\mathrm{Un}_{\mathcal{C}}(\phi) : \mathrm{Un}_{\mathcal{C}}(F) \to \mathrm{Un}_{\mathcal{C}}(G)$ maps $(X, \xi \in F(X)) \mapsto (X, \phi_X(\xi) \in G(X))$, and for morphisms $(X, \xi) \to (Y, \eta)$ in $\mathrm{Un}_{\mathcal{C}}(F)$ it maps

$$
\left(\begin{array}{c} f : X \to Y \\ u : \xi \to f^*(\eta) \end{array} \right) \mapsto \left(\begin{array}{c} f : X \to Y \\ \phi_X(u) : \phi_X(\xi) \to f^*(\phi_Y(\eta)) \end{array} \right),
$$

where we have used the fact that $\phi_X f^* = f^* \phi_Y$. This functor evidently preserves arrows in the canonical splittings. To show that this functor preserves cartesian arrows, we observe that every arrow in $\mathrm{Un}_\mathcal{C}(\mathcal{F})$ can be factored uniquely as an arrow in the splitting K followed by a morphism in a fiber:

$$\begin{pmatrix} f : X \to Y \\ u : \xi \to f^*(\eta) \end{pmatrix} = \begin{pmatrix} id_X : X \to X \\ u : \xi \to f^*(\eta) \end{pmatrix} \circ \begin{pmatrix} f : X \to Y \\ \mathrm{id} : f^*(\eta) \to f^*(\eta) \end{pmatrix}. \qquad \text{\{E:Factorization\}} \quad (2.3)$$

We leave it to the reader to use this factorization to show that a functor of fibered categories $\mathrm{Un}_\mathcal{C}(F) \to \mathrm{Un}_\mathcal{C}(G)$ which preserves the canonical splittings is of the form $\mathrm{Un}_\mathcal{C}(\phi)$ for a unique ϕ.

On the level of 2-morphisms, one can explicitly describe a natural transformation $a : \mathrm{Un}_\mathcal{C}(\phi_1) \Rightarrow \mathrm{Un}_\mathcal{C}(\phi_2)$ as follows: For every pair $(X \in \mathcal{C}, \xi \in F(X))$, a assigns a morphism $a_X(\xi) : (\phi_1)_X(\xi) \to (\phi_2)_X(\xi)$ in $G(X)$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$, any $\eta \in F(Y)$, and morphism $u : \xi \to f^*(\eta)$ in $F(X)$, the following diagram commutes

$$\begin{array}{ccc}
(\phi_1)_X(\xi) & \xrightarrow{\;a_X(\xi)\;} & (\phi_2)_X(\xi) \\
\downarrow{\scriptstyle (\phi_1)_X(u)} & & \downarrow{\scriptstyle (\phi_2)_X(u)} \\
f^*((\phi_1)_Y(\eta)) & \xrightarrow[\;f^*(a_Y(\eta))\;]{} & f^*((\phi_2)_Y(\eta))
\end{array} \qquad \text{\{E:natural\_square\}} \quad (2.4)$$

If we restrict (2.2.1) only to morphisms in the fiber over $X \in \mathcal{C}$, it is just the condition that $a_X : (\phi_1)_X \to (\phi_2)_X$ is a natural transformation. If we restrict (2.2.1) to morphisms in the canonical splitting, it is the condition that $f^* \circ a_Y = a_X \circ f^*$. Conversely, one can use the fact that any morphism in $\mathrm{Un}_\mathcal{C}(F)$ admits a unique factorization to conclude that any collection of natural transformations $\{a_X\}_{X \in \mathcal{C}}$ satisfying these compatibility conditions also satisfying the condition for any morphism in $\mathrm{Un}_\mathcal{C}(F)$. $\qquad \square$

For any 2-category $\mathcal{C}$, we let $\mathrm{Ho}(\mathcal{C})$ denote the 1-category whose objects are the same, and whose morphisms are 2-isomorphism classes of 1-morphisms in $\mathcal{C}$. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories is an equivalence if $\mathrm{Map}_\mathcal{C}(X, Y) \to \mathrm{Map}_\mathcal{D}(F(X), F(Y))$ is an equivalence of categories for any $X, Y \in \mathcal{C}$, and $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is essentially surjective.

Our main justification for working with fibered categories rather than strict presheaves of categories, or even lax presheaves, is the following:

Theorem 2.15. *The unstraightening functor*

$$\mathrm{Un}_\mathcal{C} : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}) \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}} \qquad \text{\{E:unstraightening\}} \quad (2.5)$$

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is an equivalence of 2*-categories, i.e., it induces an equivalence on mapping categories for any pair of objects, and it is essentially surjective.*

Proof. In light of Lemma 2.14, this is a consequence of two facts: Every fibered category is canonically equivalent to a split fibered category [V, Thm. 3.45]; and any functor of fibered categories $\mathrm{Un}_{\mathcal{C}}(F) \to \mathrm{Un}_{\mathcal{C}}(G)$ is 2-isomorphic to one which preserves the splitting. We leave the latter as an exercise for the reader. $\square$

Lecture 3

Stacks II

References: [S4, Tag 0238], [V], [F2], [O1, Chap. 4]
Date: 1/30/2020
Exercises: 18

I could not find references that were both accessible to beginning algebraic geometers and covered the material from the perspective I wanted to take (suggestions welcome!), so I ended up writing up proofs in a bit more detail, and much more detail than I covered in the actual lecture.

I have tried to present the theory of descent in a manner parallel to the theory of descent in ∞-categories, at least as developed in [L2]. Developing a robust theory of descent in a homotopical context was arguably one of the main motivations for developing the theory of ∞-categories, and one of the reasons these methods have been so useful recently. Below I have formulated and proved many of the ideas from ∞-categorical context, such as constructing limits of categories as categories of cartesian sections and cofinal ∞-functors, using only 1-category theory. The proofs are simpler, and I hope this introduction to descent will subtely prepare students to study the ∞-categorical generalization.

3.1   Descent

S:descent

Consider the category $\mathcal{C} = \mathrm{Sch}_{/S}$ with its étale topology. Let $p : \mathcal{F} \to \mathrm{Sch}_{/S}$ be a fibered category. A diagram of schemes over S is a category $\mathcal{I}$ and a functor $D : \mathcal{I} \to \mathrm{Sch}_{/S}$. We refer to $\mathcal{I}$ as the indexing category, and typically $\mathcal{I}$ will be small.

Example 3.1. One can consider the category Δ^1 with two objects and one non identity arrow between them. Then a commutative square of schemes is a functor $\Delta^1 \times \Delta^1 \to \mathrm{Sch}_{/S}$.

<div align="right">D:cartesian_section</div>

Definition 3.2. A *cartesian section* of p over $D : \mathfrak{I} \to \mathrm{Sch}_{/S}$ is a functor $\sigma : \mathfrak{I} \to \mathfrak{F}$ such that $p \circ \sigma = D$, and every arrow in D maps to a p-cartesian arrow of $\mathfrak{F}$. The class of cartesian sections admits the structure of a category which we denote $\Gamma_{\mathcal{C}}^{\mathrm{cart}}(D, \mathfrak{F})$, where morphisms are natural transformations $\eta : \sigma_1 \Rightarrow \sigma_2$ such that $p \circ \eta : D \Rightarrow D$ is the identity transformation.

Exercise 3.1. *If $\mathfrak{F} = \mathrm{QCoh}_{/S} \to \mathrm{Sch}_{/S}$ and D is a diagram consisting of two schemes and two non-identity arrows $f, g : X \to Y$, then the category of cartesian sections of $\mathfrak{F}$ over $\mathcal{D}$ is equivalent to the category of quasi-coherent sheaves E over Y along with an isomorphism $f^*(E) \cong g^*(E)$.*

<div align="right">EX:terminal</div>

Exercise 3.2. *Show that if the indexing category $\mathfrak{I}$ has a terminal object $*$, i.e., there is a unique morphism $X \to *$ for any $X \in \mathfrak{I}$, then restricting any diagram $D : \mathfrak{I} \to \mathcal{C}$ to $*$ defines an equivalence $\Gamma_{\mathcal{C}}^{\mathrm{cart}}(D, \mathfrak{F}) \cong \mathfrak{F}(D(*))$. (Hint: first restrict to the subcategory consisting of all objects but only the unique arrow from each X to $*$).*

Given a covering $\mathcal{U} = \{U_\alpha \to U\}_{\alpha \in I}$, we will be particularly interested in the diagram $D_{\mathcal{U}}$ consisting of schemes U_α, $U_{\alpha\beta} := U_\alpha \times_U U_\beta$, and $U_{\alpha\beta\gamma} := U_\alpha \times_U U_\beta \times_U U_\gamma$ for all $\alpha, \beta, \gamma \in I$, and arrows consisting of the canonical projection maps (satisfying the evident relations). More precisely, the indexing category $\mathfrak{I}_I$ of $D_{\mathcal{U}}$ consists of non-empty ordered lists of length ≤ 3, $[\alpha]$, $[\alpha, \beta]$ and $[\alpha, \beta, \gamma]$ and a morphism for every deletion $[\alpha, \beta, \gamma] \to [\alpha, \beta]$, $[\alpha, \beta, \gamma] \to [\alpha, \gamma]$, etc...

Definition 3.3. The category of *descent data* for $\mathfrak{F}$ on the cover $\mathcal{U}$ is defined to be the category of cartesian sections $\mathrm{Desc}_{\mathfrak{F}}(\mathcal{U}) := \Gamma_{\mathcal{C}}^{\mathrm{cart}}(D_{\mathcal{U}}, \mathfrak{F})$.

Exercise 3.3. *Show that given a cleavage K for $\mathfrak{F}$, so that we may define $f^*(\xi)$ for any $f : X \to Y$ in $\mathrm{Sch}_{/S}$ and any $\xi \in \mathfrak{F}(Y)$, the category of descent data on $\{U_\alpha \to U\}_{\alpha \in I}$ is equivalent to the category of collections $(\{\xi_\alpha\}, \{\phi_{\alpha\beta}\})$ of objects $\xi_\alpha \in \mathfrak{F}(U_\alpha)$ and isomorphisms $\phi_{\alpha\beta} : \mathrm{pr}_1^*(\xi_\beta) \cong \mathrm{pr}_0^*(\xi_\alpha)$ in $\mathfrak{F}(U_{\alpha\beta})$ such that for any $\alpha, \beta, \gamma \in I$ we have an equality of morphisms in $\mathfrak{F}(U_{\alpha\beta\gamma})$*

$$\mathrm{pr}_{02}^*(\phi_{\alpha\gamma}) = \mathrm{pr}_{01}^*(\phi_{\alpha\beta}) \circ \mathrm{pr}_{12}^*(\phi_{\beta\gamma}) : \mathrm{pr}_2^*(\xi_\gamma) \to \mathrm{pr}_0^*(\xi_\alpha).$$

where pr_{ab} is the projection onto the a and b factors, and pr_a is the projection onto the a factor. (This is the definition of descent data in [V, Def. 4.2].)

<div align="center">26</div>

Given a cover $\mathcal{U} = \{U_\alpha \to U\}_{\alpha \in I}$, let $\mathcal{I}_I^+$ denote the category obtained from $\mathcal{I}_I$ by adding a terminal object corresponding to the empty list, $\emptyset$. Note that $D_\mathcal{U} : \mathcal{I}_I \to \mathrm{Sch}_{/S}$ extends canonically to a diagram $D_\mathcal{U}^+ : \mathcal{I}_I^+ \to \mathrm{Sch}_{/S}$, such that $D_\mathcal{U}^+$ maps $\emptyset \in \mathcal{I}_I^+$ to U. We refer to $D_\mathcal{U}^+$ as the *augmented* descent diagram for $\mathcal{U}$.

<div align="right">D:stack</div>

Definition 3.4. A fibered category $\mathcal{F}/\mathcal{C}$ is a *stack* with respect to the topology on $\mathcal{C}$ if for every covering $\mathcal{U} = \{U_\alpha \to U\}_{\alpha \in I}$, the canonical restriction functor

$$\mathcal{F}(U) \cong \Gamma_\mathcal{C}^{\mathrm{cart}}(D_\mathcal{U}^+, \mathcal{F}) \to \Gamma_\mathcal{C}^{\mathrm{cart}}(D_\mathcal{U}, \mathcal{F}) \qquad \text{\{E:descent\}} \quad (3.1)$$

is an equivalence of categories, i.e., $\mathcal{F}(U)$ is equivalent to the category of descent data. The first equivalence in (3.1) is given by restriction to $\emptyset \in \mathcal{I}_I^+$, see Exercise 3.2, and holds for any $\mathcal{F}$.

Remark 3.5. This is equivalent to other formulations of descent, such as [V, Def. 4.6] (which makes use of a cleavage) or [V, Cor. 4.13] (which makes use of sieves), but this formulation breaks the descent condition into two pieces. The main piece, the fact that restriction is an equivalence $\Gamma_\mathcal{C}^{\mathrm{cart}}(D_\mathcal{U}^+, \mathcal{F}) \to \Gamma_\mathcal{C}^{\mathrm{cart}}(D_\mathcal{U}, \mathcal{F})$ actually implies that restriction is an *isomorphism*, not just an equivalence. In other words, any cartesian section of $\mathcal{F}$ over $D_\mathcal{U}$ extends *uniquely* to a cartesian section over $D_\mathcal{U}^+$. One is only forced to say "equivalence of categories" when one identifies sections over $D_\mathcal{U}^+$ with $\mathcal{F}(U)$.

One can show that the functor (3.1) is fully faithful if and only if for any $X \in \mathcal{C}$ and $\xi, \eta \in \mathcal{F}(X)$, the functor

$$\underline{\mathrm{Map}}_X(\xi, \eta) : \mathcal{C}_{/X}^{op} \to \mathrm{Set}$$

which assigns $f : T \to X$ to $\mathrm{Map}_{\mathcal{F}(T)}(f^*(\xi), f^*(\eta))$ is a sheaf (of sets) for the inherited topology on the slice category $\mathcal{C}_{/X}$ [V, Prop. 4.7] or [S4, Tag 06NT]. So a fibered category is a stack if and only if the mapping presheaves are sheaves and every descent datum is "effective," meaning it is isomorphic to the pullback of an object of $\mathcal{F}(U)$. Note also that the property of a fibered category being a stack is invariant under equivalence, because categories of cartesian sections are invariant under equivalence.

<div align="right">E:setoids</div>

Example 3.6. Specifying a category fibered in sets is equivalent to specifying a presheaf of sets [V, Sect. 3.4]. In fact under Theorem 2.15, the category of presheaves of sets, regarded as a 2-category with only identity 2-morphisms, is equivalent to the full sub 2-category of $\mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cart}}$ consisting of categories

fibered in *setoids*, i.e. fibered categories such that objects in $\mathcal{F}(X)$ have no automorphisms. A category fibered in sets (or setoids) is a stack if and only if the corresponding presheaf of sets is a sheaf.

<div align="right">EX:descent_composition</div>

Exercise 3.4. *We have noted that a composition of cartesian fibrations $\mathcal{F}' \to \mathcal{F} \to \mathcal{C}$ is again a cartesian fibration. Given a Grothendieck topology on $\mathcal{C}$, we can define a Grothendieck topology on $\mathcal{F}$ whose coverings consist of families of cartesian arrows $\{\xi_i \to \xi\}_{i \in I}$ such that $\{p(\xi_i) \to p(\xi)\}_{i \in I}$ is a covering in $\mathcal{C}$, where $p : \mathcal{F} \to \mathcal{C}$ is the fiber functor* [S4, Tag 06NT]. *(Note that fiber products of cartesian morphisms automatically exist in $\mathcal{F}$.) Show that if $\mathcal{F}$ is a stack in groupoids over $\mathcal{C}$, and $\mathcal{F}'$ is a stack over $\mathcal{F}$ with this inherited topology, then $\mathcal{F}'$ is a stack over $\mathcal{C}$.*

There is a more "canonical" way to express the descent condition using *sieves*. By definition, for any covering $\mathcal{U} = \{U_i \to U\}$ we let $\mathfrak{h}_{\mathcal{U}} \subset \mathfrak{h}_U$ denote the subfunctor of morphisms to U which factor through one of the maps $U_i \to U$ in the cover. Subfunctor $\mathcal{S} \subset \mathfrak{h}_U$ is called a *covering sieve*[V, Def. 2.41] for the topology on $\mathcal{C}$ if there is some covering $\mathcal{U} = \{U_i \to U\}$ such that $\mathfrak{h}_{\mathcal{U}} \subset \mathcal{S}$.

<div align="right">EX:covering_sieve_descent</div>

Exercise 3.5. *Show that a category $\mathcal{F}$ fibered in groupoids over a site $\mathcal{C}$ is a stack if and only if for every covering sieve $S \subset \mathfrak{h}_U$ the restriction map $\Gamma_{\mathcal{C}}(\mathfrak{h}_U, \mathcal{F}) \to \Gamma_{\mathcal{C}}(\mathcal{S}, \mathcal{F})$ is an equivalence of categories. (Hint: see* [V, Prop. 4.14]*.)*

3.2 Morphisms which preserve cartesian sections

In this section we will discuss a bit of category theory which will lead to a better conceptual understanding of the descent condition. The main idea is that, under the unstraightening equivalence Theorem 2.15, we can identify a fibered category $\mathcal{F} \to \mathcal{C}$ as a diagram of categories indexed by $\mathcal{C}^{op}$. From this perspective, the category $\Gamma_{\mathcal{C}}^{\mathrm{cart}}(\mathcal{F})$ should be regarded as the *limit* of this diagram, i.e., the category consisting of assignments of objects $\xi_X \in \mathcal{F}(X)$ for each $X \in \mathcal{C}$ which are compatible with pullback along morphisms in $\mathcal{C}$. This is formalized in Example 3.14.

The main result of the section is Proposition 3.19, which gives a criterion under which you can replace a diagram of categories with a sub-diagram without changing the limit. We have already seen an example of this in Exercise 3.2. The criterion is a higher-categorical analog of the fact in 1-category theory that if a subcategory $\mathcal{C} \subset \mathcal{D}$ is *final* then one can restrict any diagram of sets over $\mathcal{D}^{op}$ to $\mathcal{C}^{op}$ without affecting the limit.

<div align="center">28</div>

3.2.1 Pullback of fibered categories

Definition 3.7 (Pullback)**.** Given functors $\mathcal{F} \to \mathcal{D}$ and $f : \mathcal{C} \to \mathcal{D}$, let $f^{-1}(\mathcal{F})$ denote the following category over $\mathcal{C}$: objects consist of pairs $(X \in \mathcal{C}, \xi \in \mathcal{F}(f(X)))$, and a morphism from (X, ξ) to (Y, η) is a morphism $X \to Y$ and a morphism $\xi \to \eta$ in $\mathcal{F}$ lying above the morphism $f(X \to Y)$, i.e.,

$$\mathrm{Map}_{f^{-1}(\mathcal{F})}((X,\xi),(Y,\eta)) = \mathrm{Map}_{\mathcal{C}}(X,Y) \times_{\mathrm{Map}_{\mathcal{D}}(f(X),f(Y))} \mathrm{Map}_{\mathcal{F}}(\xi,\eta).$$

This definition is written in a non-symmetric notation, because in practice we are regarding $\mathcal{F}$ as a presheaf of categories over $\mathcal{D}$ and $f^{-1}(\mathcal{F})$ as the pullback presheaf, but in fact $f^{-1}(\mathcal{F})$ is just the (strict!) fiber product of categories $\mathcal{C} \times_{\mathcal{D}} \mathcal{F}$. This is consistent with our earlier notation: if we consider a fibered category $\mathcal{F} \to \mathcal{C}$ and regard an object $U \in \mathcal{C}$ as the inclusion of the one-point category $\{\mathrm{id}_U\} \subset \mathcal{C}$, then $\mathcal{F}(U) = p^{-1}(U)$ is the preimage of this subcategory.

Exercise 3.6. *Show that if $f : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{F}$ is a fibered category over $\mathcal{D}$, then $f^{-1}(\mathcal{F})$ is a fibered category over $\mathcal{C}$, where an arrow is cartesian if and only if the corresponding arrow in $\mathcal{F}$ is cartesian.*

Note given a fibered category over $\mathcal{C}$ and a diagram in $\mathcal{C}$ corresponding to a functor $D : \mathcal{I} \to \mathcal{C}$, we have defined

$$\Gamma_{\mathcal{C}}^{\mathrm{cart}}(D, \mathcal{F}) = \Gamma_{\mathcal{I}}^{\mathrm{cart}}(\mathrm{id}_{\mathcal{I}}, D^{-1}\mathcal{F}).$$

We will sometimes simplify notation by denoting $\Gamma_{\mathcal{I}}^{\mathrm{cart}}(\mathrm{id}_{\mathcal{I}}, D^{-1}\mathcal{F})$ by $\Gamma_{\mathcal{I}}^{\mathrm{cart}}(D^{-1}\mathcal{F})$ or $\Gamma^{\mathrm{cart}}(D^{-1}\mathcal{F})$, when the base is understood. In particular if you have a small diagram D in $\mathcal{C}$, the category of cartesian sections of a fibered category $\mathcal{F}$ over D is just the category of "cartesian global sections" of the pullback $D^{-1}\mathcal{F}$.

Warning 3.8. In many of our applications we will consider fibered categories over a site $\mathcal{C}$, and in this case what we are calling $f^{-1}(\mathcal{F})$ is more typically denoted $f_*(\mathcal{F})$ in the topos literature (see for instance [O1, Sect. 2.2]). For example, if $\phi : X \to Y$ is a map of topological spaces, then the functor on sites of open subsets $f : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ maps $U \mapsto \phi^{-1}(U)$. Given a presheaf or sheaf $F(-)$ on $\mathrm{Op}(X)$, the presheaf $F(f(-))$ on $\mathrm{Op}(Y)$ is $\phi_*(F)$. We are using the notation f^{-1} because it is more natural from the perspective of cartesian fibrations, and because we are reserving the notation f_* for a different construction.

3.2.2 Pushforward along a cocartesian morphism

Given a functor $\phi : \mathcal{C}_0 \to \mathcal{C}_1$, and fibered categories $\mathcal{F}_i$ over $\mathcal{C}_i$ for $i = 0, 1$ along with an equivalence $\mathcal{F}_0 \cong \phi^{-1}(\mathcal{F}_1)$, we can pullback sections to obtain two categories and a functor $\Gamma^{\mathrm{cart}}_{\mathcal{C}_1}(\mathcal{F}_1) \to \Gamma^{\mathrm{cart}}_{\mathcal{C}_0}(\mathcal{F}_0)$. More generally, one might ask if given a (covariant) diagram of categories $D : \mathcal{D} \to \mathrm{Cat}$, and a collection of fibered categories over each $D(X)$ which is sufficiently compatible with pullback, if there is a natural way to define a fibered category over $\mathcal{D}$ whose fiber over each X is $\Gamma_{D(X)}(\mathcal{F}_X)$. In this section we give such a construction.

Definition 3.9. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor. We say that an arrow in $\mathcal{C}$ is *cocartesian* if and only if the corresponding arrow in $\mathcal{C}^{\mathrm{op}}$ is cartesian relative to $\mathcal{D}^{\mathrm{op}}$. A morphism $f : \mathcal{C} \to \mathcal{D}$ is a *cocartesian fibration* if $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a cartesian fibration.

f-cocartesian morphisms in $\mathcal{C}$ satisfy a universal property which is dual to the one for cartesian morphisms **??**. Under the straightening theorem Theorem 2.15, we can identify a cocartesian fibration $\mathcal{C} \to \mathcal{D}$ with a functor $\mathcal{D} \to \mathrm{Cat}$.

D:pushforward

Definition 3.10 (Pushforward). Given a fibered category $p : \mathcal{F} \to \mathcal{C}$, and given a cocartesian fibration $f : \mathcal{C} \to \mathcal{D}$, we will define a fibered category $f_*(\mathcal{F})$ over $\mathcal{D}$. The set of objects over $X \in \mathcal{D}$ is the set of cartesian sections $\Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(X), \mathcal{F})$. If Δ^1 denotes the category with two objects $[0], [1]$ and a single non-identity morphism $[0] \to [1]$, we regard any arrow $\gamma : X \to Y$ in $\mathcal{D}$ as a functor $\varphi : \Delta^1 \to \mathcal{D}$ taking $[0] \mapsto X$ and $[1] \mapsto Y$. Then a morphism in $f_*(\mathcal{F})$ lying over γ is a *not necessarily cartesian section* in $\Gamma_{\mathcal{C}}(\varphi^{-1}(\mathcal{C}), \mathcal{F})$ whose restriction to $f^{-1}(X)$ and $f^{-1}(Y)$ are cartesian. Here we are regarding the category $\varphi^{-1}(\mathcal{C})$ of Definition 3.7 as a diagram $\varphi^{-1}(\mathcal{C}) \to \mathcal{C}$ in $\mathcal{C}$.

Let Δ^2 be the category with objects $[0], [1], [2]$, two non-identity arrows $[0] \to [1]$ and $[1] \to [2]$ and their composition $[0] \to [2]$. Specifying a functor $\varphi : \Delta^2 \to \mathcal{D}$ is equivalent to specifying a pair of composable arrows in $\mathcal{D}$. In order to show that $f_*(\mathcal{F})$ is actually a category, we have to show that composable morphisms, as defined in Definition 3.10, have a unique composition. This is a consequence of the following:

L:pushforward_is_category

Lemma 3.11. *Consider a fibered category $\mathcal{F} \to \mathcal{C}$, a cocartesian fibration $f : \mathcal{C} \to \mathcal{D}$, and a functor $\varphi : \Delta^2 \to \mathcal{D}$. If $\mathcal{C}_0 \cup \mathcal{C}_1 \subset \varphi^{-1}(\mathcal{C})$ (respectively $\mathcal{C}_1 \cup \mathcal{C}_2$) denotes the full subcategory with objects lying over $[0], [1] \in \Delta^2$ (respectively $[1], [2]$), then a pair of sections $s_{01} \in \Gamma_{\mathcal{C}}(\mathcal{C}_{01}, \mathcal{F})$ and $s_{12} \in \Gamma_{\mathcal{C}}(\mathcal{C}_{12}, \mathcal{F})$ which agree over $\mathcal{C}_1$ extend uniquely to a section in $\Gamma_{\mathcal{C}}(\varphi^{-1}(\mathcal{C}), \mathcal{F})$.*

Proof. The functor $p : \varphi^{-1}(\mathcal{C}) \to \Delta^2$ is a cocartesian fibration. Choose a cocartesian arrow $X \to Y$ over $(0) \to (1)$ for every $X \in p^{-1}(0)$ and likewise for $(1) \to (2)$, and let $K \subset \mathrm{Mor}(\varphi^{-1}(\mathcal{C}))$ denote these arrows and their compositions, which is a splitting for p. Then any arrow in $\varphi^{-1}(\mathcal{C})$ can be written uniquely as a composition $\beta \circ \alpha$, where $\alpha \in K$ and β lies in a fiber of p. This implies that giving a section $s : \varphi^{-1}(\mathcal{C}) \to \mathcal{F}$ is equivalent to giving an assignment of arrows $s(\alpha)$ for $\alpha \in K$ lying over either $(0) \to (1)$ or $(1) \to (2)$ and $s(\beta)$ for β in a fiber of p satisfying some relations. Any $\alpha \in K$ over $(0) \to (2)$ can be factored uniquely as $\alpha_2 \circ \alpha_1$ with α_1 over $(0) \to (1)$ and α_2 over $(1) \to (2)$, and the resulting sections assigns $s(\alpha) = s(\alpha_2) \circ s(\alpha_1)$. The relations the assignment $s(\alpha)$ and $s(\beta)$ must satisfy are: i) s must be a functor restricted to each $p^{-1}(i)$, and ii) for any $\alpha_1, \alpha_2 \in K$ over $(i) \to (j)$ with $(i,j) = (0,1)$ or $(1,2)$, $\beta_1 \in p^{-1}(i)$, and $\beta_2 \in p^{-1}(j)$ satisfying the equality $\alpha_1 \circ \beta_1 = \beta_2 \circ \alpha_2$, we have $s(\alpha_1) \circ s(\beta_1) = s(\beta_2) \circ s(\alpha_2)$. These relations are equivalent to saying that s restricted to $\mathcal{C}_0 \cup \mathcal{C}_1$ and $\mathcal{C}_1 \cup \mathcal{C}_2$ defines a section. $\square$

<div align="right">

L:pushforward_cartesian
</div>

Lemma 3.12. *Given a cocartesian fibration $f : \mathcal{C} \to \mathcal{D}$ and a fibered category $\mathcal{F}$ over $\mathcal{C}$, the category $f_*(\mathcal{F})$ of Definition 3.10 is a fibered category over $\mathcal{C}$, in which the cartesian morphisms over an arrow $\gamma : \Delta^1 \to \mathcal{D}$ are precisely the cartesian sections of $\mathcal{F}$ over the diagram $\gamma^{-1}(\mathcal{C}) \to \mathcal{C}$.*

Proof. To show that a cartesian section of $\mathcal{F}$ over $\gamma^{-1}(\mathcal{C}) \to \mathcal{C}$ is cartesian as a morphism of $f_*(\mathcal{F})$, we consider an arbitrary functor $\varphi : \Delta^2 \to \mathcal{D}$ mapping $(1) \to (2)$ to γ. We use the same description of $\varphi^{-1}(\mathcal{C})$ as in the proof of **??**, i.e., $\varphi^{-1}(\mathcal{C}) \to \Delta^2$ admits a splitting $K \subset \mathrm{Mor}(\varphi^{-1}(\mathcal{C}))$, and every arrow factors uniquely as an arrow in this splitting followed by an arrow in a fiber. It suffices to show that for any sections s_{12} of $\mathcal{F}$ over $\mathcal{C}_1 \cup \mathcal{C}_2$ and s_{02} over $\mathcal{C}_0 \cup \mathcal{C}_2$ such that s_{12} is cartesian, $s_{01}|_{\mathcal{C}_0}$ is cartesian, and $s_{12}|_{\mathcal{C}_2} = s_{02}|_{\mathcal{C}_2}$, there is a section over $\mathcal{C}$ extending s_{02} and s_{12}. For any $\alpha \in K$ over $(0) \to (1)$, if α' is the unique arrow in K over $(1) \to (2)$ whose source is the target of α, then the universal property for cartesian arrows in $\mathcal{F}$ over $\mathcal{C}$ guarantees that there is a unique arrow $s(\alpha)$ over α whose composition with $s_{12}(\alpha')$ is $s_{02}(\alpha' \circ \alpha)$. The uniqueness of $s(\alpha)$ allows one to check the relations needed to verify that this combined with s_{12} and s_{01} extends uniquely to a section.

So, every cartesian section of $\mathcal{F}$ over $\gamma^{-1}(\mathcal{C})$ is cartesian as a morphism in $f_*(\mathcal{F})$, and it remains to show that there are enough of these. In other words, we need to show that given a functor $\gamma : \Delta^1 \to \mathcal{D}$ and a cartesian section s_1 of $\mathcal{F}$ over $\mathcal{C}_1 \subset \gamma^{-1}(\mathcal{C})$, we can extend this to a cartesian section over $\gamma^{-1}(\mathcal{C})$. We first choose a splitting K for the cocartesian fibration $\gamma^{-1}(\mathcal{C}) \to \mathcal{C}$. Then

<div align="center">

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</div>

we choose a cartesian lift $s(\alpha) : \xi \to s_1(Y)$ in $\mathcal{F}$ of every arrow $\alpha : X \to Y$ in K. Using this we will construct a unique cartesian section $s \in \Gamma_{\mathcal{C}}(\gamma^{-1}(\mathcal{C}), \mathcal{F})$ with this assignment of $s(\alpha)$ for $\alpha \in K$ and $s(\beta) = s_1(\beta)$ for $\beta \in \mathcal{C}_1$. To define $s(\beta)$ for $\beta : X \to Y$ in $\mathcal{C}_0$, consider the unique $\alpha : Y \to Z$ in K over $(0) \to (1)$. Then we have $\alpha \circ \beta = \beta' \circ \alpha'$ for a unique $\alpha' : X \to W$ in K and $\beta' : W \to Z$ in $\mathcal{C}_1$. The fact that $s(\alpha)$ is cartesian implies that there is a unique arrow ϕ in $\mathcal{F}$ over β such that $s(\alpha) \circ \phi = s(\beta') \circ s(\alpha')$, and the fact that $s(\beta') \circ s(\alpha')$ is cartesian implies that ϕ is an isomorphism. We define $s(\beta) = \phi$. We leave it to the reader to check that this assignment $s(\beta)$ is compatible with composition of morphisms in $\mathcal{C}_0$ and morphisms lying over $(0) \to (1)$.

Finally, the fact that any section in $\Gamma_{\mathcal{C}}(\gamma^{-1}(\mathcal{C}), \mathcal{F})$ that is cartesian as a morphism in $f_*(\mathcal{F})$ is actually a cartesian section follows from the fact that any two cartesian arrows in $f_*(\mathcal{F})$ over $\gamma : \Delta^1 \to \mathcal{D}$ with the same target differ by pre-composition with an isomorphism. $\square$

Given a cocartesian fibration $f : \mathcal{C} \to \mathcal{D}$, these constructions define 2-functors $f_* : \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}} \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}$ and $f^{-1} : \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}} \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}$.

Exercise 3.7. *Show that in Definition 3.10, the fiber category of $f_*(\mathcal{F})(U)$ for any $U \in \mathcal{D}$ is isomorphic to the category $\Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(U), \mathcal{F})$ as defined in Definition 3.2. (The class of objects is the same, so the question is to identify their morphisms as well.)*

3.2.3 An "adjunction"

Our motivation for the notation f^{-1} and f_* is the following lemma, which suggests that these are "adjoint" 2-functors between 2-categories. We will not worry about formalizing this notion.

L:stack_adjunction

Lemma 3.13. *If $f : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration, $\mathcal{E}$ is a fibered category over $\mathcal{D}$, and $\mathcal{F}$ is a fibered category over $\mathcal{C}$, then there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(f^{-1}(\mathcal{E}), \mathcal{F}) \cong \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{E}, f_*(\mathcal{F})). \tag{3.2}$$

{E:pushforward_adjunction}

Proof. What we will show is that there are unit and counit functors

$$\eta_{\mathcal{E}} : \mathcal{E} \to f_*(f^{-1}(\mathcal{E})) \text{ and } \varepsilon_{\mathcal{F}} : f^{-1}(f_*(\mathcal{F})) \to \mathcal{F}$$

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which are natural in $\mathcal{E}$ and $\mathcal{F}$ respectively,[1] and such that both compositions

$$f^{-1}(\mathcal{E}) \xrightarrow{f^{-1}(\eta_{\mathcal{E}})} f^{-1}(f_*(f^{-1}(\mathcal{E}))) \xrightarrow{\varepsilon_{f^{-1}(\mathcal{E})}} f^{-1}(\mathcal{E}) \quad \text{\{E:counit\_unit\_identities\}} \tag{3.3}$$
$$f_*(\mathcal{F}) \xrightarrow{\eta_{f_*(\mathcal{F})}} f_*(f^{-1}(f_*(\mathcal{F}))) \xrightarrow{f_*(\varepsilon_{\mathcal{F}})} f_*(\mathcal{F})$$

are (naturally equivalent to) the identity functor of fibered categories. This implies that the functors

$$\phi \in \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(f^{-1}(\mathcal{E}), \mathcal{F}) \mapsto f_*(\phi) \circ \eta_{\mathcal{E}} \in \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{E}, f_*(\mathcal{F})), \text{ and}$$
$$\psi \in \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{E}, f_*(\mathcal{F})) \mapsto \varepsilon_{\mathcal{F}} \circ f^{-1}(\psi) \in \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(f^{-1}(\mathcal{E}), \mathcal{F})$$

are mutually inverse equivalences of categories, just as in the case of 1-categories the counit and unit identities imply these maps are mutually inverse bijections of sets.

The functor $\eta : \mathcal{E} \to f_*(f^{-1}(\mathcal{E}))$ takes an object $\xi \in \mathcal{E}(Y)$ to the constant section

$$s_\xi \in \Gamma^{\mathrm{cart}}(f^{-1}(Y), f^{-1}(Y) \times \mathcal{E}(Y))$$

defined by $s_\xi(X) = \xi$ and $s_\xi(X \to X') = \mathrm{id}_\xi$, where we have identified the restriction of $f^{-1}(\mathcal{E})$ to the subcategory $f^{-1}(Y) \subset \mathcal{C}$ with the constant fibered category $f^{-1}(Y) \times \mathcal{E}(Y)$ with fiber $\mathcal{E}(Y)$. Similarly, a morphism $\xi \to \xi'$ in $\mathcal{E}$ over a morphism $Y \to Y'$ corresponding to a functor $\gamma : \Delta^1 \to \mathcal{D}$ defines a constant section of $\mathcal{F}$ over $\gamma^{-1}(\mathcal{C})$ which assigns id_ξ to every morphism in the fiber over (0), $\mathrm{id}_{\xi'}$ to every morphism in the fiber over (1), and the given morphism $\xi \to \xi'$ to every morphism over $(0) \to (1)$ in Δ^1.

For the functor $\varepsilon : f^{-1}(f_*(\mathcal{F})) \to \mathcal{F}$, we identify the fiber of $f^{-1}(f_*(\mathcal{F}))$ over $X \in \mathcal{C}$ with the fiber of $f_*(\mathcal{F})$ over $f(X)$, i.e.,

$$f^{-1}(f_*(\mathcal{F}))(X) = \Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(f(X)), \mathcal{F})$$

Then on objects the functor ε takes a section s to its value $s(X) \in \mathcal{F}(X)$. To define ε on morphisms, observe that an arrow in $f^{-1}(f_*(\mathcal{F}))$ lying over an arrow $X \to X'$ corresponding to a functor $\gamma : \Delta^1 \to \mathcal{C}$ is by definition an arrow in $f_*(\mathcal{F})$ lying over $f \circ \gamma$, i.e., a section $s \in \Gamma_{\mathcal{C}}((f \circ \gamma)^{-1}(\mathcal{C}), \mathcal{F})$. Evaluating $s(X \to X')$ gives a morphism $s(X) \to s(X')$ in $\mathcal{F}$ over $X \to X'$ in $\mathcal{C}$.

It is straightforward to check that $\eta_{\mathcal{E}}$ and $\varepsilon_{\mathcal{F}}$ as defined above are morphisms of fibered categories, that they are natural in $\mathcal{E}$ and $\mathcal{F}$, and that

---

[1] By this we mean that given two fibered categories $\mathcal{F}, \mathcal{F}'$ over $\mathcal{C}$, the two functors $\mathrm{Map}(\mathcal{F}, \mathcal{F}') \to \mathrm{Map}(f^{-1}(f_*(\mathcal{F})), \mathcal{F}')$ given by $(-) \circ \varepsilon_{\mathcal{F}}$ and $\varepsilon_{\mathcal{F}'} \circ (-)$ are isomorphic, and an analogous statement holds for the unit.

compositions (3.3) are isomorphic to the identity. We leave these verifications to the reader. □

Example 3.14. Under the straightening theorem, Theorem 2.15, one can regard a fibered category $p : \mathcal{F} \to \mathcal{I}^{\mathrm{op}}$ as diagram of categories $D : \mathcal{I} \to \mathrm{Cat}$. Note also that if $f : \mathcal{I} \to \mathrm{pt}$ is the unique functor, then $f_*(\mathcal{F})$ is isomorphic to $\Gamma_{\mathcal{I}}^{\mathrm{cart}}(\mathcal{F})$. Lemma 3.13 then shows that $\Gamma_{\mathcal{I}}^{\mathrm{cart}}(\mathcal{F})$ is the 2-categorical limit of the diagram of categories D, i.e., for any category $\mathcal{C}$, the category of functors $\mathcal{C} \to \Gamma^{\mathrm{cart}}(\mathcal{F})$ is equivalent to the category of 1-morphisms in $\mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$ from the constant diagram with value $\mathcal{C}$ to the diagram D.

Example 3.15. Given cocartesian fibrations $f_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $f_2 : \mathcal{C}_2 \to \mathcal{C}_3$, one has a canonical equivalence $(f_2)_*((f_1)_*(\mathcal{E})) \cong (f_2 \circ f_1)_*(\mathcal{E})$ for any fibered category over $\mathcal{C}_1$. This follows from Lemma 3.13 and the obvious isomorphism of pullbacks $f_1^{-1}(f_2^{-1}(\mathcal{F})) \cong (f_2 \circ f_1)^{-1}(\mathcal{F})$. A special case of this, which we will use in **??**, involves computing sections of a fibered category $\mathcal{F}$ over a product of categories $\mathcal{C} \times \mathcal{D}$. In this case both projection functors $p_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $p_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ are cocartesian fibrations. The compatibility of the pushforward with composition implies that

$$\Gamma_{\mathcal{C} \times \mathcal{D}}^{\mathrm{cart}}(\mathcal{F}) \cong \Gamma_{\mathcal{C}}^{\mathrm{cart}}((p_1)_*(\mathcal{F})) \cong \Gamma_{\mathcal{D}}^{\mathrm{cart}}((p_2)_*(\mathcal{F})),$$

so we can compute teh cartesian sections first along $\mathcal{C}$ or first along $\mathcal{D}$ and get the same answer.

Exercise 3.8. *Consider the category* $\mathcal{C}$ *with three objects and two non-identity arrows*

$$X \, .$$
$$\uparrow$$
$$Y \longleftarrow Z$$

A fibered category over $\mathcal{C}$ *consists of three categories and two functors* $F : \mathcal{F}(X) \to \mathcal{F}(Z)$ *and* $G : \mathcal{F}(Y) \to \mathcal{F}(Z)$. *The limit of this diagram of categories is the 2-categorical fiber product, denoted* $\mathcal{F}(X) \times_{\mathcal{F}(Z)} \mathcal{F}(Y)$. *Show that* $\mathcal{F}(X) \times_{\mathcal{F}(Z)} \mathcal{F}(Y)$ *is equivalent to the category of triples*

$$(\xi \in \mathcal{F}(X), \eta \in \mathcal{F}(Y), \phi : F(\xi) \cong G(\eta)) \, .$$

Give an explicit description of $\mathrm{Fun}(\mathcal{C}', \mathcal{F}(X) \times_{\mathcal{F}(Z)} \mathcal{F}(Y))$ *for any category* $\mathcal{C}'$.

Remark 3.16. Note that this is another reason why given two functors $f : \mathcal{C} \to \mathcal{D}$ and $\mathcal{F} \to \mathcal{D}$, we are using the notation $f^{-1}(\mathcal{F})$ instead of $\mathcal{C} \times_{\mathcal{D}} \mathcal{F}$. It is standard in the algebraic geometry literature for the latter to refer to the 2-categorical fiber product of Exercise 3.8, also called the homotopy fiber product.

3.2.4 Criterion for preserving cartesian sections

Let us say that a category $\mathcal{C}$ has *contractible nerve* if it is non-empty and for any groupoid $\mathcal{G}$, composition with the unique functor $\mathcal{C} \to \mathrm{pt}$ to the trivial category pt, which has one object with its identity arrow, induces an equivalence of categories[2]

$$\mathcal{G} \cong \mathrm{Fun}(\mathrm{pt}, \mathcal{G}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{G}).$$

An equivalent way to say this is that if one considers the groupoid $\mathcal{C}'$ obtained from $\mathcal{C}$ by freely adjoining an inverse arrow $f^{-1} : Y \to X$ for each $f : X \to Y$ and modding out by all relations generated by $f^{-1}f = \mathrm{id}_X$ and $ff^{-1} = \mathrm{id}_Y$, then $\mathcal{C}' \cong \mathrm{pt}$.

Example 3.17. Any filtered category[3], such as a category with a terminal object, has contractible nerve. The same holds for cofiltered categories, i.e., categories $\mathcal{C}$ for which $\mathcal{C}^{op}$ is filtered.

Remark 3.18. The construction of $\mathcal{C}'$ above actually defines a functor $\mathrm{Frac} : \mathrm{Cat} \to \mathrm{Gpd}$ which is left adjoint to the fully-faithful embedding $\mathrm{Gpd} \subset \mathrm{Cat}$, in the sense that composition with the canonical embedding $\mathcal{C} \subset \mathrm{Frac}(\mathcal{C})$ induces an *isomorphism* of categories $\mathrm{Fun}(\mathrm{Frac}(\mathcal{C}), \mathcal{G}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{G})$ for any groupoid $\mathcal{G}$.

The following is our main result, which builds on and generalizes Exercise 3.2.

`P:final_infinity_functor`

Proposition 3.19. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor such that for any $Y \in \mathcal{D}$ the comma category (Y/φ), which by definition consists of pairs $(X \in \mathcal{C}, \alpha : Y \to \varphi(X))$ and morphisms induced by those in $\mathcal{C}$, has contractible nerve. If $\mathcal{F}$ is a fibered category over $\mathcal{D}$, then the restriction functor*

$$\Gamma^{\mathrm{cart}}(\mathcal{D}, \mathcal{F}) \to \Gamma^{\mathrm{cart}}(\mathcal{C}, \varphi^{-1}(\mathcal{F}))$$

is an equivalence of categories.

We will need the following lemma.

---

[2]The explanation for this somewhat strange terminology is that this categorical criterion is equivalent to contractibility of the nerve of $\mathcal{C}$, which is a topological "classifying space" for a category defined in [????].

[3]Recall that a category is said to be filtered if 1) it is non-empty, 2) for any pair of objects X, Y, there is a Z with arrows $X \to Z \leftarrow Y$, and 3) for any two morphisms $f, g : X \to Y$, there is an $h : Y \to Z$ such that $h \circ f = h \circ g$.

Lemma 3.20. *Let $f : \mathcal{C} \to \mathcal{D}$ be a cartesian fibration such that for every $D \in \mathcal{D}$ the fiber $f^{-1}(D)$ has contractible nerve. Then for any fibered category $\mathcal{F}$ over $\mathcal{D}$, the canonical pullback functor $\Gamma^{\mathrm{cart}}_{\mathcal{D}}(\mathcal{F}) \to \Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(\mathcal{F}))$ is an equivalence of categories.*

Proof. $\Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(\mathcal{F}))$ is the category of functors $\mathcal{C} \to \mathcal{F}$ over $\mathcal{D}$ which take every arrow of $\mathcal{C}$ to a cartesian arrow of $\mathcal{F}$. In particular we may replace $\mathcal{F}$ with $\mathcal{F}^{\mathrm{cart}}$, the subcategory consisting of only cartesian arrows, and therefore we may assume that $\mathcal{F}$ is fibered in groupoids over $\mathcal{D}$. What we wish to show is that the unique functor $\mathcal{C} \to \mathcal{D}$, where the latter is regarded as the terminal fibered category over $\mathcal{D}$, induces an equivalence of categories

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{D}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{C}, \mathcal{F}).$$

We can replace $\mathcal{C}$ with an equivalent fibered category which admits a splitting and thus corresponds to a functor $C : \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$, by Theorem 2.15. Then we let $\mathcal{C}'$ denote the fibered category corresponding to the composition $\mathrm{Frac} \circ C : \mathcal{D}^{\mathrm{op}} \to \mathrm{Gpd}$. We have a canonical morphism of fibered categories $\mathcal{C} \to \mathcal{C}'$ which on every fiber can be identified with the canonical embedding $\mathcal{C}(X) \subset \mathrm{Frac}(\mathcal{C}(X))$. Using this one can show that every morphism to a category fibered in groupoids $\mathcal{C} \to \mathcal{F}$ factors uniquely through the functor $\mathcal{C} \to \mathcal{C}'$ and in fact that the embedding induces an equivalence of categories

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{C}', \mathcal{F}) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}}(\mathcal{C}, \mathcal{F}).$$

If one considers the unique morphism $\mathcal{C}' \to \mathcal{D}$, where we regard $\mathcal{D}$ as the terminal fibered category over itself, then the hypothesis implies that $\mathcal{C}' \to \mathcal{D}$ is an equivalence of categories on each fiber, and thus an equivalence of fibered categories [V, Prop. 3.36]. The claim follows. $\qquad\square$

Proof of Proposition 3.19. Note that because by definition $\Gamma^{\mathrm{cart}}(\mathcal{F}) = \Gamma^{\mathrm{cart}}(\mathcal{F}^{\mathrm{cart}})$, we can replace $\mathcal{F}$ with its subcategory of cartesian arrows. We may therefore assume that $\mathcal{F}$ is a category fibered in groupoids over $\mathcal{C}$.

Consider the comma category $(\mathcal{D}/\varphi)$, which consists of $(X \in \mathcal{C}, Y \in \mathcal{D}, \gamma : Y \to \varphi(X))$ and morphisms induced by pairs of morphisms in $\mathcal{C}$ and $\mathcal{D}$ which commute with the arrows γ. Then we have three functors

$$\begin{array}{c} (\mathcal{D}/\varphi) \\ {\scriptstyle pr_{\mathcal{C}}} \swarrow \;\uparrow\; \searrow {\scriptstyle pr_{\mathcal{D}}} \\ \mathcal{C} \quad\;{\scriptstyle \sigma}\;\quad \mathcal{D} \end{array} \;,\; \text{where} \; \left\{ \begin{array}{l} pr_{\mathcal{D}}(Y \to \varphi(X)) = Y, \\ pr_{\mathcal{C}}(Y \to \varphi(X)) = X, \\ \sigma(X) = (\mathrm{id} : \varphi(X) \to \varphi(X)) \end{array} \right. \tag{3.4}$$

{E:comma_construction}

such that σ is fully faithful with left adjoint $pr_{\mathcal{C}}$, $pr_{\mathcal{C}} \circ \sigma = \sigma$, and $pr_{\mathcal{D}} \circ \sigma = \varphi$. The last identity shows that it suffices to prove the claim separately for σ and $pr_{\mathcal{D}}$.

The morphism $pr_{\mathcal{D}}$ is the cartesian fibration corresponding to the strict functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ mapping $Y \mapsto (Y/\varphi)$. The hypothesis of the proposition is that the fibers of $pr_{\mathcal{D}}$ have contractible nerve, so the claim for $pr_{\mathcal{D}}$ follows from Lemma 3.20.

The morphism $pr_{\mathcal{C}}$ is the cocartesian fibration corresponding to the strict functor $\mathcal{C} \to \mathrm{Cat}$ mapping $X \mapsto (\mathcal{D}/\varphi(X))$. To show that

$$\Gamma^{\mathrm{cart}}_{(\mathcal{D}/\varphi)}(\mathcal{F}) \to \Gamma_{\mathcal{C}}(\sigma^{-1}(\mathcal{F}))$$

is an equivalence for any fibered category $\mathcal{F}$ over $(\mathcal{D}/\varphi)$, it suffices to show that the canonical morphism[4] $(pr_{\mathcal{C}})_*(\mathcal{F}) \to \sigma^{-1}(\mathcal{F})$ is an equivalence of fibered categories over $\mathcal{C}$. For this it suffices to check that the induced functor on fibers is an equivalence. On fibers, this functor is simply the restriction of cartesian sections of $\mathcal{F}$ over $pr_{\mathcal{C}}^{-1}(X)$ to the fiber of $\mathcal{F}$ over $\sigma(X)$. Note that $\sigma(X)$ is a terminal object in $pr_{\mathcal{J}}^{-1}(X)$, so this restriction functor is an equivalence by Exercise 3.2. $\qquad\square$

Remark 3.21. The condition in Proposition 3.19 is a little bit stronger than the condition that φ is *cofinal* in the sense of category theory, which implies that limits of sets over $\mathcal{D}^{\mathrm{op}}$ can be computed after restricting to $\mathcal{C}^{\mathrm{op}}$. As discussed above, Proposition 3.19 should be interpreted as a condition which guarantees the same for limits of *categories*. In fact, the condition is just the condition that φ is cofinal in the sense of ∞-category theory, and Proposition 3.19 is inspired by the analogous result for ∞-categories, which is due to Joyal (see **??***Thm. 4.1.3.1).

Exercise 3.9. *Use Proposition 3.19 to show that if $f : \mathcal{C} \to \mathcal{D}$ is a functor which admits a left adjoint, then $\Gamma^{\mathrm{cart}}_{\mathcal{D}}(\mathcal{F}) \to \Gamma^{\mathrm{cart}}_{\mathcal{C}}(f^{-1}(\mathcal{F}))$ is an equivalence. In particular, any equivalence of categories induces an equivalence on cartesian sections.*

3.3 Techniques for studying descent

One common simplification when studying descent is to focus on coverings consisting of a single morphism $\{U' \to U\}$. This is justified by the following:

---

[4]For any morphism f with section σ, we have a canonical morphism $f_*(\mathcal{F}) \to \sigma^{-1}(\mathcal{F})$ which comes from applying f_* to the counit of adjunction $\mathcal{F} \to \sigma_* \sigma^{-1}(\mathcal{F})$ from Lemma 3.13, followed by the observation that $f_* \sigma_* = \mathrm{id}$.

Exercise 3.10. *Let $\mathcal{F}$ be a fibered category over $\mathrm{Sch}_{/S}$ such that for any set of schemes $\{U_i\}_{i \in I}$, the canonical morphism $\mathcal{F}(\bigsqcup_i U_i) \to \prod_i \mathcal{F}(U_i)$ is an equivalence of categories. Show that $\mathcal{F}$ satisfies descent with respect to a covering $\{U_i \to U\}_{i \in I}$ if and only if it satisfies descent with respect to the covering $\{U' := \bigsqcup_{i \in I} U_i \to U\}$. Use this to show that a fibered category $\mathcal{F}$ over $\mathrm{Sch}_{/S}$ is a stack if and only if $\mathcal{F}(-)$ maps disjoint unions of schemes to products of categories, and $\mathcal{F}$ satisfies descent with respect to all coverings $U' \to U$.*

3.3.1 Brief discussion of simplicial objects

For a covering $\{U_0 \to U\}$, we will simplify notation by denoting the augmented descent diagram from [Section 3.1](#) by $D^+_{U_0 \to U}$. It consists of four schemes U, U_0, $U_1 := U_0 \times_U U_0$, and $U_2 = U_0 \times_U U_0 \times_U U_0$:

$$
U_2 \begin{array}{c} \xrightarrow{d_0^2} \\[-2pt] \xrightarrow{d_1^2} \\[-2pt] \xrightarrow{d_2^2} \end{array} U_1 \begin{array}{c} \xrightarrow{d_0^1} \\[-2pt] \xrightarrow{d_1^1} \end{array} U_0 \xrightarrow{d_0^0} U \ ,
$$

where d_i^* denotes the map projecting away from the ith factor of U_0. The indexing category for this diagram is most naturally understood in the context of simplicial methods. Because we will use these methods later, we take a small digression to introduce some notation.

We let $\boldsymbol{\Delta}_+$ denote the augment simplex category. The objects are the totally ordered sets $[n] = \{0 < \cdots < n\}$ for $n \geq -1$, where by convention $[-1]$ denotes the empty set. The morphisms are order preserving maps $f : [m] \to [n]$, which can be conveniently represented by increasing sequences $(i_0 \leq \cdots \leq i_m)$, where $i_j := f(j)$. The category is generated by injective morphisms $\delta^{n,i} : [n-1] \to [n]$ for $i = 0, \ldots, n$, which skip the ith element, and the surjective morphisms $\sigma^{n,i} : [n+1] \to [n]$ for $i = 0, \ldots, n$ which hits i twice. Note that $[-1] \in \boldsymbol{\Delta}$ is initial.

We let $\boldsymbol{\Delta} \subset \boldsymbol{\Delta}_+$ be the full subcategory containing all objects except $[-1]$. A simplicial object in a category $\mathcal{C}$ is a functor $X_\bullet : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and an augmented simplicial object is a functor $\boldsymbol{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$. Concretely, a simplicial object $X_\bullet$ is given by a sequence of objects $X_n \in \mathcal{C}$ for $n \geq 0$ along with *face maps* $d_{n,i} : X_n \to X_{n-1}$ induced by $\delta^{n,i}$ and *degeneracy maps* $s_{n,i} : X_n \to X_{n+1}$ induced by $\sigma^{n,i}$ that satisfy the evident relations.

We will more often use the subcategory $\boldsymbol{\Delta}_{\mathrm{inj},+} \subset \boldsymbol{\Delta}_+$ consisting of all objects and *injective* order preserving maps $[m] \to [n]$. This category is

generated by the $\delta^{n,i}$, and the only relations are $\delta^{n,j}\delta^{n-1,i} = \delta^{n,i}\delta^{n-1,j-1}$ for all $0 \le i < j \le n$. Likewise we let $\mathbf{\Delta}_{\mathrm{inj}} \subset \mathbf{\Delta}_{\mathrm{inj},+}$ denote the full subcategory on all objects except $[-1]$. A morphism $[m] \to [n]$ in $\mathbf{\Delta}_{\mathrm{inj}}$ corresponds to a *strictly* increasing sequence $(i_0 < \cdots < i_m)$. Functors $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ are sometimes called semisimplicial objects in $\mathcal{C}$.

The augmented descent diagram $D_{U_0 \to U}^{+}$ is truncated augmented simplicial object, i.e. a functor

$$D_{U_0 \to U}^{+} : (\mathbf{\Delta}_{\mathrm{inj},+}^{\le 2})^{\mathrm{op}} \to \mathrm{Sch}_{/S},$$

where $\mathbf{\Delta}_{\mathrm{inj},+}^{\le 2} \subset \mathbf{\Delta}_{\mathrm{inj},+}$ denotes the full subcategory on objects $[n]$ with $n \le 2$. $[-1] \in \mathbf{\Delta}_{\mathrm{inj},+}^{\le 2}$ is initial, so Exercise 3.2 implies that $\Gamma^{\mathrm{cart}}(D_{U_0 \to U}^{+}, \mathcal{F}) \to \mathcal{F}(U)$ is an equivalence for any fibered category $\mathcal{F} \to \mathrm{Sch}_{/S}$. The nonaugmented descent diagram $D_{U_0 \to U}$ is the restriction of $D_{U_0 \to U}^{+}$ to the full subcategory $\mathbf{\Delta}_{\mathrm{inj}}^{\le 2} \subset \mathbf{\Delta}_{\mathrm{inj},+}^{\le 2}$ that contains all objects except $[-1]$.

We will also discuss *split* simplicial and semisimplicial objects. We let $\mathbf{\Delta}_{\perp}$ denote the category with the same objects as $\mathbf{\Delta}$ but with different morphisms. If $[m] \cup \{\infty\}$ denotes the totally ordered set obtained by adjoining a new maximal element ∞, then a morphism $[m] \to [n]$ in $\mathbf{\Delta}_{\perp}$ is defined to be an order preserving maps $f : [m] \cup \{\infty\} \to [n] \cup \{\infty\}$ which maps $\infty \mapsto \infty$.

We let $\mathbf{\Delta}_{\mathrm{inj},\perp}$ denote the same category but with only those morphisms $f : [m] \cup \{\infty\} \to [n] \cup \{\infty\}$ such that for any $j < \infty$ in $[n]$, $f^{-1}(j)$ consists of at most one element of $[m]$. A morphism $f : [m] \to [n]$ in $\mathbf{\Delta}_{\mathrm{inj},\perp}$ can be represented by an increasing sequence

$$(i_0 < \cdots < i_k < \overbrace{\infty = \cdots = \infty}^{n-k+1 \text{ times}}).$$

for some $k \le m$ and $i_k \le n$. We identify $\mathbf{\Delta}_{\mathrm{inj},+} \subset \mathbf{\Delta}_{\mathrm{inj},\perp}$ as the subcategory consisting of morphisms for which $f^{-1}(\infty) = \infty$, and we call such a morphism non-degenerate. We define a "degeneracy" morphism $\sigma^n : [n+1] \to [n]$ corresponding to the ordered list $(0 < \cdots < n < \infty = \infty)$. Any morphism $f : [m] \to [n]$ in $\Delta_{\mathrm{inj},\perp}$ can be uniquely factored as $f = f'\sigma^k \cdots \sigma^{m-1}$, where $f' : [k] \to [n]$ is non-degenerate. It follows that $\mathbf{\Delta}_{\mathrm{inj},\perp}$ is generated by morphisms in $\mathbf{\Delta}_{\mathrm{inj},+}$ along with the relations $\sigma^{i-1}\delta^{i,i} = \mathrm{id}_{[i-1]}$ and $\sigma^i \delta^{i+1,j} = \delta^{i,j}\sigma^{i-1}$ for $0 \le j \le i$. A *split augmented semi-simplicial object* in a category $\mathcal{C}$ is a functor $(\Delta_{\mathrm{inj},\perp})^{\mathrm{op}} \to \mathcal{C}$.

3.3.2 First results on descent

Lemma 3.22. *If $f : X \to Y$ is any morphism which admits a section, i.e., a morphism $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$, then any fibered category*

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$\mathcal{F} \to \mathrm{Sch}_{/S}$ *satisfies descent with respect to* $\{f : X \to Y\}$, *regardless of whether* f *is a covering map in our topology.*

Proof. We observe that if $f : X \to Y$ admits a section, then the augmented descent diagram $D^+_{X \to Y}$ extends to a split augmented truncated simplicial diagram

$$D^\perp_{X \to Y} : (\mathbf{\Delta}^{\leq 2}_{\mathrm{inj},\perp})^{\mathrm{op}} \to \mathrm{Sch}_{/S}.$$

The only additional data needed to define $D^\perp_{X \to Y}$ from $D^+_{X \to Y}$ are maps

$$X_{-1} = Y \xrightarrow{s_{-1}} X_0 = X \xrightarrow{s_0} X \times_Y X \xrightarrow{s_1} X \times_Y X \times_Y X$$

corresponding to the arrows σ^{-1}, σ^0, and σ^1 that satisfy the identities $d_{i,i}s_{i-1} = \mathrm{id}_{U_{i-1}}$ for $i = 0, 1, 2$ and $d_{i+1,j}s_i = s_{i-1}d_{i,j}$ for $0 \leq j \leq i \leq 1$. We define $D^\perp_{X \to Y}$ by assigning s_{-1} to be the section s, $s_0(x) = (x, s(f(x)))$, and $s_1(x_1, x_2) = (x_1, x_2, s(f(x_2)))$. We leave it to the reader to verify the necessary relations.

Exercise 3.2 implies that for both $\Gamma^{\mathrm{cart}}(D^\perp_{X \to Y}, \mathcal{F})$ and $\Gamma^{\mathrm{cart}}(D^+_{X \to Y}, \mathcal{F})$, restriction to $[-1]$ induces an equivalence with $\mathcal{F}(Y)$. It follows that the restriction functor

$$\Gamma^{\mathrm{cart}}_{\mathrm{Sch}_{/S}}(D^\perp_{X \to Y}, \mathcal{F}) \xrightarrow{\cong} \Gamma^{\mathrm{cart}}_{\mathrm{Sch}_{/S}}(D^+_{X \to Y}, \mathcal{F})$$

is an equivalence as well. Therefore, to prove the claim, it suffices to show that the inclusion of categories $(\Delta^{\leq 2}_{\mathrm{inj}})^{\mathrm{op}} \subset (\Delta^{\leq 2}_{\mathrm{inj},\perp})^{\mathrm{op}}$ satisfies the criterion of Proposition 3.19. We leave this as an exercise to the reader. $\square$

Exercise 3.11. *Complete the proof of Lemma 3.22 by showing that the inclusion* $\varphi : (\mathbf{\Delta}^{\leq 2}_{\mathrm{inj}})^{\mathrm{op}} \subset (\mathbf{\Delta}^{\leq 2}_{\mathrm{inj},\perp})^{\mathrm{op}}$, *which identifies the former as the subcategory of objects* $[0], [1], [2]$ *and non-degenerate morphisms, satisfies the criterion of Proposition 3.19. (Hint: If a category* $\mathcal{C}$ *has contractible nerve and* $\mathcal{C} \subset \mathcal{C}'$ *is a full subcategory, then to show that* $\mathcal{C}'$ *has contractible nerve, it suffices to show that for any composition of arrows in* $\mathrm{Frac}(\mathcal{C}') \setminus \mathrm{Frac}(\mathcal{C})$ *such that the source of the first arrow and target of the last arrow are objects of* $\mathcal{C}$, *then that composition must also lie in* $\mathrm{Frac}(\mathcal{C})$. *One can use this to show that the category* $([n]/\varphi)$ *has contractible nerve, by first showing that the full subcategory* $\mathcal{C}_0 \subset ([n]/\varphi)$ *consisting of objects for which the morphism* $\varphi([m]) \to [n]$ *in* $\mathbf{\Delta}^{\leq 2}_{\mathrm{inj},\perp}$ *is non-degenerate has contractible nerve, and then extending this to an ascending union of full subcategories of* $([n]/\varphi)$ *all of which have contractible nerve.)*

Remark 3.23. For a more direct, but less conceptual, proof of Lemma 3.22, see [O1, Prop. 4.2.10].

Note that while a Zariski cover $X \to Y$ rarely has a section, smooth covers, such as $\mathbb{A}_Y^n \to Y$, can have many sections.

Lemma 3.24. *If $X \to Y$ and $X' \to Y$ are morphisms in $\mathrm{Sch}_{/S}$, we let X_i for $i = 0, 1, 2$ denote the schemes in the diagram $D_{X \to Y}$, and likewise for X_i'. If $\mathcal{F}$ satisfies descent with respect to $X_i \times_Y X' \to X_i$ and $X \times_Y X_i' \to X_i'$ for $i = 0, 1, 2$, then $\mathcal{F}$ satisfies descent along $X' \to Y$ if and only if it satisfies descent along $X \to Y$.*

Proof. This is a very common argument. If we let X_i' for $i = 0, 1, 2$ denote the schemes in the diagram $D_{X' \to Y}$, and we let $W_{ij} := X_i \times_Y X_j'$, then we have a diagram in $\mathrm{Sch}_{/S}$

$$
\begin{array}{cccc}
W_{22} \rightrightarrows W_{12} \rightrightarrows W_{02} \longrightarrow X_2' \\
W_{21} \rightrightarrows W_{11} \rightrightarrows W_{01} \longrightarrow X_1' \\
W_{20} \rightrightarrows W_{10} \rightrightarrows W_{00} \longrightarrow X_0' \\
X_2 \rightrightarrows X_1 \rightrightarrows X_0 \longrightarrow Y
\end{array}
$$

Notice that every row and every column of this diagram is an augmented descent diagram.

$\mathcal{F}(W_{\bullet,\bullet})$ is a fibered category over $\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2} \times \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}$, and $\mathcal{F}(X_\bullet)$ and $\mathcal{F}(X_\bullet')$ are fibered categories over $\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}$. If $p_0, p_1 : \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2} \times \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2} \to \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}$ are the projections onto the two factors, then the hypothesis that $\mathcal{F}$ satisfies descent along the first three rows and first three columns implies that the canonical morphisms $\mathcal{F}(X_\bullet) \to (p_1)_*(\mathcal{F}(W_{\bullet,\bullet}))$ and $\mathcal{F}(X_\bullet') \to (p_2)_*(\mathcal{F}(W_{\bullet,\bullet}))$ are equivalences. Applying Γ^{cart} to these equivalences, we see that the canonical morphisms

$$
\Gamma^{\mathrm{cart}}_{\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}}(\mathcal{F}(X_\bullet)) \to \Gamma^{\mathrm{cart}}_{\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}}((p_1)_*(\mathcal{F}(W_{\bullet,\bullet}))) = \Gamma^{\mathrm{cart}}_{\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2} \times \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}}(\mathcal{F}(W_{\bullet,\bullet})), \text{ and}
$$
$$
\Gamma^{\mathrm{cart}}_{\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}}(\mathcal{F}(X_\bullet')) \to \Gamma^{\mathrm{cart}}_{\boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2} \times \boldsymbol{\Delta}_{\mathrm{inj}}^{\leq 2}}(\mathcal{F}(W_{\bullet,\bullet}))
$$

are equivalences. The canonical morphism $\mathcal{F}(Y) \to \Gamma^{\mathrm{cart}}(\mathcal{F}(W_{\bullet,\bullet}))$ factors through both $\Gamma^{\mathrm{cart}}(\mathcal{F}(X_\bullet))$ and $\Gamma^{\mathrm{cart}}(\mathcal{F}(X_\bullet'))$. This implies the claim. $\square$

Definition 3.25. Let $\mathcal{F} \to \mathrm{Sch}_{/S}$ be a fibered category. We say that $\mathcal{F}$ satisfies universal descent along $X \to Y$ if for any map of schemes $T \to Y$, $\mathcal{F}$ satisfies descent along $T \times_Y X \to T$.

Corollary 3.26 (Refinement of covers)**.** *If $\mathcal{F}$ is a fibered category over $\mathrm{Sch}_{/S}$ that satisfies universal descent along a morphism arising as a composition $X' \to X \to Y$, then it satisfies universal descent along $X \to Y$.*

Proof. Apply Lemma 3.24 to the morphisms $X \to Y$ and $X' \to Y$. Note that by hypothesis $\mathcal{F}$ satisfies descent along all of the morphisms $X_i \times_Y X' \to X_i$. All of the morphisms $X_i' \times_Y X \to X_i'$ admit a section, so Lemma 3.22 implies that $\mathcal{F}$ satisfies descent along these morphisms. $\qquad\square$

Exercise 3.12. *Use Corollary 3.26 to show that a fibered category over $\mathrm{Sch}_{/S}$ is a stack for the étale topology if and only if it is a stack for the smooth topology. Hence the 2-category of étale stacks is equivalent to the 2-category of smooth stacks. The* topos *associated to a site is defined to be the category of sheaves associated to that site, so this shows that the étale site and the smooth site define the same topos.*

Exercise 3.13. *Use the results of this section to show that a fibered category over $\mathrm{Sch}_{/S}$ is a stack for the étale topology if and only if it is a stack for the Zariski topology and $\mathcal{F}$ satisfies descent along étale standard smooth morphisms $\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)) \to \mathrm{Spec}(R)$. This reflects the idea that the étale topology is "generated" by Zariski covers and maps of this form. The technique of finding a small class of morphisms which generate a topology in this sense is a common method for dealing with other topologies. (See* [MV]*, Sect. 3.1] for an example of this.)*

Exercise 3.14. *Let $\mathcal{F} \to \mathrm{Sch}_{/S}$ be a stack with respect to the étale topology. Show that $\mathcal{F}$ is determined by its values on affine schemes in the following sense: For any separated scheme X, let $\mathrm{Sch}_{/X}^{\mathrm{aff,et}}$ denote the category of affine schemes along with an étale morphism $\mathrm{Spec}(A) \to X$. Regard the functor which forgets the morphism to X as a diagram $D_X : \mathrm{Sch}_{/X}^{\mathrm{aff,et}} \to \mathrm{Sch}_{/S}$. Then there is a canonical equivalence $\Gamma_{\mathrm{Sch}_{/S}}^{\mathrm{cart}}(D_X, \mathcal{F}) \cong \mathcal{F}(X)$.*

Exercise 3.15 (Locality)**.** *Use Proposition 3.19 to show that for any fibered category over $(\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \times (\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}}$, the category of cartesion sections can be computed after restricting along the diagonal embedding $(\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \times (\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}}$. Then, use an argument similar to the proof of Lemma 3.24 to show the following* locality *principal: If $X, X' \to Y$ are two morphisms and $\mathcal{F} \to \mathrm{Sch}_{/S}$ is a fibered category that satisfies universal descent along $X \to Y$ and $X' \times_Y X \to X$, then it satisfies universal descent along $X' \to Y$. (Hint: first show that $\mathcal{F}$ satisfies universal descent along $X \times_Y X' \to Y$.)*

Exercise 3.16 (Composition)**.** *Use Exercise 3.15 to show that if a fibered category $\mathcal{F} \to \mathrm{Sch}_{/S}$ satisfies universal descent along morphisms $X' \to X$ and $X \to Y$, then it satisfies universal descent along $X' \to Y$.*

3.3.3 The relationship with simplicial descent

Our formulation of descent is inspired by the ∞-categorical analog [L2, Sect. 6.1.3]. The main difference is that in the setting of ∞-categories, the descent diagram $D_{\mathcal{U}}$ is replaced with a simplicial diagram in schemes, i.e. a functor $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Sch}_{/S}$.

Example 3.27. For a covering $\mathcal{U} = \{f : X \to Y\}$, one can associate a simplicial scheme $\mathrm{Cech}(f)$, called the Cech nerve, whose n^{th} level is the fiber product of $n+1$ copies of X, $X_n := X \times_Y \cdots \times_Y X$. $\mathrm{Cech}(f)$ canonically extends to an augmented simplicial scheme $\mathrm{Cech}(f)_+ : \mathbf{\Delta}_+^{\mathrm{op}} \to \mathrm{Sch}_{/S}$ that assigns $X_{-1} = Y$.

Given a fibered category $\mathcal{F} \to \mathrm{Sch}_{/S}$ and a simplicial scheme $X_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Sch}_{/S}$, there is a commonly used notation

$$\mathrm{Tot}\{\mathcal{F}(X_\bullet)\} = \Gamma_{\mathbf{\Delta}^{\mathrm{op}}}^{\mathrm{cart}}((X_\bullet)^{-1}(\mathcal{F})) = \Gamma_{\mathcal{C}}^{cart}(X_\bullet, \mathcal{F}).$$

We say that a fibered category $\mathcal{F} \to \mathrm{Sch}_{/S}$ satisfies descent along $f : X \to Y$ if the restriction functor

$$\Gamma^{\mathrm{cart}}(\mathrm{Cech}(f)_+, \mathcal{F}) \to \Gamma^{\mathrm{cart}}(\mathrm{Cech}(f), \mathcal{F})$$

is an equivalence of categories. Because the empty set $[-1] \in \mathbf{\Delta}_+^{\mathrm{op}}$ is terminal, restriction to $[-1]$ defines an equivalence $\Gamma^{\mathrm{cart}}(\mathrm{Cech}(f)_+, \mathcal{F}) \cong \mathcal{F}(Y)$.

This simplicial definition of descent is equivalent to Definition 3.4 because of the following:

Lemma 3.28. *Let $\mathcal{F}$ be a fibered category over $\mathrm{Sch}_{/S}$, and let $X \to Y$ be a covering morphism. Then the descent diagram $D_{X \to Y} : (\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \to \mathrm{Sch}_{/S}$ is the restriction of $\mathrm{Cech}(f)$ along the canonical embedding $(\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \subset \mathbf{\Delta}^{\mathrm{op}}$, and restriction defines an equivalence*

$$\mathrm{Tot}\{\mathcal{F}(X_\bullet)\} \xrightarrow{\cong} \Gamma_{\mathrm{Sch}_{/S}}^{\mathrm{cart}}(D_{X \to Y}, \mathcal{F}).$$

Exercise 3.17. *Prove Lemma 3.28. By Theorem 2.15 you may assume that $\mathcal{F}$ admits a splitting. See [S4, Tag 023H] and [S4, Tag 0D7I] for examples of the argument for specific $\mathcal{F}$. (The embedding $(\mathbf{\Delta}_{\mathrm{inj}}^{\leq 2})^{\mathrm{op}} \subset \mathbf{\Delta}^{\mathrm{op}}$ does not satisfy the criterion of Proposition 3.19, so that is not an effective strategy.)*

The results analogous to Lemma 3.22, Lemma 3.24, and Corollary 3.26 hold in the simplicial context. The proofs are mostly the same, but with $\mathbf{\Delta}^{\mathrm{op}}$ and $\mathbf{\Delta}^{\mathrm{op}}_+$ replacing the descent diagrams $\mathbf{\Delta}^{\leq 2,\mathrm{op}}_{\mathrm{inj}}$ and $\mathbf{\Delta}^{\leq 2,\mathrm{op}}_{\mathrm{inj},+}$ respectively. In fact, the proof of Lemma 3.22 is even easier in the simplicial context, by the following:

Exercise 3.18. *Show that the inclusion $\mathbf{\Delta} \subset \mathbf{\Delta}_\perp$ taking $f : [m] \to [n]$ to the extension $\tilde{f} : [m] \cup \{\infty\} \to [n] \cup \{\infty\}$ mapping $\infty \mapsto \infty$ admits a left adjoint. Use this to conclude that if a fibered category $\mathcal{F}$ over Δ^{op} extends to a fibered category over $\Delta^{\mathrm{op}}_\perp$, then there is a canonical equivalence $\Gamma^{\mathrm{cart}}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{F}) \cong \mathcal{F}([-1])$. This can be summarized by the slogan "any coaugmented cosimplicial diagram that extends to a split coaugmented cosimplicial diagram is a limit diagram," where in this case the coaugmented cosimplicial diagram is the functor $\mathbf{\Delta}_+ \to \mathrm{Cat}$ classified by the extension of $\mathcal{F}$ to $\mathbf{\Delta}^{\mathrm{op}}_+$.*

Lecture 4

Examples of stacks

References: [V], [F2]
Date: 2/4/2020
Exercises: 3

4.1   Examples of stacks

Let S be a scheme, and let $p : \mathrm{QCoh}_{/S} \to \mathrm{Sch}_{/S}$ be the fibered category of **??**, and let $\mathrm{QCoh}_{/S} \subset \mathrm{QCoh}_{/S}$ denote the subcategory consisting of all p-cartesian arrows.

<div align="right">

`T:fpqc_descent`
</div>

Theorem 4.1. $p : \mathrm{QCoh}_{/S} \to \mathrm{Sch}_{/S}$ *is a stack for the smooth topology.*

Proof. Because the descent condition only involves cartesian arrows, it suffices to show that $(\mathrm{QCoh}_{/S})^{\mathrm{cart}} \cong (\mathrm{QCoh}^{\mathrm{op}}_{/S})^{\mathrm{cart}}$ is a stack, so we will work with the latter. We know that $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ satisfies Zariski descent, so by Exercise 3.13 it suffices to show that $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ satisfies descent along a standard smooth surjective morphism of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$, i.e., it suffices to show descent of modules along a faithfully flat ring map $R \to A$.[1] The key idea is that for any morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$, if $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is faithfully flat and $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ has descent along $\mathrm{Spec}(A \otimes_R R') \to \mathrm{Spec}(R')$, then it has descent along $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ as well. We show this in Lemma 4.2 below. If $R \to A$ is faithfully flat, we can apply the lemma to

---

[1] We are not using smoothness in this proof. What we actually show is that $\mathrm{QCoh}_{/S}$ satisfies Zariski descent and descent along faithfully flat maps of affine schemes. This shows that $\mathrm{QCoh}_{/S}$ is a stack for the "fpqc" topology, a much stronger condition [**????**].

the case where $R' = A$, so it suffices to show $\mathrm{QCoh}^{\mathrm{op}}_{/S}$ satisfies descent along $\mathrm{Spec}(A \otimes_R A) \to \mathrm{Spec}(A)$. This map admits a section, induced by the ring homomorphism $a \otimes b \mapsto ab$, so any fibered category satisfies descent along this morphism by Lemma 3.22. (See also [S4, Tag 023N]). $\qquad \square$

Lemma 4.2. *Let $R \to A$ be a ring map, and let $R \to R'$ be a faithfully flat ring map. If the fibered category of modules satisfies descent along the map $R' \to A' := R' \otimes_R A$, then it satisfies descent along the map $R \to A$ as well.*

Proof. A descent datum, after translating into algebraic terms, consists of an $N \in A$-Mod along with an isomorphism of $A \otimes_R A$-modules

$$\phi : N \otimes_R A \to A \otimes_R N$$

satisfying the cocycle condition that the following diagram commutes

,

where the subscript on ϕ indicates which tensor factors it is acting on. A homomorphism of descent data $(N, \phi) \to (N', \phi')$ is a homomorphism $f : N \to N'$ in A-Mod which intertwines the cocycles ϕ and ϕ' after tensoring with A.

We denote the category of descent data $\mathrm{Desc}(A/R)$. The pullback functor

$$F : R\text{-Mod} \to \mathrm{Desc}(A/R)$$

maps M to $A \otimes_R M$ with its *canonical* cocycle $\phi_{\mathrm{canon}} : (A \otimes_R M) \otimes_R A \cong (A \otimes_R A) \otimes_R M \cong A \otimes_R (A \otimes_R M)$. One can show that F admits a right adjoint $G : \mathrm{Desc}(A/R) \to R$-Mod which maps

$$G : (N, \phi) \mapsto \ker \left(N \xrightarrow{n \mapsto 1 \otimes n - \phi(n \otimes 1)} A \otimes_R N \right).$$

It thus follows that F is an equivalence if and only if the unit $M \to G(F(M))$ is an isomorphism for all $M \in R$-Mod, and the counit $F(G(N, \phi)) \to (N, \phi)$ is an isomorphism for all $(N, \phi) \in \mathrm{Desc}(A/R)$.

We have canonical base change functors $R' \otimes_R (-) : R\text{-Mod} \to R'\text{-Mod}$ and $R' \otimes_R (-) : \mathrm{Desc}(A/R) \to \mathrm{Desc}(A'/R')$, where the latter maps (N, ϕ) to $N \otimes_R R' \in A'$-Mod with its induced cocycle

$$N \otimes_{R'} A' \cong (N \otimes_R A) \otimes_R R' \xrightarrow{\phi \otimes 1} (A \otimes_R N) \otimes_R R' \cong A' \otimes_{R'} (R' \otimes_R N).$$

One can show that if $R \to R'$ is flat, then $R' \otimes_R (-)$ commutes with F and G in a manner which preserves the adjunction, and thus preserve the unit and counit of adjunction.

Recall that $R \to R'$ being faithfully flat means that $R' \otimes_R (-) : R\text{-Mod} \to R'\text{-Mod}$ is an exact functor of abelian categories, and a module $M \in R\text{-Mod}$ is zero if and only if $R' \otimes_R M$ is 0. It follows by considering the kernel and cokernel that a homomorphism $f : M \to M'$ in $R\text{-Mod}$ is an isomorphism if and only $R' \otimes_R f$ is. In particular the canonical equivalence

$$R' \otimes_R (M \to G(F(M))) \cong R' \otimes_R M \to G(F(R' \otimes_R M))$$

shows that the unit of adjunction being an isomorphism for A'/R' implies the same for A/R. Similary, $\mathrm{Desc}(A/R)$ is an abelian category, where kernels and cokernels are just the kernels and cokernels of the underlying A-modules along with canonical cocycles which they inherit. Because $A \to A'$ is faithfully flat, a morphism in $\mathrm{Desc}(A/R)$ is an isomorphism if and only if its pullback to $\mathrm{Desc}(A'/R')$ is faithfully flat. So the canonical equivalence

$$R' \otimes_R (F(G(N, \phi)) \to (N, \phi)) \cong F(G(R' \otimes_R (N, \phi))) \to R' \otimes_R (N, \phi).$$

implies that the counit of adjunction being an isomorphism for A'/R' implies the same for A/R. Therefore if modules descend along $R' \to A'$, the same is true for $R \to A$. $\qquad\square$

Remark 4.3. This argument has a more abstract formulation known as the Beck's monadicity theorem [M1, Chap. 6]. The adjunction between the pullback functor $F : R\text{-Mod} \to A\text{-Mod}$ and its right adjoint G give a natural transformation $\eta : F(G(-)) \Rightarrow \mathrm{id}$ which has a structure known as a "comonad." Because F reflects isomorphisms (faithfulness) and preserves finite limits (flatness), the Barr-Beck theorem implies that F induces an equivalence between $R\text{-Mod}$ and "coalgebras over the comonad" $F(G(-)) \Rightarrow \mathrm{id}$. To derive descent from this theorem, one must identify the structure of a coalgebra for the comonad on $N \in A\text{-Mod}$ with descent data on N.

Theorem 4.1 allows one to construct many other examples. The following, in the special case where X is a smooth projective curve, will be our main example for the second half of the course.

`EX:descent_bun`

Exercise 4.1. *Let $X \to S$ be a scheme over a base scheme S, and let $\mathrm{Bun}(X)$ denote the category of pairs (T, E), where T is an S-scheme and E is a locally free sheaf on $T \times_S X$, and a morphism $(T, E) \to (T', E')$ is a morphism of schemes $f : T \to T'$ along with a homomorphism of quasi-coherent sheaves*

$E' \to (f \times \mathrm{id}_X)_*(E)$ on $T' \times_S X$ which induces an isomorphism $f^*(E') \cong E$. *Show that this is a category fibered in groupoids over* $\mathrm{Sch}_{/S}$ *without appealing to* [V, Prop. 3.22]. *Show that it is a stack for the étale topology.*

Exercise 4.2. *Let* $\mathrm{Alg}(\mathrm{QCoh}_{/S})$ *denote the fibered category whose fiber over a scheme* T *is the category of quasi-coherent* $\mathcal{O}_T$-*algebras. Show that* $\mathrm{Alg}(\mathrm{QCoh}_{/S})$ *is a stack for the étale topology on* $\mathrm{Sch}_{/S}$. *Show that the same holds for the fibered category of graded quasi-coherent algebras* $\mathrm{Alg}^{\mathrm{gr}}(\mathrm{QCoh}_{/S})$. *(Hint: one way is to consider the composition* $\mathrm{Alg}(\mathrm{QCoh}_{/S}) \to \mathrm{QCoh}_{/S} \to \mathrm{Sch}_{/S}$ *and use Exercise 3.16.)*

Example 4.4. Let $\mathcal{F} \to \mathrm{Sch}_{/S}$ be the fibered category which assigns $T \to S$ to the category of flat, proper morphisms $X \to T$ along with a relatively ample invertible sheaf L on X. A morphism $(X, L) \to (X', L')$ is an isomorphism $f : X \to X'$ over T, along with an isomorphism $f^*(L') \cong L$. Using the previous exercises one can show that $\mathcal{F}$ is a stack in groupoids over $\mathrm{Sch}_{/S}$. Zariski descent is straightforward, so it suffices to show descent along a smooth morphism of affine schemes (see **??**). In this case you can use Exercise 4.2 to show that given a descent datum for $\mathcal{F}$, the homogeneous coordinate rings define a descent datum for a graded quasi-coherent algebra, which is effective by Exercise 4.2. Taking Proj shows that the original descent datum was effective.

The following is the algebraic version of our first example of a moduli problem Example 1.1:

Example 4.5. Let $\mathcal{M}_g \to \mathrm{Sch}_{/S}$ be the fibered category which assigns to any scheme $T \to S$ the category of smooth, proper morphisms $X \to T$ with connected geometric fibers which are smooth curves of genus g. $\mathcal{M}_g$ is a stack in groupoids. In fact $\mathcal{M}_g$ has a canonical "compactification" as a stack, $\overline{\mathcal{M}}_g$, which is an important object in many subjects [DM1].

Example 4.6. If $\mathcal{F} \to \mathrm{Sch}_{/S}$ denotes the fibered category which assigns $T \to S$ to the groupoid of flat, proper morphisms $X \to T$, the $\mathcal{F}$ is *not* a stack [V, Sect. 4.4.2]. The problem is there are descent data along a morphism $T' \to T$ which define a proper algebraic space over T, rather than a scheme. We will discuss algebraic spaces below.

4.2 Representable morphisms

Recall from Example 2.12 and **??** that the functor $X \mapsto \mathrm{Sch}_{/X}$ defines a fully faithful embedding $\mathrm{Sch}_{/S} \hookrightarrow \mathrm{Cat}^{\mathrm{cart}}_{/\mathrm{Sch}_{/S}}$, where $\mathrm{Sch}_{/X}$ is fibered over $\mathrm{Sch}_{/S}$

via the functor that maps $T \to X$ to the composition $T \to X \to S$. We will often abuse notation by writing X for the fibered category $\mathrm{Sch}_{/X}$.

Definition 4.7. A category fibered in groupoids over $\mathrm{Sch}_{/S}$ is *representable by a scheme* if it is isomorphic to $\mathrm{Sch}_{/X} \to \mathrm{Sch}_{/S}$ for some $X \in \mathrm{Sch}_{/S}$. A morphism of categories fibered in groupoids $\mathcal{X} \to \mathcal{Y}$ is *representable* if for any morphism of fibered categories $\mathrm{Sch}_{/X} \to \mathcal{Y}$, the 2-categorical fiber product $\mathrm{Sch}_{/X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by a scheme (see Exercise 3.8).

D:representable_properties

Definition 4.8. Let **P** be a property of a morphism of schemes which is stable under base change and local for the étale topology over the base (examples: affine, quasi-affine, separated, proper, finitely presented, open immersion, closed immersion, smooth, étale, surjective). If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks in groupoids which is representable by schemes, then we say f has property **P** if for any morphism from a scheme $T \to \mathcal{Y}$, $T \times_{\mathcal{Y}} \mathcal{X} \to T$ satisfies property **P**.

Exercise 4.3. *Consider a pair of morphisms*

$$\mathcal{X} \xrightarrow{\ f\ } \mathcal{Y} \xrightarrow{\ g\ } \mathcal{Z} \ .$$

Show that 1) if f and g are representable, then so is $g \circ f$, and 2) if g and $g \circ f$ are representable, then so is f.

Lecture 5

Algebraic spaces and stacks

References: [O1], [F2]
Date: 2/6/2020
Exercises: 3

Definition 5.1. A stack $\mathcal{X}$ over $\mathrm{Sch}_{/S}$ is an algebraic space if

1. the fibers of $\mathcal{X}$ are setoids

2. the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by schemes

3. there exists a morphism from a scheme $U \to \mathcal{X}$ which is surjective étale.

In order for the third condition to make sense, we need the fact that the morphism $U \to \mathcal{X}$ is representable. This is the result of the following:

Exercise 5.1. *Show that if $\mathcal{X}$ is a stack such that the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by schemes (resp. algebraic spaces), then any morphism from a scheme $Y \to \mathcal{X}$ is representable by schemes (resp. algebraic spaces).*

For any property **P** of a morphism of schemes $X \to T$ that is étale local on X, we can say that an algebraic space $\mathcal{X} \to T$ has property **P** if for some, and hence for all, surjective étale morphisms from a scheme $U \to \mathcal{X}$, the composition $U \to \mathcal{X} \to T$ has property **P**.

We will discuss algebraic spaces in more detail later. For now, we will just use this definition as a building block for the definition of an algebraic stack. Just like before, we can define a morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibered in

groupoids over $\mathrm{Sch}_{/S}$ is *representable by algebraic spaces* if for any $T \in \mathrm{Sch}_{/S}$ and any morphism $T \to \mathcal{Y}$, the fiber product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

Definition 5.2. A stack in groupoids $\mathcal{X}$ over $\mathrm{Sch}_{/S}$ is an algebraic (resp. Deligne-Mumford) stack if

1. the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces

2. there exists a smooth (resp. étale), surjective morphism from a scheme $U \to \mathcal{X}$ (sometimes called an atlas).

Exercise 5.2. *Show that any representable morphism $\mathcal{X} \to \mathcal{Y}$ of stacks for the étale topology on $\mathrm{Sch}_{/S}$ which is smooth and surjective in the sense of [Definition 4.8](#) is a surjective morphism of stacks in the following sense: for any $T \in \mathrm{Sch}_{/S}$ and $\xi \in \mathcal{Y}(T)$, there is an étale cover $\phi : T' \to T$ such that $\phi^*(\xi) \in \mathcal{Y}(T')$ lifts to $\mathcal{X}(T')$.*

One important property of algebraic stacks is that if $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are morphisms of algebraic stacks, then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is an algebraic stack [S4, Tag 04TD]. As a consequence, for any morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f_0} & Y \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
$$

in which the top objects are schemes, $Y \to \mathcal{Y}$ is representable, smooth, and surjective, and $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ is representable, smooth, and surjective. For any property P of a morphism of schemes which is smooth local on the source and the target, we can say that f has property P if f_0 does. This is independent of the choice of lift f_0 satisfying these conditions. See [S4, Tag 06FL] for further discussion.

Likewise, for any property P of schemes which is local in the smooth topology, we say that an algebraic stack $\mathcal{X}$ has property P if there is an atlas $X \to \mathcal{X}$ such that X has that property P. See [S4, Tag 04YH] for a list of such properties.

5.0.1 Properties of the diagonal

Many results for algebraic stacks require additional axioms on $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$, beyond the fact that it is representable by algebraic spaces. These

conditions are called *separation axioms*. There is also a relative version, where one considers the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ for a morphism $f : \mathcal{X} \to \mathcal{Y}$, which recovers the absolute version when applied to the structure morphism $\mathcal{X} \to S$. In many examples of interest in moduli theory, Δ_f is representable by schemes, and in fact by *affine schemes*. The hypothesis that Δ_f is affine simplifies many proofs.

The minimal separation axiom that is required for most of the results we will discuss is that $f : \mathcal{X} \to \mathcal{Y}$ is quasi-separated.

Definition 5.3. We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for any $\mathrm{Spec}(A) \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(A)$ admits a representable (by algebraic spaces) smooth surjection from an affine scheme $\mathrm{Spec}(B) \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(A)$.[1]

<div align="right">D:quasi-separated</div>

Definition 5.4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-separated* if the following conditions hold:

1. The double diagonal

$$\Delta_{\Delta_f} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X},$$

   which is automatically a representable, separated, locally finite type, locally quasi-finite, monomorphism by [S4, Tag 04YQ], is also **quasi-compact**.

2. The diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

An algebraic stack is quasi-separated if the structure morphism $\mathcal{X} \to S$ is so. We will often abbreviate the condition that f is quasi-compact and quasi-separated as qc.qs..

An example of a non-quasi-separated algebraic stack is provided by the algebraic space $\mathbb{A}^1_{\mathbb{Q}}/\mathbb{Z}$, where we regard $\mathbb{Z}$ as a discrete group scheme acting freely by translation. One commonly studied class of stacks which have well-behaved categories of quasi-coherent sheaves is the following:

Definition 5.5. A stack $\mathcal{X}$ is *noetherian* if it is quasi-compact and quasi-separated (over $\mathbb{Z}$) and admits a smooth surjection from a noetherian affine scheme $\mathrm{Spec}(A) \to \mathcal{X}$.

---

[1]This is equivalent to the definition in the stacks project, which is formulated using the topological space $|\mathcal{X}|$ of points of an algebraic stack $\mathcal{X}$.

5.1 Example: $B\operatorname{GL}_n$

We first show that the stack $B\operatorname{GL}_n$ parameterizing locally free sheaves of rank n defined in Equation (1.1) is an algebraic stack. To give a more precise definition, we let $B\operatorname{GL}_n \subset \operatorname{QCoh}_{/S}^{\mathrm{op}}$ denote the subcategory (see Example 2.10) of pairs (X, E) where $E \in \operatorname{QCoh}(X)$ is locally free of rank n, and which only includes cartesian morphisms. One can deduce that $B\operatorname{GL}_n$ is a stack from the fact that $\operatorname{QCoh}_{/S}^{\mathrm{op}}$ is a stack (Theorem 4.1) and the fact that a quasi-coherent sheaf on a scheme if locally free of rank n if and only if it is so after restricting along a flat surjective morphism $X' \to X$.

We first consider the diagonal $B\operatorname{GL}_n \to B\operatorname{GL}_n \times B\operatorname{GL}_n$. For any $T \in \operatorname{Sch}_{/S}$, the groupoid of maps $T \to B\operatorname{GL}_n \times B\operatorname{GL}_n$ is the groupoid of pairs (E, E') of locally free sheaves of rank n, where a morphism is a pair of isomorphisms. Let $\mathfrak{X} = T \times_{B\operatorname{GL}_n \times B\operatorname{GL}_n} B\operatorname{GL}_n$ be the fiber product, and we regard it as a fibered category over $\operatorname{Sch}_{/T}$. By definition, for any $f : T' \to T$, we have

$$
\begin{aligned}
\mathfrak{X}(T') &= \{f\} \times_{B\operatorname{GL}_n(T') \times B\operatorname{GL}_n(T')} B\operatorname{GL}_n(T') \\
&= \{F \in B\operatorname{GL}_n(T') + \text{ isomorphisms } (\varphi_0, \varphi_1) : (f^*(E), f^*(E')) \cong (F, F)\}
\end{aligned}
$$

The last expression is a groupoid, where a morphism of this data is an isomorphism $\phi : F \to F'$ of locally free sheaves on T' such that $(\phi, \phi) \circ (\varphi_0, \varphi_1) = (\varphi_0', \varphi_1')$. The map which takes (φ_0, φ_1) to the composition $\varphi_1^{-1}\varphi_0$ defines an equivalence of groupoids

$$
\mathfrak{X}(T') = \{\text{isomorphisms } \varphi : f^*(E) \cong f^*(E')\} = \operatorname{Isom}_{T'}(f^*(E), f^*(E')).
$$

The latter is a setoid, i.e. $\mathfrak{X}$ corresponds to a presheaf of sets on $\operatorname{Sch}_{/T}$. If E and E' were trivializable, then after choosing trivializations one could regard the data of an isomorphism $f^*(E) \cong f^*(E')$ as a morphism $T' \to \operatorname{GL}_n$. Therefore, in this case, $\mathfrak{X}$ would be representable by the T-scheme $\operatorname{GL}_n \times T$. In general one can find a Zariski open cover U_α of T such that E and E' are trivializable over each U_α. The previous discussion allows one to realize $\mathfrak{X}$ as copies of the T-schemes $U_\alpha \times \operatorname{GL}_n$ glued along open immersions, hence $\mathfrak{X}$ is representable by a scheme, which is in fact affine over T!

Now consider the morphism $\operatorname{pt} = \operatorname{Sch}_{/S} \to B\operatorname{GL}_n$ classifying $\mathcal{O}_S^n$. For any $T \to B\operatorname{GL}_n$, corresponding to a locally free sheaf E of rank n on T, one can verify as above that $\mathfrak{X} = T \times_{B\operatorname{GL}_n} \operatorname{pt}$, regarded as a fibered category over $\operatorname{Sch}_{/T}$ is the sheaf of sets

$$
(f : T' \to T) \mapsto \operatorname{Isom}_{T'}(\mathcal{O}_{T'}^n, f^*(E)).
$$

We have already seen that this sheaf is representable by a scheme which isomorphic to $T \times \mathrm{GL}_n$ locally over T. Hence the morphism to T is smooth and surjective, i.e., $\mathrm{pt} \to B\,\mathrm{GL}_n$ is a smooth surjective morphism. This verifies that $B\,\mathrm{GL}_n$ is an algebraic stack.

5.2 Principal G-bundles

5.2.1 Group schemes and group actions

Definition 5.6. A group scheme over S is a scheme G along with a lift of the corresponding functor $\mathfrak{h}_G$ along the forgetful functor

$$
\begin{array}{ccc}
& & \mathrm{Group} \; . \\
& \nearrow & \downarrow \\
\mathrm{Sch}^{\mathrm{op}}_{/S} & \xrightarrow{\;\mathfrak{h}_G\;} & \mathrm{Set}
\end{array}
$$

Equivalently, G has an identity section $e : \mathrm{pt} \to G$ and a multiplication map $\mu : G \times G \to G$ satisfying the associativity, identity, and invertibility axioms defining a group.

A left action of G on $X \in \mathrm{Sch}_{/S}$ is an action of the sheaf of sets $\mathfrak{h}_G$ on the sheaf $\mathfrak{h}_X$, i.e., and action of $G(T)$ on $X(T)$ for all $T \in \mathrm{Sch}_{/S}$ which is functorial in T. This is equivalent to an action morphism $\sigma : G \times X \to X$ such that the following diagrams commute

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mu \times \mathrm{id}_X} & G \times X \\
{\scriptstyle \mathrm{id}_X \times \sigma} \downarrow & & \downarrow {\scriptstyle \sigma} \\
G \times X & \xrightarrow{\;\;\sigma\;\;} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{e \times \mathrm{id}_X} & G \times X \; . \\
& {\scriptstyle \mathrm{id}_X} \searrow & \downarrow {\scriptstyle \sigma} \\
& & X
\end{array}
$$

The definition of a right action is analogous.

5.2.2 Principal bundles

Definition 5.7. Let G be a smooth affine group scheme over S, let $T \in \mathrm{Sch}_{/S}$, and let P be an S-scheme equipped with a right G action $\sigma : P \times G \to P$ and a projection $\pi : P \to T$ which is smooth and G-invariant, meaning the two compositions agree

$$
P \times G \underset{pr_1}{\overset{\sigma}{\rightrightarrows}} P \xrightarrow{\;\pi\;} T \; .
$$

Then P is a principal G-bundle if the morphism $P \times G \to P \times_T P$ that is given on T' points by $(p, g) \mapsto (p, p \cdot g)$ is an isomorphism and π is surjective. P is also called a G-torsor. Morphisms of principal G-bundles are G-equivariant maps relative to T. We call the resulting category $BG(T)$.

This is saying that the G-action on P is free and transitive relative to T. The action of G on P along with the invariant map π is equivalent to the data of an action of the base-changed group G_T on P relative to T. Using this the torsor condition is that $P \times_T G_T \to P \times_T P$, i.e. the G_T action on P is free and transitive in the category $\mathrm{Sch}_{/T}$.

Example 5.8. The simplest example is the trivial principal G-bundle $G_T = T \times G \to T$, with its right G action. The set $\mathrm{Map}_{BG(T)}(G_T, P)$ is naturally in bijection with the set of sections of the map $P \to T$. Given a section $s : T \to P$, the corresponding equivariant map is $T \times G \to P$ mapping $(t, g) \mapsto s(t) \cdot g$, and given an equivariant morphism $T \times G \to P$, the section is the restriction to $T \times \{1\}$. It follows that the sheaf of sets $\mathrm{Isom}_T(G_T, P)$ over Sch_T which maps $f : T' \to T$ to $\mathrm{Isom}_{T'}(G_{T'}, f^{-1}(P))$.

Exercise 5.3. *Show that any morphism between principal G-bundles over T is an isomorphism.*

Note that because $\pi : P \to T$ is smooth and surjective, it admits a section étale locally. Hence locally P is isomorphic to $T \times G \to T$. Because G is affine, this implies that $\pi : P \to T$ is also affine, because the fact that the canonical map $P \to \mathrm{Spec}_T(\pi_*(\mathcal{O}_P))$ is an isomorphism can be checked étale locally over T.

Example 5.9. If E is a locally free sheaf on T of rank n, then the scheme $\mathrm{Isom}_T(\mathcal{O}_T^n, E)$ is a principal GL_n-bundle. This induces an equivalence of categories between principal GL_n-bundles and locally free sheaves. To go from a prinicpal GL_n bundle P back to a locally free sheaf, one chooses an étale cover $T' \to T$ such that $P|_{T'}$ is trivializable. Then we consider the descent diagram $s, t : T' \times_T T' \rightrightarrows T'$, and the descent datum defining P from $P|_{T'}$ is an isomorphism $\phi : s^*(P|_{T'}) \to t^*(P|_{T'})$ satisfying a cocycle condition on $T' \times_T T' \times_T T'$. If we fix a trivialization $P|_{T'} \cong (\mathrm{GL}_n)_{T'}$, then this allows us to identify ϕ with an automorphism of the trivial GL_n-bundle on T'. The key observation is that the automorphism group scheme of a trivial GL_n-bundle is canonically isomorphic to the automorphism group scheme of the trivial locally free sheaf, so the cocycle defining P gives us a cocycle defining a locally free sheaf $\mathcal{E}$.

The assignment $T \mapsto BG(T)$ defines a stack on $\mathrm{Sch}_{/S}$. We can prove that BG is an algebraic stack in much the same way as for GL_n. Consider

the map $\mathrm{pt} = \mathrm{Sch}_{/S} \to BG$ classifying the trivial G-bundle on S. Then for any $T \to BG$, classifying a G-bundle P over T, we have a cartesian diagram

$$
\begin{array}{ccc}
P = \mathrm{Isom}_T(G_T, P) & \longrightarrow & \mathrm{pt} \\
\downarrow & & \downarrow \\
T & \longrightarrow & BG
\end{array}
\quad .
$$

In particular $\mathrm{pt} \to BG$ is representable by schemes and smooth. Furthermore because $P \to T$ is smooth and surjective, it admits a section étale locally over T. Hence étale locally over T, $\mathrm{Isom}_T(G_T, P)$

Likewise, for a morphism $T \to BG \times BG$ classifying a pair of G-bundles (P_1, P_2), we have a cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Isom}_T(P_1, P_2) & \longrightarrow & BG \\
\downarrow & & \downarrow \\
T & \longrightarrow & BG \times BG
\end{array}
\quad .
$$

Because P_1 and P_2 are trivializable étale locally over T, this map is representable by affine schemes. Hence we can associate to any smooth affine group scheme G an algebraic stack BG, which has an affine diagonal and admits a smooth surjection $\mathrm{pt} = \mathrm{Sch}_{/S} \to BG$.

On the other hand, let $\mathcal{X}$ be an algebraic stack over $\mathrm{Sch}_{/S}$ which admits an atlas from the terminal object $X_0 = \mathrm{pt} = \mathrm{Sch}_{/S}$. Let $G := X_1 = \mathrm{pt} \times_{\mathcal{X}} \mathrm{pt}$. Because $X_0(T) = \{*\}$ for any $T \in \mathrm{Sch}_{/S}$, $G(T)$ is a group instead of a groupoid. If $\mathcal{X}$ has affine diagonal, then G is a smooth affine group scheme.

If one applies this construction to $\mathrm{pt} \to BG$, one gets the group G back. It turns out that the same is true in the other direction: any algebraic stack with affine diagonal which admits a surjection $\mathrm{pt} \to \mathcal{X}$ is isomorphic to BG for the smooth affine group scheme $\mathrm{pt} \times_{\mathcal{X}} \mathrm{pt}$.

5.3 Groupoid scheme associated to an algebraic stack

Consider an algebraic stack $\mathcal{X}$ over $\mathrm{Sch}_{/S}$, and let $f : X_0 \to \mathcal{X}$ be an atlas. Assume for simplicity that the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by *schemes*. Then the fiber product $X_1 := X_0 \times_{\mathcal{X}} X_0$ is representable by a scheme, and by definition for any $T \in \mathrm{Sch}_{/S}$, its set of T-points is

$$
X_1(T) = \{\text{triples } (x, y \in X_0(T), \varphi : f(x) \cong f(y) \text{ in } \mathcal{X}(T)\}.
$$

We observe that $X_1(T)$ is naturally the set of arrows in a groupoid whose objects are $X_0(T)$. There are natural maps $s, t : X_1 \to X_0$ which on T-points maps any arrow to its source and target. There is also an "identity" section $e : X_0 \to X_1$ which on T-points maps $x \mapsto (x, x, \mathrm{id}_x)$. In order to encode the structure of a groupoid on $X_1(T)$ diagramatically, we consider the scheme of composable arrows $X_1 \times_{s,X_0,t} X_1 = X_0 \times_\mathfrak{X} X_0 \times_\mathfrak{X} X_0$, which is a scheme whose T-points are

$$\{(x, y, z \in X_0(T) + \text{ two arrows } f(x) \xrightarrow{\varphi_0} f(y) \xrightarrow{\varphi_1} f(z) \text{ in } \mathfrak{X}(T))\}.$$

Then the groupoid structure on $X_1(T)$ is encoded by the "composition" morphism $c : X_1 \times_{s,X_0,t} X_1 \to X_1$ which acts on T-points as $(\varphi_0, \varphi_1) \mapsto \varphi_1 \circ \varphi_0$. Putting this all together, we have a diagram of schemes

$$X_1 \times_{s,X_0,t} X_1 \xrightarrow{\ c\ } X_1 \underset{\underset{inv}{\overset{s}{\rightrightarrows}}}{\overset{\overset{e}{\underset{t}{\leftarrow}}}{}} X_0 . \qquad \text{\{E:groupoid\}} \quad (5.1)$$

These arrows satisfy some axioms:

- correct sources and targets: $s \circ c = s \circ pr_1$, $t \circ c = t \circ pr_2$, $s \circ e = \mathrm{id}_{X_0}$, $t \circ e = \mathrm{id}_{X_0}$.

- identity: $\mathrm{id}_{X_1} = c \circ (e \times \mathrm{id}_{X_1}) : X_1 = X_0 \times_{X_0,s} X_1 \to X_1$

- identity: $\mathrm{id}_{X_1} = c \circ (\mathrm{id}_{X_1} \times e) : X_1 = X_1 \times_{t,X_0} X_0 \to X_1$

- associative : $c \circ (c \times \mathrm{id}_{X_1}) = c \circ (\mathrm{id}_{X_1} \times c) : X_1 \times_{t,X_0,s} X_1 \times_{t,X_0,s} X_1 \to X_1$

- right inverses: the map $X_1 \times_{s,X_0,t} X_1 \to X_1 \times_{t,X_0,t} X_1$ mapping $(\varphi_0, \varphi_1) \to (\varphi_1, \varphi_1 \circ \varphi_0)$ is an isomorphism

- left inverses: the map $X_1 \times_{s,X_0,t} X_1 \to X_1 \times_{s,X_0,s} X_1$ mapping $(\varphi_0, \varphi_1) \mapsto (\varphi_0, \varphi_1 \circ \varphi_0)$ is an isomorphism

These are exactly the axioms which equip $X_1(T) \rightrightarrows X_0(T)$ with the structure of a groupoid for any $T \in \mathrm{Sch}_{/S}$. Note that the two axioms on inverses is equivalent to the existence of an "inverse" map $i : X_1 \to X_1$ taking every arrow to its inverse, which we have illustrated in (5.1) with a dotted arrow, but this map is not strictly part of the data.

Definition 5.10. A *groupoid scheme* is a diagram (5.1) satisfying the axioms above. Equivalently, a groupoid scheme is a strict functor $\mathrm{Sch}_{/S} \to \mathrm{Gpd}$ such that the set of objects functor and set of morphisms functor are both representable by schemes.

Our shorthand notation for a groupoid scheme is $X_1 \rightrightarrows X_0$, or even more compactly $X_\bullet$. The fact that $X_0 \to \mathcal{X}$ is smooth implies that both the source and target maps $s, t : X_1 \to X_0$ are smooth, so we refer to $X_1 \rightrightarrows X_0$ as a smooth groupoid. If $\mathcal{X}$ were a Deligne-Mumford stack, one can choose $X_0 \to \mathcal{X}$ étale, in which case s, t are étale, and we say $X_1 \rightrightarrows X_0$ is an étale groupoid.

Remark 5.11. If the diagonal of $\mathcal{X}$ is not representable by schemes, then X_1 is not a scheme, but an algebraic space. In this case $X_1 \rightrightarrows X_0$ is a groupoid in algebraic spaces. The theory is the same as above.

Our ultimate goal is to construct a correspondence between algebraic stacks and groupoid schemes up to a certain notion of equivalence.

Lecture 6

Groupoid algebraic spaces

References: [BX, Sect. 2] [O1]
Date: 2/18/2020
Exercises: 6

Last time we defined the groupoid scheme associated to an algebraic stack $\mathfrak{X}$ with a smooth surjective morphism $X_0 \to \mathfrak{X}$, where $X_1 = X_0 \times_{\mathfrak{X}} X_0$. Today we continue this discussion. When the diagonal of $\mathfrak{X}$ is not representable by schemes, X_1 will be an algebraic space instead of a scheme, so we will mostly work with the more general notion of groupoid algebraic spaces.

6.1   Baby case: algebraic spaces

An *equivalence relation* in $\mathrm{Sch}_{/S}$ is a morphism of S-schemes $R \to U \times U$ such that for any $T \in \mathrm{Sch}_{/S}$, $R(T) \to U(T) \times U(T)$ is injective and defines an equivalence relation on $U(T)$, where $x \sim y$ if and only if $(x, y) \in R(T)$. We say that $R \to U \times U$ is an étale equivalence relation if each projection $s, t : R \to U$ is étale.

Lemma 6.1. [O1, Prop. 5.2.5] *If $R \to U \times U$ is an étale equivalence relation on $\mathrm{Sch}_{/S}$, then 1) the sheafification of the presheaf $T \mapsto U(T)/R(T)$, which we call U/R, is an algebraic space, 2) the canonical morphism $U \to U/R$ is étale and surjective, and 3) the canonical map $R \to U \times_{U/R} U$ is an isomorphism.*

So we see that an étale equivalence relation determines a space. In the other direction, given an algebraic space $\mathfrak{X}$ and a surjective étale morphism $U \to \mathfrak{X}$, the scheme $R = U \times_{\mathfrak{X}} U$ defines an étale equivalence relation, but

given any surjective étale map $U' \to U$, one let R' be the preimage of R under $U' \times U' \to U \times U$, and $U'/R' \to U/R$ is an equivalence. For any two surjective étale morphisms $U, U' \to \mathcal{X}$, the fiber product $U \times_{\mathcal{X}} U' \to \mathcal{X}$ is a surjective étale morphism lying above U and U'. This shows that there is a bijection between algebraic spaces and étale equivalence relations up to pullback along a further étale covering map.

In fact, something stronger is true. If $R \to U \times U$ is an equivalence relation in schemes such that the projections $R \to U$ are flat and finitely presented, then U/R is an algebraic space [S4, Tag 04S6]. This is a nontrivial fact, because it requires one to construct a surjective étale morphism $U' \to U/R$ from thin air.

E:free_quotients

Example 6.2. We say that a group scheme G acts freely on a scheme X if $G(T)$ acts freely on $X(T)$ for any $T \in \mathrm{Sch}_{/S}$. This is equivalent to the induced map $G \times X \to X \times X$ being a monomorphism, in which case it is an equivalence relation. The previous discussion implies that if G is a flat and finitely presented group scheme over S, and G acts freely on $X \in \mathrm{Sch}_{/S}$, then X/G, the sheafification of $T \mapsto X(T)/G(T)$, is an algebraic space.

The fact that an fppf equivalence relation defines an algebraic space implies that algebraic spaces are a more natural class of objects from the perspective of descent:

Corollary 6.3 (descent for algebraic spaces). *The category fibered in groupoids over* $\mathrm{Sch}_{/S}$ *whose fiber over* T *is the category of algebraic spaces* $X \to T$ *and isomorphisms relative to* T *satisfies smooth (and even fppf) descent.*

Exercise 6.1. *Prove this corollary.*

6.2 Morita equivalence of groupoid schemes

First we generalize the notion of equivalence between étale equivalence relations above.

Note that groupoid algebraic spaces form a 2-category, in which a morphism $X_\bullet \to Y_\bullet$ is a pair of morphisms $X_0 \to Y_0$ and $X_1 \to Y_1$ which commute with the structure maps for the groupoids. A 2-morphism η between functors $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ is a lift

{E:natural_isomorphism} (6.1)

$$
\begin{array}{ccc}
 & & Y_1 \\
 & \nearrow^{\eta} & \downarrow {\scriptstyle (s,t)} \\
X_0 & \xrightarrow{(f_0,g_0)} & Y_0 \times Y_0
\end{array}
$$

which induces a natural transformation of functors of T points $f_\bullet(T) \Rightarrow g_\bullet(T)$.

Definition 6.4. A morita morphism is a functor of groupoid spaces $f_\bullet$: $X_\bullet \to Y_\bullet$ such that $X_0 \to Y_0$ is smooth and surjective, and $X_1 \to X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,f_0} X_0$ is an isomorphism (i.e. $f_\bullet(T)$ is fully faithful for all T).

We say that two groupoid algebraic spaces $X_\bullet$ and $Y_\bullet$ are *Morita equivalent* if there is a third groupoid space $Z_\bullet$ with Morita morphisms $Z_\bullet \to Y_\bullet$ and $Z_\bullet \to X_\bullet$.

Lemma 6.5. *Let $\mathfrak{X}$ be an algebraic stack. Then the groupoid algebraic space associated to any two atlases of $\mathfrak{X}$ are Morita equivalent.*

Proof. Let $X_0 \to \mathfrak{X}$ and $X_0' \to \mathfrak{X}$ be two smooth surjective morphisms. Then $X_0'' := X_0 \times_{\mathfrak{X}} X_0' \to \mathfrak{X}$ is again smooth and surjective. It therefore suffices to check that if $U_0 \to U_0' \to \mathfrak{X}$ is a composition of smooth surjective morphisms, then one naturally has an induced Morita morphism between the associated groupoids $U_\bullet \to U_\bullet'$, and we leave this to the reader. $\square$

6.3 The stackification of a smooth groupoid space

For any presheaf F on a site $\mathcal{C}$, F admits a sheafification, which is a sheaf F^a and a morphism $F \to F^a$ which is universal for maps from F to a presheaf, i.e., for any sheaf G on $\mathcal{C}$, composition with $F \to F^a$ gives a bijection

$$\mathrm{Map}(F^a, G) \to \mathrm{Map}(F, G).$$

There is an analogous construction for stacks. For any fibered category $\mathcal{F}$ over a site $\mathcal{C}$, there is a morphism of fibered categories to a stack $\alpha : \mathcal{F} \to \mathcal{F}^a$ such that for any stack $\mathcal{G}$, composition with α gives an equivalence of categories

$$\mathrm{Map}(\mathcal{F}^a, \mathcal{G}) \to \mathrm{Map}(\mathcal{F}, \mathcal{G}).$$

We refer to [O1, Thm. 4.6.5] for the case of categories fibered in groupoids, and [S4, Tag 02ZM] for the general case. $\mathcal{F}^a$ is characterized by two properties: 1) for any $\xi, \eta \in \mathcal{F}(U)$, the map

$$\mathrm{Map}_{\mathcal{F}(U)}(\xi, \eta) \to \mathrm{Map}_{\mathcal{F}^a(U)}(\alpha(\xi), \alpha(\eta))$$

identifies the right side with the sheafification of the left side, and 2) for any $\xi \in \mathcal{F}^a(U)$, there is a cover $\{U_i \to U\}$ such that the restriction of ξ to each U_i lies in the essential image of $\mathcal{F}(U_i) \to \mathcal{F}^a(U_i)$.

Although the general construction of $\mathcal{F}^a$ is not too hard, we have opted to take a more direct route to associating a stack to a smooth groupoid space. We will need the following:

Lemma 6.6. *Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibered in groupoids over $\mathcal{C}$ such that for any $U \in \mathcal{C}$*

1. *$\varphi_U : \mathcal{X}(U) \to \mathcal{Y}(U)$ is fully faithful, and*

2. *φ_U is locally surjective in the sense that $\forall \xi \in \mathcal{Y}(U)$, there is a cover $\{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i}$ lies in the essential image of $\varphi_{U_i} : \mathcal{X}(U_i) \to \mathcal{Y}(U_i)$ for all $i \in I$.*

Then for any stack in groupoids $\mathcal{Z}$ over $\mathcal{C}$, composition with φ gives an equivalence of categories

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(\mathcal{Y}, \mathcal{Z}) \xrightarrow{\cong} \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(\mathcal{X}, \mathcal{Z}).$$

Proof. Note that if $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{C}$ is the fiber functor, then

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{C}}}(\mathcal{X}, \mathcal{Z}) = \Gamma(\mathcal{X}, p_{\mathcal{X}}^{-1}(\mathcal{Z})),$$

Note also that $p_{\mathcal{X}}^{-1}(\mathcal{Z})$ is a stack for the inherited topology on $\mathcal{X}$ [S4, Tag 06NT]. It therefore suffices to regard $\mathcal{X}$ and $\mathcal{Y}$ as sites, and to show that for any continuous morphism of sites $\varphi : \mathcal{X} \to \mathcal{Y}$, i.e., a functor which takes coverings to coverings and preserves fiber products, and any stack in groupoids $\mathcal{F}$ on $\mathcal{Y}$, the canonical restriction functor

$$\Gamma(\mathcal{Y}, \mathcal{F}) \to \Gamma(\mathcal{X}, \varphi^{-1}(\mathcal{F})). \tag{6.2}$$

is an equivalence under the following hypotheses: 1) φ is fully faithful; 2) every object admits a cover by something in the essential image of φ; and 3) if there is a morphism $\eta \to \varphi(\xi)$ for any $\xi \in \mathcal{X}$ and $\eta \in \mathcal{Y}$, then η lies in the essential image of φ.

The key idea here is that showing that (6.2) is an equivalence can be reduced by a formal argument, which we explain below, to showing that objects in $\mathcal{F}(Y)$ are uniquely determined by the restriction of $\mathcal{F}$ to $\mathcal{X} \subset \mathcal{Y}$. More precisely, it suffices to show that for any $Y \in \mathcal{Y}$, if we let $(\mathcal{X}/Y)$ denote the comma category whose objects are morphisms $(U \to Y)$ with $U \in \mathcal{X}$, then the canonical restriction functor

$$\mathcal{F}(Y) \cong \Gamma_{\mathcal{Y}}((\mathcal{Y}/Y), \mathcal{F}) \to \Gamma_{\mathcal{Y}}((\mathcal{X}/Y), \mathcal{F}) \tag{6.3}$$

is an equivalence of categories. As presheaves of sets over $\mathcal{Y}$, $(\mathcal{Y}/Y)$ corresponds to the representable functor $\mathfrak{h}_Y$, and $(\mathcal{X}/Y)$ corresponds to the

subfunctor $\mathcal{S} \subset \mathfrak{h}_Y$ of maps which factor through an object of $\mathcal{X}$. The hypotheses on $\mathcal{X}$ imply that $\mathcal{S}$ is a covering sieve for the topology on $\mathcal{Y}$, so (6.3) is an equivalence by Exercise 3.5.

Showing that (6.2) *is an equivalence if* (6.3) *is an equivalence for all* $Y \in \mathcal{Y}$:

Fully faithfulness and Proposition 3.19 implies that we may replace $\mathcal{X}$ with its image in $\mathcal{Y}$, i.e., we may assume φ is the inclusion of a full subcategory. Consider the comma category $(\mathcal{X}/\mathcal{Y})$, whose objects are triples $(X \in \mathcal{X}, Y \in \mathcal{Y}, X \to Y)$, and let $pr_1 : (\mathcal{X}/\mathcal{Y}) \to \mathcal{X}$ denote the projection map taking $(X \to Y) \mapsto X$. We have a commutative diagram

$$
\begin{array}{ccc}
(\mathcal{X}/\mathcal{Y}) \xrightarrow{(\varphi/\mathrm{id})} (\mathcal{Y}/\mathcal{Y}) \xrightarrow{pr_2} \mathcal{Y} \, . \\
\Big\downarrow{pr_1} \qquad\quad \Big\downarrow{pr_1} \\
\mathcal{X} \xrightarrow{\quad\varphi\quad} \mathcal{Y}
\end{array}
$$

Consider the projection $pr_1 : (\mathcal{X}/\mathcal{Y}) \to \mathcal{X}$. If we fix $X \in \mathcal{X}$, then the comma category (X/pr_1) has an initial object $((\mathrm{id} : X \to X) \in (\mathcal{X}/\mathcal{Y}), \mathrm{id} : X \to X)$, which implies that (X/pr_1) has contractible nerve. It follows from Proposition 3.19 that the canonical map $\Gamma_{\mathcal{X}}(\varphi^{-1}(\mathcal{F})) \to \Gamma_{(\mathcal{X}/\mathcal{Y})}(pr_1^{-1}(\varphi^{-1}(\mathcal{F})))$ is an equivalence. Applying this to $(\mathcal{Y}/\mathcal{Y})$ as well, it suffices to prove the claim of the lemma for the stack in groupoids $pr_1^{-1}(\mathcal{F})$ over $(\mathcal{Y}/\mathcal{Y})$ and the full subcategory $(\mathcal{X}/\mathcal{Y}) \subset (\mathcal{Y}/\mathcal{Y})$.

Let $\mathcal{G} = pr_1^{-1}(\mathcal{F})$ over $(\mathcal{Y}/\mathcal{Y})$, so concretely $\mathcal{G}(X \to Y) = \mathcal{F}(X)$, and let $\mathcal{G}'$ denote the restriction of $\mathcal{G}$ to $(\mathcal{X}/\mathcal{Y})$. We need to show that the restriction map $\Gamma_{(\mathcal{Y}/\mathcal{Y})}(\mathcal{G}) \to \Gamma_{(\mathcal{X}/\mathcal{Y})}(\mathcal{G}')$ is an equivalence of categories. For this it suffices to show that the canonical map $pr_2(\mathcal{G}) \to pr_2(\mathcal{G}')$ is an equivalence of categories, and it suffices to verify this on the fiber over each $Y \in \mathcal{Y}$. Note that the fiber $pr_2^{-1}(Y) \subset (\mathcal{Y}/\mathcal{Y})$ is the category of morphisms $(Y' \to Y)$. Under projection onto the first object, we may regard this as a category fibered in setoids over $\mathcal{Y}$, in which case it is just the representable fibered category $\mathfrak{h}_Y$. On the other hand $pr_2^{-1}(Y) \cap (\mathcal{X}/\mathcal{Y})$, regarded as a fibered category over $\mathcal{Y}$, is the subfunctor $\mathcal{S} \subset \mathfrak{h}_Y$ consisting of morphisms $(U \to Y)$ with $U \subset \mathcal{X}$. So the claim boils down to showing that (6.3) is an equivalence of categories, as mentioned above. $\qquad\square$

Definition 6.7. Give a smooth groupoid algebraic space $X_\bullet = (X_1 \rightrightarrows X_0)$ over S an $X_\bullet$-*space over* T is a functor of groupoid algebraic spaces $\pi : P_\bullet \to (X_\bullet)_T$, where $(X_\bullet)_T := X_\bullet \times T$ denotes the groupoid space $(X_1)_T \rightrightarrows (X_0)_T$,

such that the diagram

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\ s\ } & P_0 \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\
(X_1)_T & \xrightarrow{\ s\ } & (X_0)_T
\end{array}
$$

is cartesian. A *morphism* of $X_\bullet$-spaces is a morphism of S-schemes $T \to T'$ and a commutative diagram of groupoids

$$
\begin{array}{ccc}
P_\bullet & \longrightarrow & P'_\bullet \\
\downarrow & & \downarrow \\
(X_\bullet)_T & \longrightarrow & (X_\bullet)_{T'}
\end{array} \quad .
$$

This defines a fibered category $X_\bullet$-Spc over $\mathrm{Sch}_{/S}$ whose fiber is the category of $X_\bullet$-spaces over T.

Another way to think of this is that $P_0 \to X_0 \times T$ is really two maps, a "structure" map $\pi : P_0 \to T$ and an "anchor" map $a : P_0 \to X_0$. For any test scheme $U \in \mathrm{Sch}_{/S}$, one can think of $p \in P_0(U)$ as an arrow whose source is $\pi(p) \in T(U)$ and whose target is $a(p) \in X_0(U)$. The fact that (6.4) is cartesian allows one to identify $P_1 = P_0 \times_{a,X_0,s} X_1$, and we can regard the target map $t : P_1 \to P_0$ as an "action" map which composes the "arrow" from $\pi(p)$ to $a(p)$ with an arrow $\gamma \in X_1(U)$ with $s(\gamma) = a(p)$. This composition law obeys the natural axioms for the action of the category $X_\bullet(U)$ on the set $P_0(U)$.

Exercise 6.2. *Check that the morphisms between two $X_\bullet$-spaces form a set, rather than a groupoid, and in fact it is naturally identified with the subset of morphisms $P_0 \to P'_0$ over $T \to T'$ that commute with the action of the category $X_\bullet$ in an appropriate sense. One should think of this as saying that a morphism $X_\bullet$-spaces is simply an equivariant morphism $P_0 \to P'_0$.*

Exercise 6.3. *Show that when $X_\bullet = (G \rightrightarrows \mathrm{pt})$ is a group scheme over S, then the category of $X_\bullet$-spaces over T is equivalent to the usual category whose objects are algebraic spaces with a G-action relative to T and whose morphisms are G-equivariant morphisms.*

Definition 6.8. If $X_\bullet$ is a smooth groupoid scheme, we say that a $P_\bullet \in X_\bullet$-Spc(T) is an $X_\bullet$-torsor if the structure map $P_0 \to T$ is smooth and surjective, and the canonical map $(s,t) : P_1 \to P_0 \times_T P_0$ is an isomorphism.

Definition 6.9. If $X_\bullet$ is a smooth groupoid algebraic space, we let $BX_\bullet$ denote the category of $X_\bullet$-torsors on S-schemes, regarded as a fibered category over $\mathrm{Sch}_{/S}$.

Exercise 6.4. *Show that for any smooth groupoid algebraic space $X_\bullet$, $BX_\bullet$ is a stack for the étale topology on $\mathrm{Sch}_{/S}$.*

Example 6.10. There is a trivial $X_\bullet$ torsor over X_0, which we denote Triv whose structure map is $s : X_1 \to X_0$ and anchor map is $t : X_1 \to X_0$. Given a morphism $f : T \to X_0$, one give an explicit description of the pullback $P_\bullet = f^{-1}(\mathrm{Triv})$: $P_0 = T \times_{f,X_0,s} X_1$, where the structure map is the projection $\pi : P_0 \to T$ onto the left factor, and the anchor map $a : P_0 \to X_0$ induced by projection onto the right factor, followed by the target map $X_1 \to X_0$. The two maps

$$P_0 \times_{a,X_0,s} X_1 = T \times_{f,X_0,s} X_1 \times_{t,X_0,s} X_1 \rightrightarrows P_0$$

consist of projection onto the first factor and the composition law in $X_\bullet$ respectively. We refer to $f^{-1}(\mathrm{Triv})$ as the trivial torsor associated to the map $f : T \to X_0$.

Another way to think of this: the trivial torsor on X_0 defines a morphism $X_0 \to BX_\bullet$. Then for any $f : T \to X_0$, the torsor $f^{-1}(\mathrm{Triv})$ corresponds to the composition $T \to X_0 \to BX_\bullet$.

It is not hard to show that an arrow $f \to g$ in the category $X_1(T) \rightrightarrows X_0(T)$ defines a morphism $f^{-1}(\mathrm{Triv}) \to g^{-1}(\mathrm{Triv})$ in the fiber $BX_\bullet(T)$. Thus we get a canonical map of fibered categories

$$\mathrm{Triv} : X_\bullet \to BX_\bullet, \qquad \text{\{E:trivial\_torsor\}} \tag{6.5}$$

where $X_\bullet$ denotes the fibered category whose fiber over $T \in \mathrm{Sch}_{/S}$ is $X_1(T) \rightrightarrows X_0(T)$.

Exercise 6.5. *Show that the compositions*

$$X_1 \xrightarrow{s} X_0 \to BX_\bullet \text{ and } X_1 \xrightarrow{t} X_0 \to BX_\bullet$$

are canonically isomorphic, and the induced map $X_1 \to X_0 \times_{BX_\bullet} X_0$ is an isomorphism. In other words, a pair of maps $f, g : T \to X_0$, and isomorphism between the composition of these maps with the map $X_0 \to BX_\bullet$ is precisely the same data as a map $\gamma : T \to X_1$ with $s(\gamma) = f$ and $t(\gamma) = g$.

We may summarize the previous discussion with the following:

Proposition 6.11. *For any smooth groupoid space $X_\bullet$, $BX_\bullet$ is an algebraic stack, and the canonical map* (6.5) *exhibits $BX_\bullet$ as the stackification of $X_\bullet$, i.e., for any stack $\mathcal{F}$, composition with* Triv *induces an equivalence of categories*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathrm{Sch}_{/S}}}(BX_\bullet, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathrm{Sch}_{/S}}}(X_\bullet, \mathcal{F}). \qquad \text{\{E:stackification\_torsors\}} \quad (6.6)$$

If $X_\bullet$ is the groupoid space induced by an algebraic stack $\mathcal{X}$ and a smooth surjective morphism $X_0 \to \mathcal{X}$, then the canonical morphism $BX_\bullet \to \mathcal{X}$ induced by the tautological morphism $X_\bullet \to \mathcal{X}$ is an equivalence.

Proof. Exercise 6.4 shows that $BX_\bullet$ is a stack, and Exercise 6.5 shows that $X_\bullet \to BX_\bullet$ is a fully faithful morphism of fibered categories. The claim then follows from Lemma 6.6. $\qquad\square$

Remark 6.12. As in the case of algebraic spaces, there is a strengthening of **??** which says that $BX_\bullet$ is an algebraic stack if the structure maps $s : X_1 \to X_0$ and $t : X_1 \to X_0$ are fppf, rather than smooth.

Exercise 6.6. *Using the correspondence between stacks and smooth groupoid spaces, show that an algebraic stack $\mathcal{X}$ whose diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by schemes is an algebraic space if and only if the automorphism group of any $\xi \in \mathcal{X}$ is trivial, which is equivlent to $\mathcal{X} \to \mathrm{Sch}_{/S}$ being fibered in setoids. In fact, the same conclusion holds without the condition on the diagonal of $\mathcal{X}$ [S4, Tag 04SZ]. Use this to show that a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is representable if and only if for any $\xi \in \mathcal{X}$, the induced group homomorphism $\mathrm{Aut}(\xi) \to \mathrm{Aut}(f(\xi))$ is injective.*

6.3.1 Morphisms as descent data

Let $(X_1 \rightrightarrows X_0)$ be a presentation for an algebraic stack $\mathcal{X}$. One can use Proposition 6.11 to show that for any stack $\mathcal{Y}$, $\mathrm{Map}_{\mathrm{Cat}^{\mathrm{cart}}_{/\mathrm{Sch}_{/S}}}(X_\bullet, \mathcal{Y})$ is canonically equivalent to the category of descent data, i.e., cartesian sections, for $\mathcal{Y}$ over the descent diagram

$$X_1 \times_{t,X_0,s} X_1 \overset{\overset{pr_0}{\underset{\mu}{\longrightarrow}}}{\underset{pr_1}{\longrightarrow}} X_1 \overset{t}{\underset{s}{\longrightarrow}} X_0 \ ,$$

which we interpret as a diagram $(\mathbf{\Delta}^{\leq 2}_{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Sch}_{/S}$. Concretely, this means that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is determined by a $\xi \in \mathcal{Y}(X_0)$ and an isomorphism

$\phi : s^*(\xi) \circ t^*(\xi)$ in $\mathcal{Y}(X_1)$ that satisfies a cocycle condition on $X_1 \times_{X_0} X_1$. A 2-morphism $(\xi, \phi) \to (\xi', \phi')$ is a morphism $\eta : \xi \to \xi'$ such that $t^*(\eta) \circ \phi = \phi' \circ s^*(\eta)$ in $\mathcal{Y}(X_1)$.

6.4 Morphisms of groupoids vs stacks

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of smooth groupoid spaces. Then the composition $X_\bullet \to Y_\bullet \to BY_\bullet$ factors essentially uniquely through a morphism $BX_\bullet \to BY_\bullet$, which we call $B(f_\bullet)$. In other words, applying the equivalence (6.6) when $\mathcal{F} = BY_\bullet$, allows us to define the dotted arrow in the following diagram

$$\text{Map}_{S-\text{Gpd}}(X_\bullet, Y_\bullet) \xrightarrow{\hspace{1cm}} \text{Map}_{\text{Cat}^{\text{cart}}_{/\text{Sch}/S}}(X_\bullet, BY_\bullet) \xrightarrow{\ \cong\ } \text{Map}(BX_\bullet, BY_\bullet)$$

with B the dotted arrow. {F:induced_morphism} (6.7)

L:groupoid_vs_stack_morphisms

Lemma 6.13. *For any pair of smooth groupoid algebraic spaces $X_\bullet, Y_\bullet$, the functor B in (6.7) is fully faithful, and its essential image consists of those morphisms $BX_\bullet \to BY_\bullet$ for which there exists a morphism $X_0 \to Y_0$ making the following diagram 2-commute*

$$\begin{array}{ccc} X_0 & \dashrightarrow & Y_0 \\ \downarrow & & \downarrow \\ BX_\bullet & \longrightarrow & BY_\bullet \end{array} \quad .$$

Proof. It is equivalent to prove the same claim, but with the fibered category $X_\bullet$ in place of $BX_\bullet$. It is clear that for any morphism $X_\bullet \to BY_\bullet$ induced by a functor $X_\bullet \to Y_\bullet$, the composition $X_0 \to X_\bullet \to BY_\bullet$ admits a lift $f_0 : X_0 \to Y_0$. Conversely, given such a lift, the two compositions $X_1 \to X_0 \to Y_0 \to BY_\bullet$ induced by $s, t : X_1 \to X_0$ are canonically isomorphic, which defines a morphism $X_1 \to Y_1$. Together, $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ define a functor of groupoid spaces $f_\bullet : X_\bullet \to Y_\bullet$ which induces the original morphism $X_\bullet \to BY_\bullet$.

To show fully faithfulness, consider two functors $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$, regarded as morphisms of fibered categories. We must show that any 2-isomorphism of the compositions with Triv : $Y_\bullet \to BY_\bullet$ is induced by a unique 2-isomorphism of $\eta : f_\bullet \Rightarrow g_\bullet$, which is equivalent to the data of a

natural transformation (6.1). The objects of $X_\bullet(T)$ is the set of points $X_0(T)$, so a natural isomorphism $\mathrm{Triv} \circ f_\bullet \cong \mathrm{Triv} \circ g_\bullet$ is just a function mapping $x \in X_0(T)$ to the set of isomorphisms between the $Y_\bullet$-torsors over T classified by $f_0(x) \in Y_0(T)$ and $g_0(x) \in Y_0(T)$ respectively. But by Exercise 6.5, two T-points of Y_0 and an isomorphism between their image in $BY_\bullet$ is equivalent to the data of a T-point of Y_1 whose source is $f_0(x)$ and whose target is $g_0(x)$. This equivalence is compatible with pullback along a morphism $T' \to T$. To complete the proof, one must show that the resulting map of algebraic spaces $X_0 \to Y_1$ over $Y_0 \times Y_0$ is actually a natural transformation. We leave this to the reader. $\qquad\square$

E:induced_principal_bundle

Example 6.14. Let G and H be smooth group schemes, and let $G_\bullet$ and $H_\bullet$ denote the corresponding groupoid spaces, where $X_0 = \mathrm{pt}$. In this case a morphism of groupoid spaces $f_\bullet : G_\bullet \to H_\bullet$ is just a map $f_1 : G \to H$ which is a homomorphism of group schemes. We can describe the induced morphism $B(f_\bullet) : BG \to BH$ more explicitly:

If $P \to T$ is a principal G-bundle, then $P \times H$ admits a left G action by the formula $g \cdot (p, h) = (pg^{-1}, f_1(g)h)$. This G action is free, because the G action on P is free. It follows from Example 6.2 that $P' = (P \times H)/G$ is an algebraic space over T. Choose an étale cover $T' \to T$ for which there exists a G-equivariant isomorphism $P_{T'} \cong G_{T'}$ relative to T'. One can check that this induces an H-equivariant isomorphism $P'_{T'} \cong H_{T'}$ over T', which shows that $P' \to T$ was a principal H-bundle. This functor from G-bundles over T to H-bundles over T is the induced morphism $B(f_\bullet) : BG \to BH$.

Lecture 7

Quotient stacks and quasi-coherent sheaves: I

References: [LMB], [O1]
Date: 3/5/2020
Exercises: 7

7.1 Quotient stacks

We have already discussed the general notion of a group-scheme over S, which we defined to be a group object in $\mathrm{Sch}_{/S}$, as well as the notion a (left) action of G on an algebraic space X over S. If G is smooth over S, then from this data we can define the *quotient stack* X/G. By definition X/G is the fibered category whose fiber over $T \in \mathrm{Sch}_{/S}$ is the category of pairs (P, u), where $P \to T$ is a principal G-bundle, and $u : P \to X$ is a G-equivariant map, i.e., $u(p \cdot g) = g^{-1} \cdot u(p)$ on T'-points for any $T' \in \mathrm{Sch}_{/T}$. Note that by convention G acts on P on the right and on X on the left, so this formula is compatible with the multiplication law $u(p \cdot gh) = u((p \cdot g) \cdot h) = h^{-1} \cdot u(p \cdot g) = h^{-1}g^{-1} \cdot u(p) = (gh)^{-1} \cdot u(p)$. A map in the category X/G is a map of principal G-bundles which commutes with the map to X.

Exercise 7.1. *Given a principal G-bundle $P \to T$ and a G-space X, we can define $P \times^G X := (P \times X)/G$, where G acts by $g \cdot (p, x) = (pg^{-1}, gx)$. By ?? this is an algebraic space. Show that an equivariant map $P \to X$ is the same*

thing as a section of the projection $P \times^G X \to T$. This gives an alternative description of X/G.

Exercise 7.2. *There is a canonical morphism $X/G \to BG$ which forgets the equivariant map u. There is also a canonical morphism $X \to X/G$ which maps a T-point $\phi : T \to X$ to the trivial G-bundle $G_T \to T$ along with the unique equivariant map $u : G_T \to X$ whose restriction to $\{1\} \times T$ is ϕ, i.e., $u(g, t) = g \cdot \phi(t)$. Show that if $\mathrm{pt} \to BG$ classifies the trivial G-bundle, then the diagram of stacks*

$$
\begin{array}{ccc}
X & \longrightarrow & X/G \\
\downarrow & & \downarrow \\
\mathrm{pt} & \longrightarrow & BG
\end{array}
$$

is cartesian, i.e., a fiber product square.

Exercise 7.3. *Show that for any morphism $T \to X/G$, classifying a principal G-bundle P and a map $u : P \to X$, the fiber product $T \times_{X/G} X \cong P$ over T, and under this isomorphism u corresponds to projection to the second factor. Show that X/G is an algebraic stack, and $X \to X/G$ is representable, smooth, and surjective.*

The morphism $X \to X/G$ classifies the trivial G-bundle $G \times X$ along with the map $u : G \times X \to X$ which maps $(g, x) \mapsto g \cdot x$. The previous exercise then shows that $X \times_{X/G} X \cong G \times X$, and furthermore that X/G is the stack associated to the groupoid

$$
G \times X \rightrightarrows X
$$

where on T-points for any $T \in \mathrm{Sch}_{/S}$ the source map is the forgetful map $(g, x) \mapsto x$ and the target map is the composition $(g, x) \mapsto gx$. The composition of the arrow (g, x) with (h, gx) is the arrow (hg, x).

Exercise 7.4. *Show that $X/G \to BG$ is representable, and in fact any representable morphism $\mathcal{X} \to BG$ is of the form X/G for the algebraic space $X := \mathcal{X} \times_{BG} \mathrm{pt}$. In fact, show that for any two representable morphisms $\mathcal{X}, \mathcal{Y} \to BG$, any morphism $f : \mathcal{X} \to \mathcal{Y}$ of fibered categories over BG has no non-trivial 2-automorphisms, and that the category of representable stacks in groupoids over BG is equivalent to the category G-Spc.*

Remark 7.1. We make the following *definition*: a G-action on a stack $\mathcal{X}$ is a morphism of stacks $\mathcal{Y} \to BG$ along with an isomorphism $\mathcal{X} \cong \mathrm{pt} \times_{BG} \mathcal{Y}$. Exercise 7.4 shows that this is consistent with our previous definition of an

action of G on an algebraic space X. One could try to define an action more concretely as a morphism of stacks $G \times \mathcal{X} \to \mathcal{X}$ satisfying associativity and identity conditions. However, the associativity and identity conditions will only hold up to 2-isomorphism, and the coherence conditions between these 2-isomorphisms becomes complicated.

Consider two group schemes G and H and a homomorphism between them $\phi : G \to H$. Given algebraic spaces $X \in G$-Spc and $Y \in H$-Spc, we say that a map of spaces $f : X \to Y$ is *equivariant with respect to* ϕ if it is equivariant when we regard Y as G-space via the homomorphism ϕ. On T points for $T \in \mathrm{Sch}_{/S}$, this means $f(g \cdot x) = \phi(g) \cdot f(x)$. ϕ and f together induce a morphism of groupoid spaces

$$(G \times X \rightrightarrows X) \to (H \times Y \rightrightarrows Y),$$

where $(g, x) \mapsto (\phi(g), f(x))$. This induces a morphism of algebraic stacks $X/G \to Y/G$ by the universal property of the stackification (see (6.7)).

We can describe the induced morphism $X/G \to Y/H$ more explicitly as the map on T-points for any $T \in \mathrm{Sch}_{/S}$

$$
\left(
\begin{array}{c}
G\text{-bundle } P \to T \\
\text{and } u : P \to X
\end{array}
\right)
\mapsto
\left(
\begin{array}{c}
H\text{-bundle } P \times^G H \to T \\
\text{and } u' : P \times^G H \to Y
\end{array}
\right),
$$

where $P \times^G H$ is the induced H-bundle of Example 6.14, and $u' : P \times^G H \to Y$ is the map induced on T-points by the map $(p, h) \mapsto h^{-1} \cdot f(u(p))$, which is $G(T)$-invariant because $(p \cdot g^{-1}, \phi(g)h)$ maps to

$$(\phi(g)h)^{-1} \cdot f(u(pg^{-1})) = h^{-1}\phi(g^{-1}) \cdot f(g \cdot u(p)) = h^{-1} \cdot f(u(p)).$$

7.1.1 Inertia and stabilizers

Definition 7.2. Let $\mathcal{X}$ be an algebraic stack. The *inertia stack* $I_{\mathcal{X}}$ is the fiber product $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$, where both maps are the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$. Regarded as a stack over $\mathcal{X}$ by the left projection $I_{\mathcal{X}} \to \mathcal{X}$ is representable by group algebraic spaces over $\mathcal{X}$. By definition, the fiber of $I_{\mathcal{X}}(T) \to \mathcal{X}(T)$ over a point $\xi \in \mathcal{X}(T)$ is the group of automorphisms of ξ in $\mathcal{X}(T)$.

The inertia stack plays an important role. For instance, Exercise 6.6 shows that an algebraic stack is an algebraic space if and only if $I_{\mathcal{X}} \to \mathcal{X}$ is an isomorphism. Note also that for any T-point $\xi : T \to \mathcal{X}$,

$$\mathrm{Aut}_{\mathcal{X}}(\xi) := T \times_{\mathcal{X}} I_{\mathcal{X}}$$

is a group algebraic space over T. In practice, we will be mostly interested in stacks with affine diagonal, so $\mathrm{Aut}_{\mathfrak{X}}(\xi)$ will be an affine group scheme over T. This can be very useful: if you have an object which you know corresponds to a k-point in an algebraic stack, then the automorphism group of that object is canonically the group of k points of some group scheme over k.

Remark 7.3. There is also a relative version of [Definition 7.2](#) associated to a morphism $f : \mathfrak{X} \to \mathfrak{Y}$. By definition $I_f := \mathfrak{X} \times_{\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}} \times \mathfrak{X}$, where the fiber product is with respect to the diagonal morphism $\Delta_f : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$. Again $I_f \to \mathfrak{X}$ is a group algebraic space over $\mathfrak{X}$, and it is representable and affine over $\mathfrak{X}$ if Δ_f is affine. The formation of I_f is compatible with base change along a map $\mathfrak{Y}' \to \mathfrak{Y}$, and it follows that f is representable by algebraic spaces if and only if $I_f \to \mathfrak{X}$ is an isomorphism.

This construction has a more concrete incarnation for quotient stacks. For any $f : T \to X$, we define stabilizer of f to be the algebraic space over T,

$$\mathrm{Stab}_G(f) := T \times_{T \times X} (T \times G),$$

where the map $T \to T \times X$ is (id_T, f) and the map $T \times G \to T \times X$ is $(t, g) \mapsto (t, gf(t))$. $\mathrm{Stab}_G(f)$ is a group algebraic space over T, and it is not hard to see from the functor of points that $\mathrm{Stab}_G(f)$ is a sub-group-scheme of the constant group scheme $G \times T$ over T.

$$
\begin{array}{ccc}
\mathrm{Stab}_G(f) & \longrightarrow & I_{X/G} \\
\downarrow & & \downarrow \\
T & \longrightarrow & X/G
\end{array}
$$

is cartesian, where $T \to X/G$ is the composition of f with the canonical map $X \to X/G$.

Exercise 7.5. *Compute describe the stabilizer of the canonical morphism $p : X \to X/G$ as a sub-group-scheme of $X \times G$ over X. If we let G act on $X \times G$ by the given action on X and the adjoint action on G, show that $\mathrm{Stab}_G(p) \hookrightarrow X \times G$ is G-equivariant. Use this and smooth descent to show that $I_{X/G} = \mathrm{Stab}_G(p)/G$ over X/G.*

Exercise 7.6. *Use [Exercise 6.6](#) to show that if $X \in G$-Spc and $Y \in H$-Spc, and $f : X \to Y$ is a morphism of spaces which is equivariant with respect to a group homomorphism $\phi : G \to H$, then the induced map of stacks $X/G \to Y/H$ is representable if and only if for any $\varphi : T \to X$, the induced homomorphism $\mathrm{Stab}_G(\varphi) \to \mathrm{Stab}_H(f \circ \varphi)$ is injective.*

It follows from Exercise 7.6 that if $\phi : G \to H$ is an injective homomorphism, then *any* morphism $f : X \to Y$ which is equivariant with respect to ϕ is representable. This is clearest when $\phi = \mathrm{id} : G \to G$. In this case we already saw in Exercise 7.4 that the category of stacks which are representable over BG is equivalent to the category of G-spaces. The fiber of $X/G \to Y/G$ over a T-point $T \to Y/G$ that corresponds to principal G-bundle P and a G-equivariant morphism $u : P \to Y$ is the fiber product $P \times_Y X$.

In the general case, where $G \subset H$ is a subgroup, we can get an explicit description as follows: consider the space $H \times^G X$, where $G(T)$ acts on T-points of $H \times X$ by $g \cdot (h, x) = (hg^{-1}, g \cdot x)$. $H \times^G X$ is an algebraic space because the G-action is free (see Example 6.2). The morphism $f : X \to Y$, which is equivariant with respect to $\phi : G \hookrightarrow H$, factors as the ϕ-equivariant morphism $X \to H \times^G X$ taking $x \mapsto (1, x)$ followed by the H-equivariant morphism $H \times^G X \to Y$ mapping $(h, x) \mapsto h \cdot f(x)$.

<div align="right">L:schur</div>

Lemma 7.4 (Schur's lemma)**.** *For any G-space X, the ϕ-equivariant morphism $X \to H \times^G X$ induces an isomorphism of stacks $X/G \to (H \times^G X)/H$.*

This shows that for the ϕ-equivariant morphism $f : X \to Y$, the induced morphism of stacks factors as an equivalence followed by an H-equivariant map $X/G \cong H \times^G X/H \to Y/H$.

Schur's lemma has many applications. Generally, it implies that G-equivariant geometric structures on X are equivalent to H-equivariant geometric structures on $H \times^G X$. For instance, it implies that the category of G-equivariant quasi-coherent sheaves on X is equivalent to the category of H-equivariant quasi-coherent sheaves on $H \times^G X$ (see [?E:schur_qcoh] below).

7.2 Quasi-coherent sheaves

<div align="right">D:quasi-coherent_sheaves</div>

Definition 7.5. Let $\mathfrak{X}$ be an algebraic stack over $\mathrm{Sch}_{/S}$. We define the category of *quasi-coherent sheaves* on $\mathfrak{X}$ to be the category

$$\mathrm{QCoh}(\mathfrak{X}) := \Gamma^{cart}_{\mathrm{Sch}_{/S}}(\mathfrak{X}, \mathrm{QCoh}_{/S}) \cong \mathrm{Map}_{\mathrm{Cat}^{cart}_{/\mathcal{C}}}(\mathfrak{X}, \mathrm{QCoh}_{/S}),$$

where we are regarding $\mathfrak{X} \to \mathrm{Sch}_{/S}$ as a diagram on the left side. (These categories are equivalent because every morphism in $\mathfrak{X}$ is cartesian.) In other words, a quasi-coherent sheaf E is an assignment of a quasi-coherent sheaf $E_\xi \in \mathrm{QCoh}(T)$ to any $\xi \in \mathfrak{X}$ lying over $T \in \mathrm{Sch}_{/S}$ in a way that is compatible with pullbacks for any morphism $\xi \to \xi'$ in $\mathfrak{X}$.

This is a "large" definition of what a quasi-coherent sheaf is, but if $X_\bullet$ is a groupoid algebraic space presenting $\mathfrak{X}$, then we can regard $X_\bullet$ as a diagram (5.1) in schemes that consists of just three schemes. Then Proposition 6.11 implies that the restriction functor

$$\Gamma^{cart}_{\mathrm{Sch}_{/S}}(\mathfrak{X}, \mathrm{QCoh}_{/S}) \to \Gamma_{\mathrm{Sch}_{/S}}(X_\bullet, \mathrm{QCoh}_{/S})$$

is an equivalence of categories. In particular, one could alternatively *define* $\mathrm{QCoh}(\mathfrak{X})$ to be $\mathrm{QCoh}(X_\bullet)$ for some groupoid presentation of $\mathfrak{X}$, and then show that any morita morphism of groupoids $X_\bullet \to Y_\bullet$ induces an equivalence of categories $\mathrm{QCoh}(Y_\bullet) \to \mathrm{QCoh}(X_\bullet)$.

Exercise 7.7. *Show that the assignment $X_\bullet \mapsto \mathrm{QCoh}(X_\bullet)$ is Morita invariant directly, using the method of proof of Lemma 3.24. In other words, show that given a Morita morphism of groupoid spaces $X_\bullet \to Y_\bullet$, the pullback functor $\mathrm{QCoh}(Y_\bullet) \to \mathrm{QCoh}(X_\bullet)$ is an equivalence of catgories.*

Example 7.6. The category $\mathrm{QCoh}(G \times X \rightrightarrows X)$ is the correct notion of a "G-equivariant quasi-coherent sheaf on X." An object of this category is a descent datum on the groupoid $G \times X \rightrightarrows X$. Explicitly, this consists of a quasi-coherent sheaf $E \in \mathrm{QCoh}(X)$ and an isomorphism $\varphi : s^*(E) \to t^*(E)$ satisfying a cocycle condition on $X_1 \times_{X_0} X_1 \cong G \times G \times X$.

For a general smooth groupoid space $X_\bullet$, $\mathrm{QCoh}(X_\bullet)$ has an identical description in terms of descent data. Objects are pairs $E \in \mathrm{QCoh}(X_0)$ with a cocycle $\varphi_E : s^*(E) \cong t^*(E)$. A homomorphism of quasi-coherent sheaves is just a homomorphism $f : E \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} s^*(E) & \xrightarrow{s^*(f)} & s^*(F) \\ \downarrow{\scriptstyle \varphi_E} & & \downarrow{\scriptstyle \varphi_F} \\ t^*(E) & \xrightarrow{t^*(f)} & t^*(F) \end{array} \ .$$

It is not hard to show that in this case the kernel and cokernel of f inherit canonical cocycles, and that these objects are kernel and cokernel of f in $\mathrm{QCoh}(X_\bullet)$. In addition, $E \otimes F \in \mathrm{QCoh}(X_0)$ inherits a canonical cocycle as well. In fact $\mathrm{QCoh}(X_\bullet)$ is a symmetric monoidal abelian category, in which the kernel, cokernel, and tensor product are just the corresponding objects computed in $\mathrm{QCoh}(X_0)$ along with their induced cocyles.

In addition to the cokernel, one can show that colimits in $\mathrm{QCoh}(\mathfrak{X})$ are just to colimits in $\mathrm{QCoh}(X_0)$ along with an induced cocycle. Choosing an atlas X_0 that is affine, we see that $\mathrm{QCoh}(\mathfrak{X})$ is cocompletem, and that filtered colimits are exact. An even stronger statement is the following:

Theorem 7.7. [S4, Tag 0781] T:grothendieck_category *If $\mathcal{X}$ is an algebraic stack, then $\mathcal{A} = \mathrm{QCoh}(\mathcal{X})$ is a Grothendieck abelian category. By definition this means that $\mathcal{A}$ has arbitrary direct sums (hence all colimits), filtered colimits are exact, and $\mathcal{A}$ admits a generator, i.e., an object $G \in \mathcal{A}$ such that for any subobject $N \subsetneq M$, there is a morphism $G \to M$ which does not factor through N.*

The category of modules over any ring is a Grothendieck abelian category (with generator R). In general, the definition of a generator implies that $\mathrm{Hom}(G, -) : \mathcal{A} \to R\text{-Mod}$, where $R = \mathrm{End}(G)$ is a faithful embedding, and in fact the Gabriel-Popescu theorem says that this functor is fully faithful and admits an exact left adjoint, hence $\mathcal{A}$ is closely related to a category of modules.

Grothendieck categories have many other nice properties. For instance a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ is representable by an object of $\mathcal{A}$ if and only if F takes colimits to limits [S4, Tag 07D7]. Using this one can show that any colimit preserving functor between Grothendieck categories $f^* : \mathcal{A} \to \mathcal{B}$ admits a right adjoint.

In addition, any Grothendieck category has enough injectives [S4, Tag 079H], so homological algebra works nicely. More generally any complex $C^\bullet$ admits a quasi-isomorphism $C^\bullet \to I^\bullet$ to a complex $I^\bullet$ which is K-injective [S4, Tag 079P], which allows one to define the unbounded derived category $D(\mathcal{A})$. In particular, given an additive functor between Grothendieck categories $F : \mathcal{A} \to \mathcal{B}$, the right-derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ exists and can be computed in the usual way, i.e., $RF(C^\bullet) \cong F(I^\bullet)$ for any quasi-isomorphism $C^\bullet \to I^\bullet$ with $I^\bullet$ K-injective (see [S4, Tag 0156], for instance).

Remark 7.8. The more traditional definition of quasi-coherent sheaves in [LMB] and [O1] associates a certain site, the Lisse-étale site, to an algebraic stack, and then defines quasi-coherent sheaves as a subcategory of all sheaves on this site. One can then use the fact that quasi-coherent sheaves, in this definition, satisfy smooth descent to show that the category is equivalent to the one we give in Definition 7.5. We have chosen this approach because it follows nicely from the theory of descent, without introducing additional technicalities.

In fact if S is separated, Exercise 3.14 implies that one could equivalently define quasi-coherent sheaves on a stack $\mathcal{X}$ over $\mathrm{Sch}_{/S}$ as the category of morphisms

$$\mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}_{/S}} \to \mathrm{Mod}_{/S}$$

where $\mathrm{Sch}^{\mathrm{aff}}_{/S}$ denotes the subcategory of *affine* schemes over S, and $\mathrm{Mod}_{/S} \cong \mathrm{QCoh}_{/S}|_{\mathrm{Sch}^{\mathrm{aff}}_{/S}}$ is the fibered category which associates $\mathrm{Spec}(A)/S \mapsto A\text{-Mod}$.

Thus one can define quasi-coherent sheaves on a stack with nothing more than the theory of fibered categories and the theory of modules over commutative rings.

7.2.1 Pushforward and pullback

Given a morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks in groupoids over $\mathrm{Sch}_{/S}$, the pullback functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X}),$$

is defined to be simply the precomposition of a quasi-coherent sheaf $E : \mathcal{X} \to \mathrm{QCoh}_{/S}$ with f. More concretely for algebraic stacks, one can choose atlases $X_0 \twoheadrightarrow \mathcal{X}$ and $Y_0 \twoheadrightarrow \mathcal{Y}$ such that f lifts to a map $f_0 : X_0 \to Y_0$, so that f is induced by a functor of groupoid spaces $f_\bullet : X_\bullet \to Y_\bullet$ (see Lemma 6.13). Then given $(E, \phi) \in \mathrm{QCoh}(Y_\bullet)$, $f^*(E, \phi) \in \mathrm{QCoh}(X_\bullet)$ is just $f_0^*(E)$ with an induced cocycle. Note that this functor commutes with colimits, because f_0 does, so Theorem 7.7 implies that f^* admits a right adjoint $f_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$.

In general f_* can be poorly behaved, but we have

`L:quasi-coherent_pushforward`

Lemma 7.9. *If $f : \mathcal{X} \to \mathcal{Y}$ is quasi-compact and quasi-separated (see Definition 5.4), then f_* commutes with filtered colimits and satisfies flat base change, i.e., given a cartesian diagram in which g is flat*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\;g'\;} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{\;g\;} & \mathcal{Y} \end{array} \;,$$

the canonical natural transformation $g^* f_* \Rightarrow (f')_*(g')^*$ is an equivalence.

Proof outline. To prove this, one can use faithfully flat descent to reduce to the case when $\mathcal{Y} = \mathrm{Spec}(A)$ and $\mathcal{Y}' = \mathrm{Spec}(A')$ are affine. For instance, if $\mathcal{Y}$ is a separated scheme, one can choose a cover $\mathrm{Spec}(A) \to \mathcal{Y}$, and $\mathrm{Spec}(A) \times_\mathcal{Y} \mathrm{Spec}(A) = \mathrm{Spec}(B)$ will be affine as well. Then you first construct the pushforward along $\mathcal{X}_A \to \mathrm{Spec}(A)$, use flat base change along the two maps $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to equip the pushforward of any object with a cocycle, and show that the resulting object of $\mathrm{QCoh}(\mathcal{Y})$ satisfies the universal property of the pushforward. Then one can bootstrap from separated schemes to schemes, then to algebraic spaces, and finally to algebraic stacks. The details of this bootstrap procedure are a bit tedious, so we will content ourselves to prove the claim over an affine base.

Choose an atlas $p : X_0 \to \mathcal{X}$, leading to a presentation $X_1 \rightrightarrows X_0$ for $\mathcal{X}$. Without appealing to Theorem 7.7, let us first imagine that p_* and f_* exist. For any descent datum (E, ϕ), representing an object in $\mathrm{QCoh}(\mathcal{X})$, there are canonical maps induced by applying p_* to the units of adjunction

$$a : p_*(E) \to p_*(t_*(t^*(E))), \text{ and}$$
$$b : P_*(E) \to p_*(s_*(s^*(E))) \cong p_*(t_*(t^*(E))),$$

where the equivalence on the second line is induced by the cocycle $\phi : s^*(E) \cong t^*(E)$ and the canonical equivalence $p_* \circ t_* \cong (p \circ t)_* \cong (p \circ s)_* \cong p_* \circ s_*$. Faithfully flat descent should imply that

$$(E, \phi) = \ker(a - b : p_*(E) \to p_*(t^*(t_*(E)))).$$

The formal properties of right adjoints would then dictate that f_* commutes with the formation of this kernel, and that $f_* \circ p_* \cong (f \circ p)_*$. The latter map $f \circ p$, however, is just a quasi-compact map of schemes, so we know that f_* exists. This suggests a definition of $f_*(E, \phi)$.

We let $f_0 := f \circ p : X_0 \to \mathrm{Spec}(A)$, and observe that $f_0 \circ s = f_0 \circ t$. We define $f_*(a)$ to be the canonical map $(f_0)_*(E) \to (f_0)_*(t_*(t^*(E)))$ and define $f_*(b) : (f_0)_*(E) \to (f_0)_*(s_*(s^*(E))) \cong (f_0)_*(t_*(t^*(E)))$ using the cocycle $\phi : s^*(E) \cong t^*(E)$ as above. We then *define*

$$f_*(E, \phi) := \ker(f_*(a) - f_*(b) : (f_0)_*(E) \to (f_0)_*(t_*(t^*(E)))). \quad \text{{E:formal\_pushforward}} \quad (7.1)$$

One can check directly that a map of A-modules $M \to f_*(E, \phi)$ is the same as a map of descent data $(f_0^*(M), \psi) \to (E, \phi)$, where $\psi : s^*(f_0^*(M)) \cong t^*(f_0^*(M))$ is the canonical cocycle induced by the fact that $f_0 \circ s = f_0 \circ t$. So we have given an explicit construction of an adjoint $f_* : \mathrm{QCoh}(X_\bullet) \to A\text{-Mod}$. It is also clear from this definition, and flat base change for the map $f_0 : X_0 \to \mathrm{Spec}(A)$, that the formation of $f_*(E, \phi)$ commutes with base change along a flat map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$, because if A' is a flat A-algebra, then $A' \otimes_A (-)$ commutes with the formation of the kernel in (7.1).

$\square$

Remark 7.10. Unlike in the case of schemes, f_* does not have finite cohomological dimension. The basic example is $B(\mathbb{Z}/2)$ in over a field of characteristic 2. This implies that Rf_* does not commute with filtered colimits in the unbounded derived category $D(\mathrm{QCoh}(\mathcal{X}))$, but it does commute with uniformly cohomologically bounded below filtered colimits.

7.2.2 The noetherian case

An algebraic stack $\mathfrak{X}$ is defined to be *noetherian* if it admits a smooth cover by a noetherian scheme and the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is quasi-compact. This is equivalent to $\mathfrak{X}$ admitting a presentation by a smooth groupoid space $X_\bullet$ for which X_0 is noetherian and $X_1 \to X_0 \times X_0$ is quasi-compact.

Definition 7.11. We say that $E \in \mathrm{QCoh}(X_\bullet)$ is coherent if its restriction to X_0 is coherent, and we let $\mathrm{Coh}(\mathfrak{X}) \subset \mathrm{QCoh}(\mathfrak{X})$ denote the full subcategory of coherent sheaves.

If $\mathfrak{X}$ is noetherian, then the coherent sheaves form a full abelian subcategory (this can be verified locally, and thus reduces the affine noetherian case).

P:noetherian_generation

Proposition 7.12. [LMB, Prop. 15.4] *If $\mathfrak{X}$ is a noetherian algebraic stack, then $\mathrm{Coh}(\mathfrak{X}) \subset \mathrm{QCoh}(\mathfrak{X})$ is the subcategory of compact objects, i.e., for any $E \in \mathrm{Coh}(\mathfrak{X})$ and any filtered system $\{F_\alpha\}_{\alpha \in I}$ in $\mathrm{QCoh}(\mathfrak{X})$,*

$$\mathrm{colim}_{\alpha \in I} \mathrm{Hom}(E, F_\alpha) \to \mathrm{Hom}(E, \mathrm{colim}_{\alpha \in I} F_\alpha)$$

is an isomorphism. Furthermore, every quasi-coherent sheaf is a filtered union of coherent subsheaves.

Proof. Consider an atlas $p : \mathrm{Spec}(A) \to \mathfrak{X}$, where A is noetherian. It follows formally from the fact that p_* preserves filtered colimits that for any compact object $E \in \mathrm{QCoh}(\mathfrak{X})$, $p^*(E) \in A\text{-Mod}$ is compact and hence coherent. Conversely, faithfully flat descent implies that for any filtered system $\{F_\alpha\}_{\alpha \in I}$, there is a filtered system $\{F'_\alpha\}_{\alpha \in I}$ in A-Mod and a map of filtered systems $p_*(p^*(F_\alpha)) \to p_*(F'_\alpha)$ such that

$$F_\alpha = \ker(p_*(p^*(F_\alpha)) \to p_*(F'_\alpha)).$$

It we apply $\mathrm{Hom}(E, -)$ to this expression and to its colimit, and use the fact that the formation of filtered colimits is exact, one sees that it suffices to show that $\mathrm{Hom}(E, -)$ preserves the colimit of a filtered system which is pushed forward from $\mathrm{Spec}(A)$. By adjunction this holds if $p^*(E)$ is coherent, and hence if E is coherent.

To show that any $M \in \mathrm{QCoh}(\mathfrak{X})$ is a filtered union of its coherent subsheaves, first write $p^*(M) = \bigcup_\alpha N_\alpha$ as a filtered union with N_α coherent. Then $p_*(p^*(M)) = \bigcup_\alpha p_*(N_\alpha)$, and faithfully flat descent implies that the canonical morphism

$$M \to p_*(p^*(M)) = \bigcup_\alpha p_*(N_\alpha)$$

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is injective. We define M_α to be the preimage of $p_*(N_\alpha) \subset p_*(p^*(M))$ in M under this map, i.e., $M_\alpha := \ker(M \oplus p_*(N_\alpha) \to p_*(p^*(M))) \subset M$. By construction $M = \bigcup_\alpha M_\alpha$. Applying the adjunction between p_* and p^*, we get a commutative square in A-Mod

$$
\begin{array}{ccc}
p^*(M_\alpha) & \longrightarrow & N_\alpha \\
\downarrow & & \downarrow \\
p^*(M) & \!\!=\!\!=\!\!=\!\! & p^*(M)
\end{array} \quad .
$$

The left vertical arrow is injective because p is flat, and the right vertical arrow is injective by construction, so $p^*(M_\alpha) \to N_\alpha$ is injective. This implies that M_α is coherent. $\qquad \square$

This says that there are "enough" coherent sheaves on noetherian algebraic stacks. On the other hand, we will see that a noetherian algebraic stacks only has "enough" vector bundles if $\mathcal{X}$ is a global quotient stack.

Lecture 8

Quotient stacks and quasi-coherent sheaves II

References: [M3, Chap. 3,4], [T2], [G1]
Date: 3/10/2020
Exercises: 8

8.1 Quotients of quasi-projective schemes by linear algebraic groups

In this section, we attempt to put some of the previous discussion in a much more concrete context. To this end, we narrow our context to the case where $S = \mathrm{Spec}(k)$ for a field k, and we study quotients of a locally closed subschemes of $\mathbb{P}^n_k$ by a linear action of smooth affine group scheme over k.[1] The results of this section remain true, with the same proofs, for an arbitrary base scheme S, when one studies linearized actions of a smooth affine S-group scheme on schemes which are quasi-projective over S.

In this case $G = \mathrm{Spec}(\mathcal{O}_G)$, where $\mathcal{O}_G$ is a flat quasi-coherent k-algebra. The product $G \times G \to G$ and unit $\mathrm{Spec}(k) \to G$ are given respectively by maps of k-algebras $\mu^* : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$, called the comultiplication, and $\epsilon : \mathcal{O}_G \to k$, called the coaugmentation. These maps equip $\mathcal{O}_G$ with a *coalgebra* structure, which means that they satisfy the associativity and identity axioms dual to those of an algebra (or equivalently equip $\mathcal{O}_G$ with

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[1]Over a field, a finite type group scheme G is smooth if and only if it is reduced, by generic smoothness and a "spreading out" trick using the action of G on itself.

the structure of an algebra in $(\mathrm{Alg}_k)^{\mathrm{op}}$. The algebra and coalgebra structure, together with the involution $a : \mathcal{O}_G \to \mathcal{O}_G$ induced by the inverse map on G, is known as a *Hopf algebra* structure on $\mathcal{O}_G$.

Example 8.1. For $\mathbb{G}_m = \mathrm{Spec}(\mathcal{O}_S[t^{\pm 1}])$, the comultiplication $\Delta : \mathcal{O}_S[t^{\pm 1}] \to \mathcal{O}_S[t_1^{\pm 1}, t_2^{\pm 1}]$ maps $t \mapsto t_1 \otimes t_2$.

Exercise 8.1. *Describe the comultiplication and counit for* $\mathrm{GL}_n = \mathrm{Spec}(\mathcal{O}_S[x_{ij}][\det^{-1}])$.

Definition 8.2. A comodule for $\mathcal{O}_G$ is a vector space V over k along with a linear map $\alpha : V \mapsto \mathcal{O}_G \otimes_k V$ satisfying the following axioms:

- $V \xrightarrow{\alpha} \mathcal{O}_G \otimes V \xrightarrow{\epsilon \otimes \mathrm{id}_V} V$ is the identity, and

- the following diagram commutes

$$
\begin{array}{ccc}
V & \xrightarrow{\quad \alpha \quad} & \mathcal{O}_G \otimes V \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mu^* \otimes \mathrm{id}_V} \\
\mathcal{O}_G \otimes V & \xrightarrow{\mathrm{id}_{\mathcal{O}_G} \otimes \alpha} & \mathcal{O}_G \otimes \mathcal{O}_G \otimes V
\end{array}
\quad .
$$

Example 8.3. A affine group scheme G over a field k is a called a torus of rank n if $G \times_k \bar{k} \cong (\mathbb{G}_m^n)_{\bar{k}}$, and it is called split if one has an isomorphism over k. For a split torus $G = (\mathbb{G}_m)_k^n$, an $\mathcal{O}_G$-comodule structure on V corresponds to a linear map

$$
\alpha : V \mapsto \mathcal{O}_G \otimes V \cong \bigoplus_{(\chi_1, \ldots, \chi_n) \in \mathbb{Z}^n} z_1^{\chi_1} \cdots z_n^{\chi_n} \cdot V,
$$

where $z_1, \ldots, z_n$ are the coordinate functions on $\mathbb{G}_m^n$, and the notation simply denotes a direct sum of one copy of V for each monomial z^χ in the monomial basis for $\mathcal{O}_G$. For any $\chi \in \mathbb{Z}^n$, let $\Pi_\chi : V \to V$ denote the composition of α with the projection onto the z^χ-summand in $\mathcal{O}_G \otimes V$. One can use the associativity axiom of a comodule to show that

$$
\Pi_\chi \Pi_\rho = \begin{cases} \Pi_\chi, & \text{if } \chi = \rho, \\ 0, & \text{otherwise} \end{cases} \quad ,
$$

and the identity axiom implies that $\sum_\chi \Pi_\chi = \mathrm{id}_V$. (Note that although the latter sum is infinite, it is well define because for any v, $\alpha(v)$ can only lie in finitely many monomial summands of $\mathcal{O}_G$ by definition.) Thus the Π_χ are mutually orthogonal projectors with the property that $\Pi_\chi(v) = 0$ for all but finitely many $\chi \in \mathbb{Z}^n$, and these projectors therefore define a direct sum

decomposition $V = \bigoplus_\chi V_\chi$, where $V_\chi := \Pi_\chi(V)$. One can show that a linear map $V \to W$ is a map of comodules if and only if it maps $V_\chi \to W_\chi$ for all $\chi \in \mathbb{Z}^n$. Thus the category of $\mathcal{O}_G$-comodules is equivalent to the category of *graded vector spaces*.

Conversely for any graded vector space, one obtains family of projectors Π_χ with the properties above. The map

$$\alpha(v) = \sum_{\chi \in \mathbb{Z}^n} z^\chi \cdot \Pi_\chi(v) \in \mathcal{O}_G \otimes V$$

defines a comodule structure on V, and this construction is left and right inverse to the construction above. We therefore arrive at the important result that the category of $\mathcal{O}_G$-comodules is equivalent to the category of graded vector spaces.

Exercise 8.2. *An interesting non-split example of a torus is the* Deligne *torus $\mathbb{S}$ over $\mathbb{R}$, which is defined by $\mathbb{S}(A) = \mathbb{G}_m(\mathbb{C} \otimes_\mathbb{R} A) = (\mathbb{C} \otimes_\mathbb{R} A)^\times$ for any $\mathbb{R}$-algebra A, i.e., $\mathbb{S}$ is the* Weil restriction *of $(\mathbb{G}_m)_\mathbb{C}$ along $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$. Observe that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, but $\mathbb{S}_\mathbb{C} := \mathbb{S} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong \mathbb{G}_m \times \mathbb{G}_m$, so $\mathbb{S}$ is a rank two torus. Compatibility of Weil restriction with étale base change implies that $\mathbb{S} = \mathrm{Spec}(\mathbb{C}[z^{\pm 1}, \bar{z}^{\pm 1}]^\sigma)$, where σ is the involution of the $\mathbb{R}$-algebra given by complex conjugation on coefficients and $z \mapsto \bar{z}$, and $(\bullet)^\sigma$ denotes the invariant subspace for σ. Write down the comultiplication on $\mathcal{O}_\mathbb{S}$ and use étale descent along $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ to construct an equivalence between the category of $\mathcal{O}_\mathbb{S}$-comodules and the category of real Hodge structures, i.e., the category in which an object is real vector space H along with a direct sum decomposition $H \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p,q} H^{p,q}$ with $\overline{H^{p,q}} = H^{q,p}$, and a morphism is an $\mathbb{R}$-linear map respecting the decompositions.*

The category of $\mathcal{O}_G$-comodules has a symmetric monoidal structure, in which $V \otimes W$ is the vector space $V \otimes_k W$ along with the new coaction map defined as the composition

$$\rho_{V \otimes W} : V \otimes W \xrightarrow{\rho_V \otimes \rho_W} \mathcal{O}_G \otimes \mathcal{O}_G \otimes V \otimes W \to \mathcal{O}_G \otimes V \otimes W,$$

where the second arrow is given by the multiplication on $\mathcal{O}_G$, which corresponds to the diagonal $G \to G \times G$. Our motivation for studying the category of $\mathcal{O}_G$-comodules is the following:

<div align="right"><code>L:comodule</code></div>

Lemma 8.4. *If one considers the groupoid $G \rightrightarrows \mathrm{Spec}(k)$ presenting BG, then $\mathrm{QCoh}(BG) \cong \mathrm{QCoh}(G \rightrightarrows \mathrm{Spec}(k))$ consists of a k vector space V along with a cocycle*

$$\varphi : \mathcal{O}_G \otimes_k V \to \mathcal{O}_G \otimes_k V.$$

φ is by definition a map of $\mathcal{O}_G$-modules and therefore is uniquely defined by the map of vector spaces

$$\alpha := \varphi|_{k \cdot 1 \otimes V} : V \to \mathcal{O}_G \otimes V$$

This construction $\varphi \mapsto \alpha$ defines an equivalence of symmetric monoidal abelian categories $\mathrm{QCoh}(BG) \cong \mathcal{O}_G\text{-Comod}$.

Exercise 8.3. *Prove Lemma 8.4.*

This leads to the following:

D:representation

Definition 8.5. A *representation* of G is an $\mathcal{O}_G$-comodule, or equivalently a quasi-coherent sheaf on BG. We refer to the abelian category of representations as $\mathrm{Rep}(G)$.

Note that one could equivalently define a finite dimensional representation as a linear action of G on the scheme $\mathbb{A}_k^n$, as in [MFK], where the equivalence takes $V \in \mathcal{O}_G\text{-Comod}$ to $\mathrm{Spec}(\mathrm{Sym}(V^*))$. Definition 8.5, however, allows one to treat infinite dimensional representations just as easily as finite dimensional ones.

Exercise 8.4. *Combining Lemma 8.4 with Proposition 7.12 implies that any comodule of $\mathcal{O}_G$ is a union of its finite dimensional sub-comodules. Show this directly by considering a comodule given by a coaction map $\rho : V \to V \otimes_k \mathcal{O}_G$, and explicitly describing, in terms of ρ, a finite dimensional sub-comodule containing any vector v.*

It is interesting to consider pushforward and pullback of quasi-coherent sheaves along the projection $f : BG \to \mathrm{Spec}(k)$ using Lemma 8.4. One can show that $f^*(V)$ corresponds to V with its trivial comodule structure $\alpha(v) = 1 \otimes v \in \mathcal{O}_G \otimes V$. Likewise, using the explicit construction of f_* in the proof of Lemma 7.9, one can show that

$$f_*(V, \alpha) = \ker(\alpha - 1 \otimes (-) : V \to \mathcal{O}_G \otimes V).$$

This is defined to be the *invariant subspace* $V^G \subset V$.

Example 8.6 (Borel-Weil theory). ¡¡TODO:¿¿

8.1.1 From G-modules to affine G-schemes

One has the following general principal:

Lemma 8.7. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a representable (by schemes) qc.qs. morphism of algebraic stacks. Then $\pi_*(\mathcal{O}_{\mathcal{X}}) \in \mathrm{QCoh}(\mathcal{Y})$ is a commutative algebra object in the symmetric monoidal category $\mathrm{QCoh}(\mathcal{Y})$, and this construction results in a functor*

$$\pi_* : \{\text{representable qc.qs. morphisms } \mathcal{X} \to \mathcal{Y}\} \to \mathrm{Alg}(\mathrm{QCoh}(\mathcal{Y}))^{\mathrm{op}}$$

which admits a fully faithful right adjoint, denoted $\mathrm{Spec}_{\mathcal{Y}}$*, whose essential image is the full subcategory of affine morphisms $\mathcal{X} \to \mathcal{Y}$,i.e., morphisms representable by* affine *morphisms of schemes.*

In addition, π_* can be canonically enriched to a functor

$$\mathrm{QCoh}(\mathcal{X}) \to \pi_*(\mathcal{O}_{\mathcal{X}})\text{-Mod}(\mathrm{QCoh}(\mathcal{Y})). \tag{8.1}$$

{E:pushforward_algebra_modules}

If $\pi : \mathcal{X} \to \mathcal{Y}$ is affine, then the canonical morphism $\mathcal{X} \to \mathrm{Spec}_{\mathcal{Y}}(\pi_*(\mathcal{O}_{\mathcal{X}}))$ is an isomorphism of stacks, and the canonical functor (8.1) is an equivalence of categories.

Proof. These follow from the corresponding claims in the case of schemes, and smooth descent for $\mathrm{QCoh}_{/S}$. $\square$

This implies that a stack $\mathcal{X}$ that admits an affine morphism $\mathcal{X} \to BG$ is of the form $\mathrm{Spec}(A)/G$ for some algebra $A \in \mathrm{QCoh}(BG)$. Using Lemma 8.4, we can identify this as a usual algebra A over k, equipped with the structure of an $\mathcal{O}_G$-comodule such that the multiplication map $A \otimes A \to A$ and multiplicative identity map $k \to A$ are maps of $\mathcal{O}_G$-modules.

Lemma 8.4 also identifies $\mathrm{QCoh}(\mathrm{Spec}(A)/G)$ with the category of A-modules M which additionally have the structure of an $\mathcal{O}_G$-comodule and such that the homomorphism $A \otimes M \to M$ defining the A-modules structure is also a map of $\mathcal{O}_G$-comodules. Kernels, cokernels, and tensor products in $\mathrm{QCoh}(\mathrm{Spec}(A)/G)$ are just kernels, cokernels, and tensor products of the underlying A-modules, with induced $\mathcal{O}_G$-comodule structure. One can think of this as a consequence of the fact that $\mathrm{Spec}(A) \to \mathrm{Spec}(A)/G$ is faithfully flat. To summarize, we have the following:

Corollary 8.8. $\mathrm{QCoh}(\mathrm{Spec}(A)/G)$ *is canonically equivalent, under* Lemma 8.7, *to the category of G-equivariant A-modules.*

Example 8.9. Given a comodule $V \in \mathcal{O}_G$-Comod, the A-module $A \otimes_k V$ is canonically G-equivariant. This defines the pullback functor along the morphism $f : \mathrm{Spec}(A)/G \to BG$. The pushforward f_* corresponds to the

functor which forgets the A-modules structure, but remembers the comodule structure. Combining this with our previous discussion, the pushforward along $\mathrm{Spec}(A)/G \to \mathrm{Spec}(k)$ maps a G-equivariant A-module M to the vector space M^G.

For any $M \in A\text{-Mod}(\mathcal{O}_G\text{-Comod}) \cong \mathrm{QCoh}(\mathrm{Spec}(A)/G)$, one the canonical map $A \otimes_k M \to M$, where on the left we regard M just as a comodule, is surjective. It follows that any M can be written as the cokernel of a map $A \otimes_k V \to A \otimes_k W$ for two G-representations V, W. The objects $A \otimes_k W$ are important examples of locally free sheaves on $\mathrm{Spec}(A)/G$.

Example 8.10. Let us continue Example 8.3, in which $G = (\mathbb{G}_m)_k^n$. Under the equivalence with graded vector spaces from Lemma 8.4, the tensor product of $\mathcal{O}_G$-comodules corresponds to

$$(V \otimes W)_\chi = \bigoplus_{\mu + \rho = \chi \in \mathbb{Z}^n} V_\mu \otimes W_\rho.$$

Then an algebra object is just an algebra A with a $\mathbb{Z}^n$-grading such that $1 \in A$ is homogeneous of weight 0 and $A_\mu \cdot A_\rho \subset A_{\mu+\rho}$. By Lemma 8.7, every stack that is affine over $B(\mathbb{G}_m)_k^n$ is of the form $\mathrm{Spec}(A)/\mathbb{G}_m^n$ for some graded k-algebra A, and $\mathrm{QCoh}(\mathrm{Spec}(A)/G)$ is identified with the symmetric monoidal abelian category of graded A-modules.

Exercise 8.5. *Use Lemma 8.7 to show that the category of closed immersions $\mathcal{Z} \hookrightarrow \mathcal{X}$ for a fixed algebraic stack $\mathcal{X}$ is equivalent to the category opposite to the category of quasi-coherent subsheaves $I \subset \mathcal{O}_\mathcal{X}$, in which morphisms are inclusions $I \subset I'$.*

Exercise 8.6. *An affine morphism $\mathcal{X} \to BG$ is finite type if and only if the corresponding G-equivariant algebra A is finite type as a k-algebra. Show that every finite type affine morphism $\mathcal{X} \to BG$ factors through a closed immersion $\mathcal{X} \hookrightarrow \mathbb{A}_k^n/G$, where the G action on $\mathbb{A}_k^n$ is linear, i.e., corresponds to a finite dimensional $\mathcal{O}_G$-comodule.*

Now consider a *linearizable* action of a smooth affine k-group G on a quasi-projective k-scheme X. By definition this means that there is a finite dimensional representation of G, call it V, and G-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$, where the G-action on $\mathbb{P}(V)$ is induced by the $\mathcal{O}_G$-comodule structure on $\mathrm{Sym}(V^*)$. A useful observation is that

$$\mathbb{P}(V)/G \cong (\mathbb{A}(V) \setminus \{0\})/(G \times \mathbb{G}_m) \subset \mathbb{A}(V)/(G \times \mathbb{G}_m)$$

is an open substack, where $\mathbb{G}_m$ acts by scaling on $\mathbb{A}(V)$. We can thus identify X/G with the locally closed stack $X'/G \times \mathbb{G}_m \subset \mathbb{A}(V)/G \times \mathbb{G}_m$,

where $X' = \mathrm{Tot}_X \, \mathcal{O}_X(-1)$. The closure of X' is a G-equivariant closed subscheme $\mathrm{Spec}(A) \subset \mathbb{A}(V)$. So to the class of stacks arising as quotients of G-linearizable quasi-projective G-schemes (these are sometimes called G-quasi-projective G-schemes) coincides with the class of stacks arising as quotients of finite type quasi-affine G-schemes.

We will therefore restrict our attention to quotients of a quasi-affine scheme by a smooth affine group scheme.

Remark 8.11. Not every G-action on a quasi-projective k-scheme is linearizable. For an example, take the $\mathbb{G}_m$-action on $\mathbb{P}^1_k$ by scaling one of the coordinates. This has two fixed points $0, \infty \in \mathbb{P}^1_k$. Then let X be the scheme obtained from $\mathbb{P}^1_k$ by identifying these two points. One can show that X does not admit an equivariant embedding in some $\mathbb{P}(V)$, even though it is quasi-projective. On the other hand, every action of G on a normal quasi-projective k-scheme is linearizable [MFK].

We also note that every G-action on a finite type quasi-affine scheme U is linearizable, because we can write $\Gamma(U, \mathcal{O}_U)$ as a filtered union of finite dimensional representations, and eventually these give equivariant locally closed immersions $U \hookrightarrow \mathbb{A}(V)$.

8.1.2 Quasi-coherent sheaves on quotients of quasi-affine schemes.

Let $\mathrm{Spec}(A)$ be an affine G-scheme, and let $j : U \subset \mathrm{Spec}(A)$ be a G-equivariant open subscheme. The complement $Z = \mathrm{Spec}(A) \setminus U$ can be equipped with its reduced subscheme structure, which is automatically G-equivariant, and we let $I \subset A$ be the corresponding G-equivariant ideal. We already have a concrete description of $\mathrm{QCoh}(\mathrm{Spec}(A)/G)$ as G-equivariant A-modules, or more precisely $A\text{-Mod}(\mathcal{O}_G\text{-Comod})$, and we would like to extend this to $\mathrm{QCoh}(U/G)$.

From faithfully flat descent combined with flat base change (Lemma 7.9), we see that the functors

$$j_* : \mathrm{QCoh}(U/G) \to \mathrm{QCoh}(\mathrm{Spec}(A)/G), \text{ and}$$
$$j^* : \mathrm{QCoh}(\mathrm{Spec}(A)/G) \to \mathrm{QCoh}(U/G)$$

can be computed by equipping the usual pushforward and pullback along $j : U \to \mathrm{Spec}(A)$ with canonical equivariant structures. In particular, $j^* j_* = \mathrm{id}_{\mathrm{QCoh}(U/G)}$, and an object maps to 0 under j^* if and only if it is I-torsion as an A-module. We can summarize this with the following

Lemma 8.12. *The pullback functor* $j^* : A\text{-Mod}(\mathcal{O}_G\text{-Comod}) \to \mathrm{QCoh}(U/G)$ *is an exact localization of abelian categories, and identifies* $\mathrm{QCoh}(U/G)$ *with*

the quotient of the abelian category A-Mod($\mathcal{O}_G$-Comod) by the Serre subcategory (see, e.g., [S4, Tag 02MN]*) of objects M whose underlying A-module is I-torsion.*

Example 8.13. We have already discussed that $\mathbb{P}^n \cong \mathbb{A}^n/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by scaling (this can be seen directly from the functor of points of $\mathbb{P}^n$). We have also seen that $\mathrm{QCoh}(\mathbb{A}^n/\mathbb{G}_m)$ is equivalent to the category of graded modules over the polynomial ring $k[x_1, \ldots, x_n]$, where all x_i have degree 1. One can identify the restriction functor $\mathrm{QCoh}(\mathbb{A}^n/\mathbb{G}_m) \to \mathrm{QCoh}(\mathbb{P}^n)$ with the usual functor from graded modules to quasi-coherent sheaves, as in [H3, Sect. 2.5].

8.2   Recognizing basic quotient stacks

One consequence of the discussion above is that on a quotient of a quasi-affine scheme, every quasi-coherent sheaf is a quotient of a locally free sheaf (see Example 8.9). It turns out that in much greater generality, this gives a criterion for an algebraic stack to be a quotient stack. These ideas go back to [T2] and earlier [EHKV], but the definitive general statement that we give below is from [G1].

D:basic

Definition 8.14. We call an algebraic stack $\mathcal{X}$ *basic* if it is of the form U/GL_n for some quasi-affine scheme U.

D:resolution_property_absolute

Definition 8.15. An algebraic stack $\mathcal{X}$ has the *resolution property* if it is qc.qs. (see Definition 5.4) and there exists a family $\{G_i\}_{i \in I}$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules which are compact[2] as objects of $\mathrm{QCoh}(\mathcal{X})$ and which *generate* in the following sense: every object $M \in \mathrm{QCoh}(\mathcal{X})$ admits a surjection $\bigoplus_{i \in I} G_i^{n_1} \twoheadrightarrow M$ for some $n_i \in \mathbb{Z}_{\geq 0}$.

Theorem 8.16. [G1] *Let $\mathcal{X}$ be a qc.qs. algebraic stack (over $\mathbb{Z}$) whose closed points (or alternatively, geometric points) have affine automorphism groups. Then $\mathcal{X}$ has the resolution property if and only if it has the form $\mathcal{X} \cong U/\mathrm{GL}_n$, where U is a quasi-affine scheme with a GL_n action for some $n \geq 0$.*

Note that a consequence is that any qc.qs. algebraic stack with the resolution property has affine diagonal. [G1] actually has a relative version of this statement:

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[2]Reminder: this means that $\mathrm{Hom}(G_i, -)$ commutes with filtered colimits. In the abelian categories literature, this is also known as "finitely presented," because for any ring R it coincides with the usual notion of finite presentation in the category R-Mod. On a noetherian stack, any locally free sheaf is compact, by **??**.

Definition 8.17. A morphism of algebraic stacks $f : \mathfrak{X} \to \mathcal{Y}$ has the *resolution property* if it is qc.qs. and there exists a family of locally free $\mathcal{O}_X$-modules $\{G_i\}_{i \in I}$ such that each $G_i \in \mathrm{QCoh}(\mathfrak{X})$ is finitely presented, and for any map from an affine scheme $\mathrm{Spec}(R) \to \mathcal{Y}$, the restriction of $\{G_i\}_{i \in I}$ to $\mathfrak{X}_R := \mathfrak{X} \times_{\mathcal{Y}} \mathrm{Spec}(R)$ is a generating family in the sense of Definition 8.15.

The property of having the relative resolution property has nice formal properties and alternate equivalent definitions (see [G1]).

Theorem 8.18. [G1, Thm. 6.10] *Let $\mathfrak{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is qc.qs., with $\mathcal{Y}$ quasi-compact. Then the following conditions are equivalent:*

1. *f has the resolution property, and the relative inertia stack $I_f \to \mathfrak{X}$ has affine fibers (i.e., points have affine relative stabilizer groups).*

2. *f admits a factorization for some $n \geq 0$*

$$\mathfrak{X} \xrightarrow{g} \mathcal{Y} \times B\,\mathrm{GL}_n$$
$$\searrow^{f} \qquad \downarrow$$
$$\mathcal{Y}$$

where g is quasi-affine.

Recall that a locally free sheaf $E \in \mathrm{QCoh}(\mathfrak{X})$ of rank n defines a morphism $\mathfrak{X} \to B\,\mathrm{GL}_n$ classifying the principal GL_n-bundle that we denote

$$P_E := \mathrm{pt} \times_{B\,\mathrm{GL}_n} \mathfrak{X} \cong \mathrm{Isom}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}^n, E). \tag{8.2}$$

If P_E is representable by an algebraic space, then this gives an isomorphism $\mathfrak{X} \cong P_E / \mathrm{GL}_n$. It turns out that this is relatively easy: a locally free sheaf E induces for every point $\xi : \mathrm{Spec}(K) \to \mathfrak{X}$ for some field K a homomorphism of K-groups $\psi_\xi : \mathrm{Aut}_{\mathfrak{X}}(\xi) \to \mathrm{GL}(E_\xi) \cong \mathrm{GL}_{n,K}$. The kernel of ψ_K can be identified with the automorphism groups of points in P_E, so P_E is an algebraic space if and only if ψ_ξ is injective for any such ξ and any K.

Now consider $\xi : \mathrm{Spec}(K) \to \mathfrak{X}$. By hypothesis $G := \mathrm{Aut}_{\mathfrak{X}}(\xi)^{\mathrm{red}}$ is a smooth affine group scheme, so we can choose a linear embedding $G \subset \mathrm{GL}(V)$ corresponding to a locally free sheaf $V \in \mathrm{QCoh}(BG)$. For any surjection $W \twoheadrightarrow V$, the resulting map is also an embedding $G \subset \mathrm{GL}(W)$. Using ξ one can define a canonical morphism of stacks

$$f : BG \to \mathfrak{X}$$

inducing identity on automorphism groups, and this map is quasi-affine (one can show this using a stratification by gerbes, discussed in the next section). By the adjunction $f^* \dashv f_*$, a map of locally free sheaves $\phi : E \to f_*(V)$ corresponds to the map

$$f^*(E) \xrightarrow{f^*(\phi)} f^*(f_*(V)) \to V,$$

and the counit $f^*(f_*(V)) \to V$ is surjective because f is quasi-affine, so if ϕ is surjective then so is the corresponding map $f^*(E) \to V$.

It follows that if one can find a surjection $E \to f_*(V)$, then P_E has trivial automorphism groups for points lying over $\xi \in \mathcal{X}(K)$. From here, one must show a "semi-continuity" result to deduce that the automorphism groups of P_E are trivial in an open neighborhood of this point, which means that after repeating this procedure finitely many times one gets that $P_{E_1 \oplus \cdots \oplus E_k}$ is an algebraic space.

On the other hand, arranging for P_E in (8.2) to be quasi-affine, is harder. Both Totaro and Gross's argument use a version of the following.

T:finite_cover

Theorem 8.19. [R, Thm. B] *Any qc.qs. algebraic space admits a finite, finitely presented surjective morphism from a scheme $Z \to X$ that is flat over a dense open subspace of X.*

Gross uses this to strengthen a classical characterization of quasi-affine schemes (see, for instance, [S4, Tag 01Q3] and [S4, Tag 01QE]) to a criterion for a qc.qs. morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ to be quasi-affine, which is independently interesting: a qc.qs. morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is quasi-affine if and only if

1. $\{\mathcal{O}_X\}$ is a relative generating set in the sense of Definition 8.17, and

2. f has affine relative stabilizer groups at geometric points. (e.g., Δ_f is affine).

Then he uses a slick limit argument: Given a generating family $\{G_i\}_{i \in I}$ as in Definition 8.17, first consider the inverse limit over finite subsets $J \subset I$

$$\mathcal{F} := \lim_{J \subset I} \prod_{i \in J} F(G_i),$$

where $F(G_i)$ denotes the frame bundle $F(G_i) := \mathrm{Isom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}^n, G_i)$, which is affine over $\mathcal{X}$ (the limit of stacks which are affine over $\mathcal{X}$ exists and can be computed as a colimit of corresponding algebras by Lemma 8.7). The generation criterion implies that $\mathcal{F} \to \mathcal{Y}$ is quasi-affine, and this implies that some $\prod_{i \in J} F(\mathcal{V}_i) \to \mathcal{Y}$ had to already be quasi-affine [R, Thm. C].

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Remark 8.20. [T2] also shows that over a field, any basic stack can be written as $\mathrm{Spec}(A)/G$ for some smooth affine k-group G. Thus there are two alternatives: work with affine G-schemes at the expense of considering arbitrary G, or restrict to reductive group schemes at the expense of working with quasi-affine varieties.

Exercise 8.7. *Let $\mathfrak{X}$ be the stack, in fact a scheme, obtained by glueing two copies of $\mathbb{A}^2$ along the open subscheme $\mathbb{A}^2 \setminus \{0\}$. Show that $\mathfrak{X}$ is not basic.*

Exercise 8.8. *Consider the smooth discrete group scheme $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{A}^2}$ over $\mathbb{A}^2$. Let $G \subset (\mathbb{Z}/2\mathbb{Z})_{\mathbb{A}^2}$ be the open complement of the non-identity point over $\{0\}$. Show that G is an open sub-group scheme over $\mathbb{A}^2$ and thus admits a classify stack $BG \to \mathbb{A}^2$. Is BG basic?*

8.3   Stratification by basic quotient stacks

Many algebraic stacks are close to being quotient stacks in the following sense.

Proposition 8.21. [HR1, Prop. 2.6] *Let $\mathfrak{X}$ be a qc.qs. algebraic stack whose points have affine automorphism groups (it suffices to consider only geometric points). Then $\mathfrak{X}$ can be written as a set-theoretic disjoint union $\mathfrak{X} = \bigcup_i \mathcal{Y}_i$ of* reduced *finitely presented locally closed substacks $\mathcal{Y}_i$ which are basic, i.e., $\mathcal{Y}_i \cong U_i/\mathrm{GL}_{n_i}$ for some reduced quasi-affine scheme U_i.*

By "set-theoretic disjoint union" above, we mean that for any field k, any k point of $\mathfrak{X}$ lies in a unique $\mathcal{Y}_i$. This result is useful, for instance, in the theory of motives, where it can be used to describe the Grothendieck ring of algebraic stacks [E]. We will see other applications later in this course.

The proof uses similar ideas to the proof of Theorem 8.18. We will not give the full proof, but let us explain the key ideas in the special case where $\mathfrak{X}$ is noetherian. A key tool is existence of a *stratification by gerbes*.

Definition 8.22. A *gerbe* over a site $\mathcal{C}$ is a stack in groupoids $\mathfrak{X}$ over $\mathcal{C}$ such that any $U \in \mathcal{C}$ admits a covering $\{U_i \to U\}$ such that $\mathfrak{X}(U_i)$ is non-empty for all i, and for any pair $x, y \in \mathfrak{X}(U)$, there is a covering $\{U_i \to U\}$ such that $x|_{U_i} \cong y|_{U_i}$ in $\mathfrak{X}(U_i)$ for all i. A morphism of stacks in groupoids $\mathfrak{X} \to \mathcal{Y}$ over $\mathcal{C}$ is a gerbe if $\mathfrak{X}$ is a gerbe as a stack over $\mathcal{Y}$, for the inherited topology on the category $\mathcal{Y}$.

We will say that an algebraic stack $\mathfrak{X}$ is a gerbe if there is a map to an algebraic space $\mathfrak{X} \to X$ which is a gerbe in the sense of Definition 8.22,

using the fppf topology. If a gerbe $\mathcal{X} \to X$ admits a section $s : X \to \mathcal{X}$, then $\mathrm{Aut}_{\mathcal{X}}(s)$ is a flat and finitely presented group scheme over X and $\mathcal{X} \cong B\,\mathrm{Aut}_{\mathcal{X}}(s)$ over X. By definition any gerbe $\mathcal{X} \to X$ admits a section after passing to an fppf cover of X, so one can think of a gerbe as a morphism which is locally the classifying stack for a group.

A noetherian stack is a gerbe if and only if $I_{\mathcal{X}} \to \mathcal{X}$ is flat by [S4, Tag 06QJ] (because $I_{\mathcal{X}} \to \mathcal{X}$ is automatically finitely presented when $\mathcal{X}$ is noetherian). If $\mathcal{X}$ is reduced, one can use this to find a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe [S4, Tag 06RC]. One can then show inductively that $\mathcal{X}$ admits a stratification by locally closed substacks that are gerbes.

Proof of Proposition 8.21 when $\mathcal{X}$ is noetherian. You can replace $\mathcal{X}$ with its underlying reduced closed substack, i.e. choose a presentation $X_1 \rightrightarrows X_0$ for $\mathcal{X}$ and replace it with $X_1^{\mathrm{red}} \rightrightarrows X_0^{\mathrm{red}}$, and therefore we may assume that $\mathcal{X}$ is reduced. Also, it suffices by noetherian induction to find a dense open substack that is basic.

The idea, in brief:

The automorphism group G of the generic point of any irreducible component admits a faithful representation $G \hookrightarrow (\mathrm{GL}_n)_K$, and it is always the case that the induced morphism $BG \to B(\mathrm{GL}_n)_K$ is quasi-projective, i.e., $(\mathrm{GL}_n)_K/G$ is quasi-projective with linearizable GL_n-action.[3] We can push this locally free sheaf on BG forward to a quasi-coherent sheaf on $\mathcal{X}$, which we may then write as a filtered union of its coherent subsheaves E_α by Proposition 7.12. G will act faithfully on the generic fiber of one of these E_α. On some open neighborhood $\mathcal{U} \subset \mathcal{X}$ of the generic point, E_α is a locally free sheaf (because $\mathcal{X}$ is reduced). This defines a morphism $f : \mathcal{U} \to B\,\mathrm{GL}_n$ which is quasi-projective at the generic point, and we wish to "spread this out" to argue that the morphism is quasi-projective when restricted to some Zariski-open substack of $\mathcal{U}$. This is accomplished using stratification by gerbes and a limit argument.

In more detail:

Let $\xi \in \mathcal{X}(K)$ be a codimension 0 point. Then because $\mathrm{Aut}_{\mathcal{X}}(\xi)$ is an affine K-group, we can choose a faithful finite dimensional representation $V \in \mathrm{QCoh}(B\,\mathrm{Aut}_{\mathcal{X}}(\xi))$. If $f : B\,\mathrm{Aut}_{\mathcal{X}}(\xi) \to \mathcal{X}$ denotes the canonical map

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[3]The correct definition of a quasi-projective morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is that f is representable by schemes and finite type, and there exists an invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ on $\mathcal{X}$ that is relatively ample for f. In particular given a quasi-projective G-scheme X, the morphism $X/G \to BG$ will be quasi-projective in this sense if and only if the action of G on X is linearizable.

inducing the identity on automorphisms groups, then because $\mathcal{X}$ is noetherian we can write $f_*(V) = \bigcup_\alpha E_\alpha$ as a filtered union of coherent subsheaves. It follows (as in the last section) that for some α the map $E_\alpha \to f_*(V)$ and hence the map $f^*(E_\alpha) \to V$ is surjective. This implies that $\operatorname{Aut}_\mathcal{X}(\xi) \to \operatorname{GL}((E_\alpha)_\xi)$ is injective and therefore $B\operatorname{Aut}_\mathcal{X}(\xi) \to B\operatorname{GL}((E_\alpha)_\xi)$ is quasi-projective.

We now use the discussion above to find a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over some algebraic space Y. Because every algebraic space Y admits a maximal open subspace which is actually a scheme, and this maximal open subspace is dense if Y is quasi-separated [S4, Tag 03JG], we can actually find an open substack $\mathcal{U} \subset \mathcal{X}$ containing ξ that is a gerbe over an affine noetherian scheme. Replacing $\mathcal{U}$ with an open substack containing ξ, we may assume that E_α is locally free of some constant rank N, because $\mathcal{X}$ is reduced. Hence E_α defines a morphism $\mathcal{U} \to B\operatorname{GL}_N$.

To summarize, we have found an open substack $\mathcal{U} \subset \mathcal{X}$ containing ξ that is a gerbe over a noetherian affine scheme $\pi : \mathcal{U} \to \operatorname{Spec}(A)$ and a morphism $\mathcal{U} \to B(\operatorname{GL}_N)_A$ over $\operatorname{Spec}(A)$ such that if $p = \pi(\xi) \in \operatorname{Spec}(A)$, then

$$\mathcal{U} \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k(p)) \to B(\operatorname{GL}_N)_{k(p)}$$

is quasi-projective. In fact, we can assume A is integral by restricting to a further open substack. p is a generic point of $\operatorname{Spec}(A)$, so in fact $\operatorname{Spec}(k(p))$ is a co-filtered intersection of all the open subschemes of $\operatorname{Spec}(A)$ containing p. The fact that

$$\varprojlim_{\xi \in U \subset \operatorname{Spec}(A)} (\mathcal{U} \times_{\operatorname{Spec}(A)} U) \to B\operatorname{GL}_N$$

is quasi-projective implies that some morphism in the limit $\mathcal{U} \times_{\operatorname{Spec}(A)} U \to B\operatorname{GL}_N$ is quasi-projective (we will prove this in **??**). This provides an open substack of the form $Z/\operatorname{GL}_N \subset \mathcal{X}$ containing ξ and with Z quasi-projective with linearizable GL_N action. One can get down to a quasi-affineh Repeating this for every generic point of $\mathcal{X}$ gives a dense open substack that is basic. $\qquad\square$

Remark 8.23. The fact that $\mathcal{Y}_i$ are reduced is important in Proposition 8.21. It is not even known whether a stack with affine automorphism groups that just a single point is a quotient stack.

8.4 Appendix: important facts about linear algebraic groups

We will need to use some basic facts about smooth affine k-group varieties at various points. All of what we need, and more, is available in [M3]. For

the reader's convenience, we collect some important facts below:

1. G admits a faithful representation, i.e., an embedding as a closed subgroup scheme $G \hookrightarrow \mathrm{GL}_n$. In fact GL_n admits a finite representation V such that G is the stabilizer of a k-point of $\mathbb{P}(V^*)$, hence GL_n /G is quasi-projective. It can even be arranged that GL_n /G is quasi-affine [T2, Lem. 3.1].

2. Jordan decomposition: for any $g \in G(\bar{k})$, $g = g_{ss} \cdot g_u$, where g_{ss} is semistable and g_u is unipotent in some (and in fact all) linear embeddings, and g_{ss} and g_u commute. This decomposition is functorial under homomorphisms of groups.

3. $\exists!$ maximal unipotent connected normal subgroup $R_u(G) \hookrightarrow G$ called the *unipotent radical*. G is defined to be *reductive* G is smooth over k and $R_u(G_{\bar{k}}) = \{1\}$ as a group scheme. If $k = \bar{k}$, then $G/R_u(G)$ is reductive. Over a field of characteristic 0, reductive is equivalent to being *linearly reductive* which means that the category $\mathrm{QCoh}(BG)$ is semisimple. In characteristic p, a result of Nagata shows that a smooth affine group scheme is reductive if and only if the identity compontent $G^\circ \cong (\mathbb{G}_m^n)_k$ and $|G/G^\circ|$ is prime to p.

4. There exists a torus $T \subset G$ such that $T_{\bar{k}}$ is maximal in $G_{\bar{k}}$, and all maximal tori in $G_{\bar{k}}$ are conjugate. A parabolic subgroup variety $P \subset G$ is one such that G/P is proper, and a *Borel* subgroup $B \subset G$ is a connected solvable parabolic subgroup. If one exists, then G is called quasi-split, and any two Borel subgroups are conjugate over $\bar{k}$.

5. (Matsushima's theorem) If $G \hookrightarrow H$ and H is reductive, then G is reductive if and only if H/G is affine.

6. $G_{\bar{k}}$ is rational, i.e., birationally equivalent to $\mathbb{A}_{\bar{k}}^n$, as a variety.

Lecture 9

Deformation theory

References: [H4] (for an accessible introduction to deformation theory) [M2] (for contangent complex of a map of dg-algebras), [MT, Part. II, Lect. 4] (for some discussion of simplicial commutative rings), [O2] (for some discussion of deformations of morphisms of stacks), [S4, Tag 0162] (For a discussion of some of the simplicial methods which arise in algebraic geometry), [LMB] (for the full classical construction of the cotangent complex of an algebraic stack)
Date: 3/12/2020
Exercises: 7

The next few lectures will build towards a discussion of Artin's criteria for a category fibered in groupoids to be an algebraic stack [A4]. The criteria involve the "infinitesimal deformations" of points of $\mathfrak{X}$. More precisely, we will need to address the question: given a surjective ring homomorphism $A' \to A$ with nilpotent kernel, how does one study the fiber of

$$\mathfrak{X}(\mathrm{Spec}(A')) \to \mathfrak{X}(\mathrm{Spec}(A))$$

over a given morphism $\mathrm{Spec}(A) \to \mathfrak{X}$? This is the basic problem one studies in *deformation theory*. This deformation problem is controlled by a certain complex of coherent A-modules, the *cotangent complex*, and one of Artin's criteria will be the existence of such a complex.

One can easily teach an entire course on deformation theory. Our goal here is to give an overview, sufficient for our uses, and provide references to more complete discussions. We will

1. Introduce the cotangent complex, and discuss it's key formal properties;

2. Discuss the construction of the cotangent complex for a map of affine schemes from a derived perspective; and

3. Compute the cotangent complex of a basic quotient stack (a.k.a., a quotient of a quasi-affine scheme by GL_N).

9.1 The cotangent complex

9.1.1 Some remarks on homological algebra:

When we use cohomological conventions for chain complexes, we will use superscripts, e.g., $C^\bullet = (\cdots \to C^i \to C^{i+1} \to \cdots)$, and when we use homological conventions we use subscripts, e.g., $C_\bullet = (\cdots C_{i+1} \to C_i \to \cdots)$. They are related by defining $C_\bullet = C^{-\bullet}$. For instance, $\tau^{\leq n}(C^\bullet)$ denotes the complex with $H^i = 0$ for $i > n$ and which admits a chain map $\tau^{\leq n}(C^\bullet) \to C^\bullet$ that induces an isomorphism on H^i for $i \leq n$. The n-fold suspension is given by $(C^\bullet[n])^i = C^{n+i}$ and differential $(-1)^n d_C$ with cohomological indexing, and $(C_\bullet[n])_i = C_{i-n}$ with homological indexing.

We have discussed that because $\mathrm{QCoh}(\mathfrak{X})$ is a Grothendieck category for any algebraic stack $\mathfrak{X}$, and complex $C^\bullet$ admits a quasi-isomorphism $C^\bullet \to I^\bullet$, where $I^\bullet$ is K-injective, meaning that the Hom-complex[1] $\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)$ has trivial homology whenever $A^\bullet$ does. A bounded below complex of injectives is K-injective, and in general the map $C^\bullet \to I^\bullet$ plays the role of injective resolutions in the more familiar definition of $D^+(\mathrm{QCoh}(\mathfrak{X}))$ from [H3, Chap. III]. By analogy with the bounded-below case, we construct the unbounded derived category $D(\mathrm{QCoh}(\mathfrak{X}))$ as the homotopy category of K-injective complexes in $\mathrm{QCoh}(\mathfrak{X})$. We can define the derived pushforward $Rf_* : D(\mathrm{QCoh}(\mathfrak{X})) \to D(\mathrm{QCoh}(\mathfrak{X}))$ as the functor which applies f_* to a K-injective complex, and in general applies f_* to a K-injective replacement.

We will define $Lf^* : D(\mathrm{QCoh}(\mathfrak{Y})) \to D(\mathrm{QCoh}(\mathfrak{X}))$ to be the left adjoint of Rf_*. Showing its existence in general requires some methods that we have not discussed, so we refer the reader to [S4, Tag 07BD] for the proof of existence of Lf^* for qc.qs. morphisms, and take its existence as granted from this point forward. When $f : \mathfrak{X} \to \mathfrak{Y}$ is flat, f^* is exact, and so $Lf^* = f^*$ is given by simply pulling back complexes. Using this one can show that Lf^* is right exact, i.e., Lf^* maps $D(\mathrm{QCoh}(\mathfrak{Y}))^{\leq 0}$ to $D(\mathrm{QCoh}(\mathfrak{X}))^{\leq 0}$. When $\mathfrak{Y}$ is basic, the expected construction of Lf^* works, at least for cohomologically

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[1]By definition, for two complexes $C^\bullet, D^\bullet$ in an abelian category, the Hom-complex is defined $\mathrm{Hom}^n(C^\bullet, D^\bullet)$ is the group of homomorphisms of graded objects $\varphi : C^\bullet \to D^\bullet[n]$, with differential $\delta\varphi = d_D \circ \varphi - (-1)^n \varphi \circ d_C$.

bounded above complexes: replace $C^\bullet$ with a quasi-isomorphic complex $P^\bullet$ that is bounded above and whose terms are locally flat (e.g., locally free of possibly infinite rank), then $Lf^*(C^\bullet) \sim f^*(P^\bullet)$.

9.1.2 Formal properties of the cotangent complex

Given an algebraic stack $\mathcal{X}$ over a base scheme X, the *cotangent complex* is a canonical object

$$\mathbb{L}_{\mathcal{X}} \in D(\mathrm{QCoh}(\mathcal{X})).$$

In fact, for any *morphism* of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, there is a canonical relative cotangent complex

$$\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D(\mathrm{QCoh}(\mathcal{X})),$$

which we sometimes denote $\mathbb{L}_f$ if we wish to emphasize the morphism f. The cotangent complex $\mathbb{L}_{\mathcal{X}}$ is just the relative cotangent complex $\mathbb{L}_{\mathcal{X}/S}$ for the structure map $\mathcal{X} \to S$.

The cotangent complex of a morphism $f : \mathcal{X} \to \mathcal{Y}$ has the following properties:

1. *Smallness:* If the morphism f is locally of finite presentation, then $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ is *pseudo-coherent.* By definition this means that for any smooth morphism $\mathrm{Spec}(A) \to \mathcal{X}$, the pullback of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ to $\mathrm{Spec}(A)$ is quasi-isomorphic to a bounded above complex of finite free A-modules. If $\mathcal{X}$ is noetherian, this is equivalent to the condition that $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ has coherent homology sheaves.

2. *Degree bounds:* The relative cotangent complex always lies in cohomological degree ≤ 1, i.e.,

   $$\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D(\mathrm{QCoh}(\mathcal{X}))^{\leq 1}.$$

   If f is representable, then $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D(\mathrm{QCoh}(\mathcal{X}))^{\leq 0}$, and in this case f is unramified (e.g., a locally closed immersion) if and only if $\mathcal{L}_{\mathcal{X}/\mathcal{Y}} \in D(\mathrm{QCoh}(\mathcal{X}))^{\leq -1}$ and f is finite type.

Before continuing with the properties of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$, we must recall the sheaf of Kaehler differentials. Given a ring map $A \to B$, the module of Kaehler

differentials is defined as the quotient of a free B-module on the symbols db for $b \in B$,

$$\Omega_{B/A} := \bigoplus_{b \in B} B \cdot db / \left(\begin{array}{c} d(b_1 + ab_2) - db_1 - a \cdot db_2 \\ d(b_1 b_2) - b_1 \cdot db_2 - b_2 \cdot b_1 \end{array} \middle| \forall a \in A, b_1, b_2 \in B \right).$$

<div align="right">{E:kaehler_differentials} (9.1)</div>

$\Omega_{B/A}$ is the universal B-module admitting an A-linear derivation $d : B \to \Omega_{B/A}$. The key properties are that the formation of $\Omega_{B/A}$ commutes with étale base change in B and arbitrary base change in A. More precisely, for any commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array},$$

the canonical map of B'-modules $B' \otimes_B \Omega_{B/A} \to \Omega_{B'/A'}$ taking $b'_1 \otimes db_2 \mapsto b'_1 \cdot d\phi(b_2)$ is an isomorphism if either 1) the diagram is a tensor product diagram, or 2) $A = A'$ and $B \to B'$ is étale.

For any separated scheme $X \to \mathrm{Spec}(A)$, one can construct a sheaf of relative Kaehler differentials $\Omega_{X/\mathrm{Spec}(A)} \in \mathrm{QCoh}(X)$, by defining $\Omega_{\mathrm{Spec}(B)/\mathrm{Spec}(A)} = \Omega_{B/A}$ for any étale map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and using the canonical base change map to descend this to a quasi-coherent sheaf on X, and the construction of $\Omega_{X/A}$ commutes with arbitrary base change in A. Then one can "bootstrap" this to a definition of $\Omega_{X/\mathrm{Spec}(A)}$ when X is an arbitrary scheme or algebraic space. Finally one can define $\Omega_{\mathcal{X}/\mathcal{Y}}$ for any morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ that is representable by algebraic spaces, using smooth descent to reduce to the case where $\mathcal{Y} = \mathrm{Spec}(A)$.

The next properties of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ are:

3. *Relation to Kaehler differentials:* If $\mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces, there is a canonical morphism $\mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}}$ that induces an isomorphism $H^0(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}) \cong \Omega_{\mathcal{X}/\mathcal{Z}}$, and when $\mathcal{X} \to \mathcal{Z}$ is smooth this map is an isomorphism $\mathbb{L}_{\mathcal{X}/\mathcal{Z}} \cong \Omega_{\mathcal{X}/\mathcal{Z}}$.

4. *Canonical triangle:* Given a composition of morphisms

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z},$$

there is a canonical exact triangle[2] in $D(\mathrm{QCoh}(\mathcal{X}))$

$$Lf^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}) \xrightarrow{Df} \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to .$$

<div align="right">{E:canonical_triangle} (9.2)</div>

---

[2]Some basic introductions to derived functors, such as [H3], omit the notion of an exact

The map Df is natural in the sense that if f and g are morphisms over another algebraic stack $\mathcal{W}$, then the canonical map $D(f \circ f)$ is homotopic to the composition

$$L(g \circ f)^*(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}) \cong Lf^*Lg^*(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}) \xrightarrow{Lf^*(Dg)} Lf^*(\mathbb{L}_{\mathcal{Y}/\mathcal{W}}) \xrightarrow{Df} \mathbb{L}_{\mathcal{X}/\mathcal{W}}.$$

If both f and g are representable by algebraic spaces, then applying $H^0(-)$ to (9.2) gives an exact sequence $H_0(f^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}})) \to H^0(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}) \to H^0(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}) \to$ that agrees under the isomorphism from (3) with the canonical exact sequence

$$f^*\Omega_{\mathcal{Y}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$

Before stating the last property, we will need to recall the following definition: a cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi'} & \mathcal{X} \\ \downarrow{f'} & & \downarrow{f} \\ \mathcal{Y}' & \xrightarrow{\pi} & \mathcal{Y} \end{array} \quad,$$

is *Tor-independent* if after base change along a smooth morphism $\mathrm{Spec}(A) \to \mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}$ admit local smooth covers by affine schemes $\mathrm{Spec}(A')$ and $\mathrm{Spec}(B)$ such that

$$\mathrm{Tor}^i_A(B, A') = 0, \text{ for } i > 0.$$

(This is equivalent to saying that the *derived* tensory product and classical tensor product agree, $A' \otimes^L B = A' \otimes_A B$.) This condition holds in either of the following situations: 1) either π or f is flat, or 2) both are regular closed immersions that meet transversely in $\mathcal{X}$. The last property is:

---

triangle in the derived category. If $\mathcal{A}$ is a Grothendieck category, an exact triangle in $D(\mathcal{A})$ is a pair of maps $A^\bullet \to B^\bullet \to C^\bullet$ that up to quasi-isomorphism can be presented as a short exact sequence of complexes. So $B^\bullet$ is an extension of $C^\bullet$ by $A^\bullet$. The remarkable thing is that in this situation one can construct a complex $\mathrm{Cone}(A^\bullet \to B^\bullet)$ that is canonically quasi-isomorphic to $C^\bullet$, but is an extension of $A^\bullet[1]$ by $B^\bullet$. So we can "rotate" an exact triangle to get a new triangle $B^\bullet \to C^\bullet \to A^\bullet[1]$. Using this one can show that an exact triangle leads to a long exact sequence in homology $\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^b(C^\bullet) \to H^{n+1}(A^\bullet) \to \cdots$.

5. *Tor-independent base change:* Given a Tor-independent cartesian square as above, the canonical maps give a quasi-isomorphism

$$L(\pi')^*(\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}) \oplus L(f')^*(\mathbb{L}_{\mathfrak{Y}'/\mathfrak{Y}}) \xrightarrow{D\pi' \oplus Df'} \mathbb{L}_{\mathfrak{X}'/\mathfrak{Y}}.$$

Note that combined with the canonical exact triangle for the composition $\mathfrak{X}' \to \mathfrak{Y}' \to \mathfrak{Y}$, this implies that the composition of canonical morphisms gives a quasi-isomorphism

$$L(\pi')^*(\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}) \underbrace{\xrightarrow{D\pi'} \mathbb{L}_{\mathfrak{X}'/\mathfrak{Y}} \longrightarrow}_{\cong} \mathbb{L}_{\mathfrak{X}'/\mathfrak{Y}'} \ .$$

We will see in **??** that these formal properties, along with the definition of the cotangent complex for a map of affine schemes, are enough to compute the cotangent complex in many examples.

Exercise 9.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finitely presented, smooth, not necessarily representable morphism of algebraic stacks. Show that $\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}$ has homology in cohomological degree 0 and 1 only, and that it is a perfect complex, meaning it is locally quasi-isomorphic to a finite complex of projective modules.*

Exercise 9.2. *Show that for any morphism of algebraic stacks $f : \mathfrak{X} \to \mathfrak{Y}$, $\mathbb{L}_f \in D(\mathrm{QCoh}(\mathfrak{X}))^{\leq 1}$ using: 1) the fact that $\mathbb{L}_{\mathrm{Spec}(B)/\mathrm{Spec}(A)} \in D(\mathrm{QCoh}(B\text{-Mod}))^{\leq 0}$, which is immediate from the construction below; 2) the canonical exact triangle associated to a composition of morphisms; and 3) the fact that the formation of the relative cotangent complex commutes with smooth base change.*

9.1.3 Deformation theory

Geometrically, the cotangent complex controls deformation theory problems of various kinds. The most basic example is the following:

Let $\mathfrak{Y}$ be an algebraic stack, let A be a ring, and consider a morphism $\xi : \mathrm{Spec}(A) \to \mathfrak{Y}$. We will use the common abbreviation $\mathfrak{Y}(A)$ for $\mathfrak{Y}(\mathrm{Spec}(A))$. Given an A-module I, we give $A \oplus I$ the ring structure in which multiplication by I is 0. The question is what is the fiber of the map of groupoids $\mathfrak{Y}(A \oplus I) \to \mathfrak{Y}(A)$ over the point ξ? The answer is the following formula, which we will explain:

$$\mathfrak{Y}(A \oplus I) \times_{\mathfrak{Y}(A)} \{\xi\} \cong \tau^{\leq 0}(\mathrm{RHom}_{A\text{-Mod}}(L\xi^*(\mathbb{L}_{\mathfrak{Y}}), I)). \qquad \text{\{E:def\_A\}} \tag{9.3}$$

The left-hand-side of (9.3) denotes the homotopy fiber product, i.e. it is the groupoid whose objects are points $\xi' \in \mathcal{Y}(A \oplus I)$ along with an isomorphism $\xi'|_{\mathrm{Spec}(A)} \cong \xi$. Due to the degree bounds on $\mathbb{L}_\mathcal{Y}$, the right-hand-side of (9.3) is a complex of abelian groups with homology in cohomological degree 0 and -1 only. We regard the right-hand-side of (9.3) as a groupoid via the following construction, applied to $E = \tau^{\leq 0}(\mathrm{RHom}_{A\text{-Mod}}(L\xi^*(\mathbb{L}_\mathcal{Y}), I))$:

<div align="right"><code>D:picard_groupoid</code></div>

Construction 9.1 (Picard groupoids)**.** Given an element $E^\bullet \in D(A\text{-Mod})$ with homology in cohomological degree 0 and -1 only, choose an explicit presentation as a two-term complex, say $E^\bullet \sim [C^{-1} \to C^0]$. Then we regard $E^\bullet$ as the quotient groupoid for the action of C^{-1} on C^0 given by $x \cdot y = d(x) + y$. Choosing a quasi-isomorphic two-term complex results in an equivalent groupoid. Note that the set of isomorphism classes is $H^0(E^\bullet)$, and the automorphisms of any object are $H^{-1}(E^\bullet)$.

Remark 9.2. One consequence of Equation (9.3) is that the set of isomorphism classes in $\mathcal{Y}(A \oplus I) \times_{\mathcal{Y}(A)} \{\xi\}$ has the structure of an abelian group. This abelian group structure can be described intrinsically, as we will see in **??** below.

This can be generalized in several directions. First we can replace $\mathrm{Spec}(A)$ with an arbitrary algebraic stack $\mathcal{X}$, and we consider an arbitrary square-zero extension:

Definition 9.3. A *square-zero* extension of an algebraic stack $\mathcal{X}$ by $I \in \mathrm{QCoh}(\mathcal{X})$ is a closed immersion $i : \mathcal{X} \to \mathcal{X}'$ along with an isomorphism

$$i_*(I) \cong \ker(\mathcal{O}_{\mathcal{X}'} \to i_*(\mathcal{O}_\mathcal{X})).$$

Note that the fact that the kernel $\mathcal{O}_{\mathcal{X}'} \to \mathcal{O}_\mathcal{X}$ is the pushforward of a quasi-coherent sheaf on $\mathcal{X}$ implies that $I^2 = 0 \subset \mathcal{O}_{\mathcal{X}'}$, and $\mathcal{X}'$ is a nilpotent thickening of $\mathcal{X}$.

Example 9.4. The *trivial* square-zero extension by $I \in \mathrm{QCoh}(\mathcal{X})$ is $i : \mathcal{X} \hookrightarrow \mathcal{X}' := \mathrm{Spec}_\mathcal{X}(\mathcal{O}_\mathcal{X} \oplus I)$, where $\mathcal{O}_\mathcal{X} \oplus I$ is given the structure of a $\mathcal{O}_\mathcal{X}$-algebra by declaring multiplication by I to be the zero map. Note that in this case there is also a projection $f : \mathcal{X}' \to \mathcal{X}$ such that $f \circ i = \mathrm{id}_\mathcal{X}$, corresponding to the embedding of $\mathcal{O}_\mathcal{X}$-algebras $\mathcal{O}_\mathcal{X} \hookrightarrow \mathcal{O}_\mathcal{X} \oplus I$.

Given a commutative diagram of algebraic stacks of the following form:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ i\ } & \mathcal{X}' \\
& f & \\
& \mathcal{Y} \xrightarrow{\ j\ } \mathcal{Y}' & \\
& & \\
& \mathcal{Z} \longrightarrow \mathcal{Z}' &
\end{array}
\qquad ,
$$

where the horizontal arrows are all square zero extensions. We let $\mathcal{F}$ denote the groupoid of dotted arrows filling this diagram, and we let $I = \ker(\mathcal{O}_{\mathcal{X}'} \to i_*(\mathcal{O}_{\mathcal{X}})) \in \mathrm{QCoh}(\mathcal{X})$.

Theorem 9.5. *There is a canonical obstruction class*

$$
\mathrm{ob}(f,i,j) \in \mathrm{Ext}^1(Lf^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}), I) := \mathrm{Hom}(Lf^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}), I[1])
$$

such that $\mathcal{F}$ is non-empty if and only if $\mathrm{ob}(f,i,j) = 0$. If $\mathcal{F}$ is non-empty, then

$$
\mathcal{F} \cong \tau^{\leq 0}(\mathrm{RHom}(f^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}), I)). \qquad \text{\{E:def\_B\}} \quad (9.4)
$$

This [O2, Thm. 1.5], where it is proved when f is representable, but it is now known that the same holds for arbitrary morphisms. (9.4) is the deformation problem that is most relevant to Artin's criteria.

Exercise 9.3 (Formal smoothness). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth (non necessarily representable) morphism of algebraic stacks, and consider a map $\mathrm{Spec}(A) \to \mathcal{Y}$. Show that for any surjective ring map $A' \to A$ with kernel I such that $I^n = 0$, any commutative diagram of the following form:*

$$
\begin{array}{ccc}
\mathrm{Spec}(A) & \longrightarrow & \mathcal{X} \\
\downarrow & \nearrow & \downarrow f \\
\mathrm{Spec}(A') & \longrightarrow & \mathcal{Y}
\end{array}
\qquad ,
$$

There exists a dotted arrow making the diagram commute. Show by example that this arrow need not be unique. This lifting condition is known as formal smoothness, *and if f is of finite presentation, then it is* equivalent *to f being smooth* [S4, Tag 00TN].

There are other kinds of deformation problems controlled by the cotangent complex, but we will not discuss them in detail:

- Given a morphism $f : \mathcal{X} \to \mathcal{Y}$, the cotangent complex $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ can be used to classify square zero extensions of $\mathcal{X}$ over $\mathcal{Y}$ [O2, Prob. 1], and

- If $\mathcal{X} \to \mathcal{Y}$ is a flat representable morphism, then flat extensions of $\mathcal{X}$ over a given square-zero extension of $\mathcal{Y}$ are classified by $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ [O2, Prob. 2].

9.2 The cotangent complex of a ring map

We will now compute $\mathbb{L}_{\mathrm{Spec}(B)/\mathrm{Spec}(A)}$, which we abbreviate $\mathbb{L}_{B/A}$, for a map of rings $A \to B$. To define it, one needs to use some simplicial methods (we discussed this briefly in **??**). The category of *simplicial A-modules* is the category
$$\mathrm{sMod}_A := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, A\text{-Mod}).$$

It inherits the structure of a symmetric monodial abelian category from A-Mod, where kernels, cokernels, and tensor products are taken level-wise. Given an $M_\bullet \in \mathrm{sMod}_A$, the Moore complex $s(M_\bullet) \in \mathrm{Ch}(A\text{-Mod})_{\geq 0}$ is the chain complex with M_n in homological degree n and differential $d_n = \sum_{i=0}^n d_{n,i} : M_n \to M_{n-1}$. A map $f : M_\bullet \to P_\bullet$ in sMod_A is called a *weak equivalence* if the induced chain map $s(M_\bullet) \to s(P_\bullet)$ is an isomorphism on homology.

The category of *simplicial A-algebras*, which we denote sAlg_A, is the category of commutative algebra objects in sMod_A. This is equivalent to simplicial objects in Alg_A, i.e.,

$$\mathrm{sAlg}_A = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Alg}_A).$$

Concretely, $R_\bullet \in \mathrm{sAlg}_A$ consists of an A-algebra R_n for all $n \geq 0$ along with maps of A-algebras $d_{n,i} : R_n \to R_{n-1}$ and $\sigma_{n,i} : R_n \to R_{n+1}$ satisfying the simplicial identities.

Given $R_\bullet \in \mathrm{sAlg}_R$, we define $R_\bullet$-Mod to be the category of $R_\bullet$-module objects in sMod_A. Concretely, an $M_\bullet \in R_\bullet$-Mod consists of an R_n-modules M_n for all $n \geq 0$, and for any map $\phi : [m] \to [n]$, a map of R_n-modules $\phi^* : M_n \to M_m$, where R_n acts on M_m via $\phi^* : R_n \to R_m$.

We now regard B as a simplicial object where $B_n = B$ and $\phi^* = \mathrm{id}$ for any $\phi : [m] \to [n]$. Giving a map of simplicial A-modules $R_\bullet \to B$ is equivalent to extending $R_\bullet$ to an *augmented* simplicial object $\tilde{R}_\bullet : \mathbf{\Delta}_+ \to \mathrm{Alg}_A$ with $R_{-1} = B$, and this is in turn equivalent to giving an A-algebra map $\alpha : R_0 \to B$ such that $\alpha \circ d_0 = \alpha \circ d_1 : R_1 \to B$. We have a base change functor $B \otimes (-) : R_\bullet$-Mod $\to \mathrm{sMod}_B$ on level n takes $M_n \mapsto B \otimes_{R_n} M_n$.

Definition 9.6. Choose a map of simplicial A-algebras $P_\bullet \to B$ such that 1) $s(P_\bullet) \to s(B) \cong B$ is an isomorphism on homology, and 2) P_n is a polynomial algebra over A on a possibly infinite generating set. Then the *cotangent complex* is

$$\mathbb{L}_{B/A} := s(B \otimes \Omega_{P_\bullet/A}) \in \mathrm{Ch}(B\text{-Mod})_{\geq 0},$$

where $s(-)$ denotes the Moore complex, and $\Omega_{P_\bullet/A} \in P_\bullet$-Mod denotes the $P_\bullet$-module which on level n is the free P_n-modules $\Omega_{P_n/A}$.

There is always a canonical weak equivalence $P_\bullet \to B$ as in Definition 9.6 in which $P_0 = A[B]$ is the free algebra on the underlying set of B and at every level $P_n = A[P_{n-1}]$ [S4, Tag 08PL]. This $P_\bullet$ tends to have enormous entries, so in practice there are methods to construct $P_\bullet$ by hand using "skeleton and coskeleton" functors [S4, Tag 08PX]. Discussing this in detail, and the fact that $\mathbb{L}_{B/A}$ does not depend up to quasi-isomorphism on the choice of $P_\bullet$, is a bit too much of a detour for our purposes, but we refer the reader to [S4, Tag 08P5] for a full account.

9.2.1 Comparision with differential graded algebras

One can often simplify the study of simplicial A-modules using the Dold-Kan correspondence, which is an equivalence of categories

$$\mathbf{N} : \mathrm{sMod}_A \to \mathrm{Ch}(A\text{-Mod})_{\geq 0}\{\texttt{E:normalized\_chains}\}$$
$$\mathbf{N}(M_\bullet)_n = \cap_{i=0}^{n-1} \ker(d_{n,i}) \text{ with differential } d_{n,n}, \tag{9.5}$$

where the latter denotes the abelian category of homologically non-negatively graded chain complexes. $\mathbf{N}(M_\bullet)$ is sometimes referred to as the normalized chain complex associated to $M_\bullet$.

Both sMod_A and $\mathrm{Ch}(A\text{-Mod})_{\geq 0}$ have canonical symmetric monoidal structures, and the functor $\mathbf{N}$ is right-lax symmetric monoidal, meaning there are canonical morphisms $\mathbf{N}(M_\bullet) \otimes \mathbf{N}(P_\bullet) \to \mathbf{N}(M_\bullet \otimes P_\bullet)$ and a map between monoidal units $A \to \mathbf{N}(A)$ satisfying associativity and monoidal unit identities. This implies that $\mathbf{N}$ takes algebra objects of sMod_A, i.e., simplicial A-algebras, to algebra objects of $\mathrm{Ch}(A\text{-Mod})_{\geq 0}$. The latter is the category *commutative differential graded A-algebras*, typically abbreviated CDGA's.

A CDGA over A is a homologically non-negatively graded A-algebra $R_\bullet$ along with a differential $d : R_n \to R_{n-1}$ with $d^2 = 0$. The algebra is required to be graded-commutative, that is $ab = (-1)^{|a||b|} ba$ where $a, b \in R$

are homogeneous of degree $|a|$ and $|b|$ respectively. Also, the differential satisfies the Leibniz identity

$$d(ab) = d(a)b + (-1)^{|a|}ad(b).$$

The fact that $\mathbf{N}$ is right-lax symmetric monoidal also implies that for any $R_\bullet \in \mathrm{sAlg}_A$, $\mathbf{N}$ extends canonically to an equivalence between the corresponding categories of module objects

$$\mathbf{N} : R_\bullet\text{-Mod}(\mathrm{sMod}_A) \to \mathbf{N}(A_\bullet)\text{-Mod}(\mathrm{Ch}(A\text{-Mod})_{\geq 0}).$$

A module over a CDGA $R_\bullet$ is defined to be a graded $R_\bullet$-module $M_\bullet$ along with a differential $d_M : M_n \to M_{n-1}$ satisfying $d^2 = 0$ and such that for homogeneous $a \in R_\bullet$ and $m \in M_\bullet$, $d_M(a{\cdot}m) = d_R(a){\cdot}m + (-1)^{|a|}a \cdots d_M(m)$. A map of CDGA's is a called a *quasi-isomorphism* if it is a quasi-isomorphism on underlying complexes.

9.2.2 The cotangent complex when A has characteristic 0

When A has characteristic 0, the normalized chain complex functor $\mathbf{N}$ of (9.5) is a Quillen equivalence for the standard model category structures on sAlg_A and the category of CDGA's over A. It follows that one can compute the cotangent complex using only CDGA's over A. This is described nicely in [M2].

Given a ring homomorphism $A \to B$, one can regard B as a trivial CDGA over A. To construct the cotangent complex, one chooses a quasi-isomorphism $P_\bullet \to B$ of CDGA's over A such that $P_\bullet = A[\{x_i\}_{i \in I}]$ is a free graded-commutative algebra on some set of homogeneous generators x_i,[3] and such that I admits a filtration $I = \bigcup_{n \geq 0} I_{<n}$ such that $I_{<0} = \emptyset$ and

$$d_P(x_i) = f_i(x) \in A[\{x_i\}_{i \in I_{<n-1}}], \forall i \in I_{<n}.$$

CDGA's of this form are sometimes called *semi-free*, and $P_\bullet \to B$ is called a semi-free resolution of B. As in the simplicial case, one then defines

$$\mathbb{L}_{B/A} = B \otimes_{P_\bullet} \Omega_{P_\bullet/A} \in D(B\text{-Mod}),$$

where $\Omega_{P_\bullet/A}$ denotes the usual module of Kaehler differentials (9.1) for the free algebra $P_\bullet$ over A, but equipped with a grading and differential induced by that of $P_\bullet$.

---

[3]Note that for odd generators $x_i x_j = -x_j x_i$, and thus $x_i^2 = 0$. So, a free graded-commutative algebra is the tensor product of the symmetric algebra on the even generators with the exterior algebra on the odd generators.

More concretely, $\Omega_{P_\bullet/A}$ is the free $P_\bullet$-module with formal generators δx_i of the same degree as x_i. δ can actually be extended uniquely to a degree zero derivation $\delta : P_\bullet \to \Omega_{P_\bullet/A}$. The differential on $\Omega_{P_\bullet/A}$ is determined by the Leibniz rule and the formula

$$d(\delta x_i) = -\delta d_P(x_i) = -\delta f_i = -\sum_{j \in I} (-1)^\# \frac{\partial f_i}{\partial j} \delta x_j$$

The sign in the last expression just keeps track of the number of times two odd symbols were permuted past one another. For instance, $\delta(x_1 x_2 x_3) = (-1)^{|x_1|(|x_2|+|x_3|)} x_2 x_3 \delta x_1 + (-1)^{|x_2||x_3|} x_1 x_3 \delta x_2 + x_1 x_2 \delta x_3$.

Remark 9.7. Let $J \subset I$ be the subset of generators that have degree 0. Then taking the tensor product $B \otimes_{P_\bullet} \Omega_{P_\bullet/A}$ has the effect of assigning $x_i \mapsto 0$ for $i \in I \setminus J$, and $x_j \mapsto b_j \in B$ for $j \in J$. In particular $\mathbb{L}_{B/A}$ is the free graded B-modules on the generators δx_i for $i \in I$, with the differential $d(\delta x_i)$ only depending on those terms in f_i which depend at most *linearly* on the x_k with $k \in I \setminus J$.

A semi-free resolution $P_\bullet \to B$ is easier to compute explicitly for CDGA's than in the simplicial context, assuming for simplicity that A is noetherian and $A \to B$ is finitely generated:

One starts with a surjection $A[x_0, \ldots, x_{n_0}] \twoheadrightarrow B$, where the variables x_i have degree 0. Which is surjective but not injective in degree 0. Then one chooses generators $f_1(x), \ldots, f_{n_1}(x) \in \ker(A[x] \to B)$, and adjoins formal variables $y_0, \ldots, y_{n_1}$ in degree 1 with the differential $dy_i = f_i(x)$ for all i. This results in a map of CDGA's $A[x, y; d] \to B$ which is an isomorphism on H_0 and surjective on H_1.

One then chooses $g_i(x, y) \in A[x, y]_1$ for $i = 0, \ldots, n_2$ that are cycles, i.e., $dg_i = 0$, and that generate $H_1(A[x, y])$. Then one adjoins variables z_i for $i = 0, \ldots, n_2$ of degree 2 with differential $dz_i = g_i$. One can continue this process to arrive at a sequence of semi-free CDGA's between A and B

$$A \to A[x] \to A[x, y; d] \to A[x, y, z; d] \to \cdots \to B.$$

Passing to the colimit, this gives a semi-free resolution of B.

E:quasi-smooth

Exercise 9.4. *Assume A and B have characteristic 0. Let $A \twoheadrightarrow B$ be a surjection whose kernel is generated by a regular sequence $f_1, \ldots, f_n$, i.e. f_i is a non zero-divisor in $A/(f_1, \ldots, f_{i-1})$ for all i. Show that $\mathbb{L}_{B/A}$ is a free module of rank n concentrated in homological degree 1 (hint: investigate the theory of Koszul complexes). Use this to show that if $A \to B$ is a*

complete intersection *ring map, meaning it factors through an isomorphism* $B \cong A[x_1, \ldots, x_m]/I$, *where I is generated by a regular sequence, then* $\mathbb{L}_{B/A}$ *is quasi-isomorphic to the 2-term complex of free B-modules*

$$I/I^2 \xrightarrow{f \mapsto \sum(\partial f/\partial x_i)dx_i} \bigoplus_{i=1}^{m} B \cdot dx_i.$$

Exercise 9.5. *Show the following converse of the previous exercise: if* $\mathbb{L}_{B/A}$ *is quasi-isomorphic to a two-term complex of projective R-modules, then for any surjection $A[x_1, \ldots, x_m] \twoheadrightarrow B$, the kernel I is generated by a regular sequence Zariski-locally on $\mathbb{A}_A^m$, i.e., there are elements $g_1, \ldots, g_k \in A[x_1, \ldots, x_m]$ which generate the unit ideal and such that I_{g_i} is generated by a regular sequence for all $i = 1, \ldots, k$.*

Remark 9.8. In fact, there is a stronger converse: a ring map $A \to B$ is locally a complete intersection if and only if B admits a finite resolution by flat A-modules, and $\mathbb{L}_{B/A}$ admits a presentation as a finite complex of flat B-modules [A5]. Note that this condition implies that $\mathbb{L}_{B/A}$ has bounded homology, but it is a strictly stronger condition of B is not regular.

9.2.3 Connection to deformation theory
S:derived_deformation

One might wonder *why* this is the correct construction of Definition 9.6, when our ultimate goal is to study deformation problems. The most natural answer for this question comes in the context of derived algebraic geometry. This subsection can be skipped by readers who are not interested in derived algebraic geometry. We will be somewhat informal, and assume some familiarity with algebraic topology, and refer the reader to [MT, Part II] for a nice introduction to these ideas.

A derived moduli functor over some fixed base ring R is a functor with values in topological spaces

$$F : \mathrm{sAlg}_R \to \mathrm{Top}$$

that takes weak equivalences of R-algebras to weak homotopy equivalences of topological spaces.[4] In order to see this as a generalization of our previous notion of a moduli functor

$$F : (\mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Spec}(R)})^{\mathrm{op}} \to \mathrm{Gpd},$$

---

[4]In practice, it is technically easier to use the category of simplicial sets sSet instead of Top. This gives an equivalent theory, because sSet is Quillen equivalent to Top for a certain canonical model structure.

we regard sAlg_R as a slight enlargement of opposite category of affine R-schemes, and we regard groupoids as the full subcategory of the weak homotopy category of topological spaces X for which $\pi_n(X, x) = 0$ for any $i > 1$ and any base point $x \in X$. We can restrict to affine schemes because any stack is uniquely determined by its restriction to the category of affine schemes.

For any $B_\bullet \in \mathrm{sAlg}_R$ and $M_\bullet \in B_\bullet$-Mod, one can form the trivial square zero extension $B_\bullet \oplus M_\bullet \in \mathrm{sAlg}_R$, which is the just the usual construction $B_n \oplus M_n$ on each level. Then the natural extension of our basic deformation theory problem to the simplicial context is, given a point $\xi \in F(B_\bullet)$, to compute the *homotopy* fiber

$$\mathrm{Def}_{F,\xi}(M) := \{\xi\} \times^h_{F(B_\bullet)} F(R_\bullet \oplus B_\bullet).$$

Now consider an $A_\bullet \in \mathrm{sAlg}_R$, which in the previous discussion was the constant simplicial R-algebra corresponding to some $A \in \mathrm{Alg}_R$. Then $A_\bullet$ represents a moduli functor

$$F(B_\bullet) := |R\,\mathrm{Map}_{\mathrm{sAlg}_R}(A_\bullet, B_\bullet)_\bullet| \in \mathrm{Top},$$

where we are using a canonical simplicial model structure on sAlg_R to form this mapping space. Without getting into the details, $F(-)$ is a functor that preserves weak equivalences, and such that $\pi_0 F(B_\bullet)$ is canonically the set of maps $A_\bullet \to B_\bullet$ in sAlg_R up to a notion of "weak homotopy equivalence."

In the classical context, given a ring map $\xi : A \to B$ and $M \in B$-Mod, the fiber of $\mathrm{Map}(A, B \oplus M) \to \mathrm{Map}(A, B)$ over ξ, i.e. the set of splittings of $B \oplus M \to B$ as an A-algebra, can be identified with the set of derivations $B \to M$ over A. This, by definition, is in bijection with $\mathrm{Hom}_B(\Omega_{B/A}, M)$. In other words, the sheaf of Kaehler differentials corepresents a functor on B-Mod.

In order to compute the homotopy fiber $\mathrm{Def}_{F,\xi}(M_\bullet)$ for a map $\xi : A_\bullet \to B_\bullet$ and $M_\bullet \in B_\bullet$-Mod, one must choose a weak equivalence $P_\bullet \to B_\bullet$, were P_n is a polynomial A_n-algebra for all n. Then one can show

$$\mathrm{Def}_{F,\xi}(M_\bullet) \sim |R\,\mathrm{Map}_{B_\bullet\text{-Mod}}(B_\bullet \otimes_{P_\bullet} \Omega_{P_\bullet/A_\bullet}, M_\bullet)_\bullet|$$
$$= |R\,\mathrm{Map}_{B_\bullet\text{-Mod}}(\mathbb{L}_{B_\bullet/A_\bullet}, M_\bullet)_\bullet|$$

where as before we are making use of a canonical simplicial model structure on the category $B_\bullet$-Mod. Thus just as for the sheaf of Kaehler differentials, $\mathbb{L}_{B_\bullet/A_\bullet}$ corepresents the functor on the weak homotopy category of $B_\bullet$-Mod which takes $M_\bullet \in B_\bullet$-Mod to the space of trivial square-zero deformations of the ring map $A_\bullet \to B_\bullet$ by $M_\bullet$.

This definition can be generalized to an arbitrary map $F \to F'$ of derived moduli functors $F, F' : \mathrm{sAlg}_R \to \mathrm{Top}$. The slogan is that in derived algebraic geometry, the cotangent complex of a map of stacks $\mathcal{X} \to \mathcal{Y}$ corepresents a functor on $D(\mathrm{QCoh}(\mathcal{X}))_{\geq 0}$, and thus it can be defined in a manner that is independent of any particular construction.

9.2.4   Comparison with the naive cotangent complex

The cotangent complex of a ring map $\alpha : A \to B$ is closely related to the *naive cotanget complex* $\mathbb{NL}(\alpha)$ introduced in (2.1). In fact, we have the following

Lemma 9.9. [S4, Tag 08RB] *Given a ring map $\alpha : A \to B$, there is a canonical quasi-isomorphism $\tau^{\geq -1}(\mathbb{L}_{B/A}) \sim \mathbb{NL}(\alpha)$, thus $H^i(\mathbb{L}_{B/A}) \cong H^i(\mathbb{NL}(\alpha))$ for $i = 0, -1$.*

In particular, we have seen that $\alpha : A \to B$ is smooth (respectively étale) if and only if $H^{-1}(\mathbb{NL}(\alpha)) = 0$ and $H^0(\mathbb{NL}(\alpha))$ is a projective B-module (respectively, $H^0(\mathbb{NL}(\alpha)) = 0$), so the same criterion can be used to detect smooth and étale ring maps using $\mathbb{L}_{B/A}$.

9.3   The cotangent complex of a quotient stack

Finally, we will put our discussion in the previous section together to compute the cotangent complex of any basic algebraic stack, i.e., a stack of the form U/GL_n for some quasi-affine scheme U. For simplicity we assume $S = \mathrm{Spec}(R)$ is affine.

Example 9.10 (The cotangent complex of BG). We will compute the cotangent complex of $BG \to \mathrm{Spec}(R)$ for a smooth R-group G by considering the diagonal morphism $\Delta : BG \to BG \times BG$. Let $p_1, p_2 : BG \times BG \to BG$ denote projection onto the first and second factor. Because $BG \to \mathrm{Spec}(R)$ is flat, Tor-independent base change gives an isomorphism

$$p_1^*(\mathbb{L}_{BG}) \oplus p_2^*(\mathbb{L}_{BG}) \xrightarrow{Dp_1 \oplus Dp_2} \mathbb{L}_{BG \times BG}.$$

Note that $\mathrm{id}_{BG} \cong p_1 \circ \Delta \cong p_2 \circ \Delta$, so $L\Delta^*(p_i^*(\mathbb{L}_{BG})) \cong \mathbb{L}_{BG}$. The naturality of the derivative map implies that the composition

$$\mathbb{L}_{BG} \oplus \mathbb{L}_{BG} \xrightarrow[\cong]{L\Delta^* \circ (p_1^* \oplus p_2^*)} L\Delta^*(\mathbb{L}_{BG \times BG}) \xrightarrow{D\Delta} \mathbb{L}_B G$$

is homotopic to the map $\mathrm{id} \oplus \mathrm{id} : \mathbb{L}_{BG} \oplus \mathbb{L}_{BG} \to \mathbb{L}_{BG}$. Therefore the canonical exact triangle associated to the composition $BG \to BG \times BG \to \mathrm{Spec}(R)$ is equivalent to

$$\mathbb{L}_{BG} \oplus \mathbb{L}_{BG} \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathbb{L}_{BG} \to \mathbb{L}_\Delta.$$

This induces an isomorphism

$$\mathbb{L}_\Delta \sim \mathrm{Cone}\left(L\Delta^*(\mathbb{L}_{BG \times BG}) \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathbb{L}_{BG}\right) \sim \mathbb{L}_{BG}[1].$$

So we have $\mathbb{L}_{BG} \sim \mathbb{L}_\Delta[-1]$. The advantage of this description is that Δ is representable and in fact smooth, so by hypothesis $\mathbb{L}_\Delta$ is just the relative Kaehler differentials by property (3) of the cotangent complex.

We compute this explicitly as follows: Let $G \times G$ act on G by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$. Then the identity section $e : \mathrm{Spec}(R) \hookrightarrow G$ is equivariant with respect to the diagonal embedding of groups $G \to G \times G$, and the associated map $\psi : BG \to G/(G \times G)$ is an equivalence by Lemma 7.4. Furthermore, Δ factors as the equivalence ψ followed by the $G \times G$-equivariant projection

$$G/G \times G \to \mathrm{pt}/(G \times G) \cong BG \times BG.$$

The morphism $G \to \mathrm{pt}$ is smooth, as mentioned above, so $\mathbb{L}_\Delta \cong \Omega_{BG/(BG \times BG)}$. Although we do not have smooth descent for general complexes, we do have smooth descent for quasi-coherent sheaves. Using this it is not hard to identify $\Omega_{BG/(BG \times BG)}$ as the locally free sheaf $\Omega_{G/R}$ equipped with its canonical $G \times G$-equivariant structure. Under the equivalence $\psi^* : \mathrm{QCoh}(\mathrm{pt}/G) \cong \mathrm{QCoh}(G/(G \times G))$, this corresponds to the locally free sheaf

$$\mathfrak{g}^* := e^*(\Omega_{G/R}) \in \mathrm{QCoh}(\mathrm{Spec}(R)),$$

along with the canonical G equivariant structure induced by conjugation (which fixes the identity section e). We have used the notation $\mathfrak{g}^*$ because when k is a field, this G-representation is dual to the adjoint representation. It follows that $\mathbb{L}_{BG} \sim \mathfrak{g}^*[-1]$ with the G-equivariant structure induced by conjugation.

Remark 9.11. The first part of the previous argument applies to any algebraic stack that is flat over a ring R, and gives a quasi-isomorphism $\mathbb{L}_{\mathcal{X}} \cong \mathbb{L}_\Delta[-1]$ where $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal.

To go further, we need one additional fact: if G is a smooth closed subgroup of $B(\mathrm{GL}_n)_R$ and $A \to B$ is a morphism in $\mathrm{Alg}(\mathrm{Rep}(G))$, then the relative cotangent complex of the representable morphism $f : \mathrm{Spec}(B)/G \to \mathrm{Spec}(A)/G$ is the cotangent complex $s(B \otimes \Omega_{P_\bullet/A})$ as in Definition 9.6, but

where $P_\bullet \to B$ is a map of simplicial algebra objects in $\mathrm{Rep}(G)$ whose underlying map in sAlg_R is a weak-equivalence. In characteristic 0, one can likewise replace B with a quasi-isomorphic semi-free CDGA $P_\bullet : A[U_0, U_1, \ldots; d] \to B$ where each U_i is a direct sum of locally free G-modules concentrated in homological degree i, then

$$\mathbb{L}_f \sim B \otimes_{P_\bullet} \Omega_{P_\bullet/A} \in D(B\text{-Mod}(\mathrm{Rep}(G))),$$

which is just $\mathbb{L}_{B/A}$ with its induced structure of a G-representation. This almost follows from the flat base along $\mathrm{Spec}(A) \to \mathrm{Spec}(A)/G$, but we have not developed sufficient machinery for faithfully flat descent of complexes.

Example 9.12 (A quotient of an affine scheme). Let $\mathfrak{X} = \mathrm{Spec}(A)/G$. Choose a weak equivalence $P_\bullet \to A$ of G-equivariant simplicial commutative R-algebras in which each $P_n \cong R[U_n]$ for a G-representation U_n that is a direct sum of projective R-modules. Alternatively, when R has characteristic 0, $P_\bullet$ can be a G-equivariant semifree CDGA of the form $R[U_0, U_1, \ldots; d]$ where each U_i is a direct sum of G-equivariant projective R-modules. As we just discussed, $\mathbb{L}_{\mathfrak{X}/BG} \cong s(A \otimes_{P_\bullet} \Omega_{P_\bullet/R}) \in A\text{-Mod}(\mathrm{Rep}(G))$. The canonical exact triangle for the composition $\mathfrak{X} \to BG \to \mathrm{Spec}(R)$ gives an equivalence

$$\mathbb{L}_{\mathfrak{X}}[1] \cong \mathrm{Cone}(\mathbb{L}_{\mathfrak{X}/BG} \to \mathbb{L}_{BG}[1]). \tag{9.6}$$

We have already computed that $\mathbb{L}_{BG}[1] \cong \mathfrak{g}^*$, and by property (2) of the cotangent complex $\mathbb{L}_{\mathfrak{X}/BG} \in D(\mathrm{QCoh}(\mathfrak{X}))^{\leq 0}$, so the map $\mathbb{L}_{\mathbb{X}/BG} \to \mathbb{L}_{BG}[1]$ factors uniquely through the canonical map $\mathbb{L}_{\mathfrak{X}/BG} \to \Omega_{\mathfrak{X}/BG}$ that induces an isomorphism of H^0. More concretely, we have an explicit presentation

$$\mathbb{L}_{\mathfrak{X}/BG} \cong (\cdots \to A \otimes_{P_2} \Omega_{P_2/R} \to A \otimes_{P_1} \Omega_{P_1/R} \to A \otimes_{P_0} \Omega_{P_0/R}),$$

where the differentials are induced by the alternating sum of face maps. Combining this with (9.6), and identifying $\Omega_{P_n/R} \cong P_n \otimes_R U_n$, we see that

$$\mathbb{L}_{\mathfrak{X}} \cong (\cdots \to A \otimes_R U_2 \to A \otimes_R U_1 \to A \otimes_R U_0 \xrightarrow{\beta} A \otimes_R \mathfrak{g}^*),$$

where $\mathfrak{g}^*$ lies in cohomological degree 1. The map $\beta : A \otimes_R U_0 \to A \otimes_R \mathfrak{g}^*$ factors uniquely through the canonical surjection $A \otimes_{P_0} \Omega_{P_0/R} \to \Omega_{A/R}$. The resulting map $\alpha_A : \Omega_{A/R} \to A \otimes \mathfrak{g}^*$ is the "infinitesimal coaction" of G on $\mathrm{Spec}(A)$. Considering the canonical triangle for the composition $\mathrm{Spec}(A)/G \to \mathrm{Spec}(P_0)/G \to BG$, one can show that the map β above is the restriction of the infinitesimal coaction $\alpha_{P_0} : \Omega_{P_0/R} \to P_0 \otimes_R \mathfrak{g}^*$ for the action of G on $\mathrm{Spec}(P_0)$ along the map $\mathrm{Spec}(B) \to \mathrm{Spec}(P_0)$.

Exercise 9.6. *Let $R = k$ be a field, and let A be a finite type G-equivariant k-algebra, i.e., $A \in \mathrm{Alg}(\mathrm{Rep}(G))$. Complete the description of $\mathbb{L}_{\mathrm{Spec}(A)/G}$ in the previous example by describing the infinitesimal coaction map $\alpha_A : \Omega_{A/k} \to A \otimes_k \mathfrak{g}^*$ explicitly in terms of the coaction of G on A. (Hint: we saw that α_A is determined uniquely the infinitesimal coaction $\alpha_{P_0} : \Omega_{P_0/k} \to P_0 \otimes_k \mathfrak{g}^*$, where $P_0 = k[U_0] \twoheadrightarrow A$ is a surjection from a free G-equivariant k-algebra on some $U_0 \in \mathrm{Rep}(G)$. The latter can be computed using the formal properties of the cotangent complex.)*

Example 9.13 (The case of a basic quotient stack). Now consider a stack $\mathcal{X}$ over R which admits an open immersion $j : \mathcal{X} \hookrightarrow \mathrm{Spec}(A)/G$ for some smooth R-group scheme G and G-equivariant R-algebra $A \in \mathrm{Alg}(\mathrm{Rep}(G))$. We claim that $\mathbb{L}_j = 0$ because j is étale. This follows from smooth base change for $\mathbb{L}_j$, which allows one to reduce to the case of an open subscheme of an affine scheme, along with the comparison with the naive cotangent complex in the affine case, Lemma 9.9. It follows that the exact triangle for a composition of morphisms gives a canonical equivalence

$$j^*(\mathbb{L}_{\mathrm{Spec}(A)/G}) \sim \mathbb{L}_{\mathcal{X}}.$$

Exercise 9.7. *Let $X \hookrightarrow \mathbb{P}^n$ be a hypersurface defined by the vanishing of a degree d homogeneous polynomial $f(x_0, \ldots, x_n)$. Use the methods of this section to compute $\mathbb{L}_X$.*

EX:bung_def

Exercise 9.8. *Let $X \to S = \mathrm{Spec}(R)$ be a flat projective morphism of schemes, and let G be smooth group scheme over S. Consider the moduli functor which assigns to $T \in \mathrm{Sch}_{/S}$*

$$\mathrm{Bun}_G(X)(T) := \mathrm{Map}_S(X_T, BG) = \{\text{principal } G\text{-bundles on } X_T\}.$$

We can associate for any $\xi : \mathrm{Spec}(A) \to \mathcal{X}$, corresponding to a morphism $f : X_A \to BG$, a deformation functor $\mathrm{Def}_\xi : A\text{-Mod} \to \mathrm{Gpd}$ defined by

$$\mathrm{Def}_\xi(M) = \{\xi\} \times_{\mathcal{X}(A)} \mathcal{X}(A \oplus M).$$

Show that

$$\mathrm{Def}_\xi(M) \cong \tau^{\leq 0}(R\Gamma(X_A, f^*(\mathfrak{g}^*)[-1])),$$

where $f^*(\mathfrak{g}^*)$ is the coadjoint bundle associated to the principal G-bundle classified by f.

Lecture 10

Approximation theorems

References: [R], [TT], [G2]
Date: 3/19/2020
Exercises: 4

A typical question in moduli theory is the following: given a moduli problem $\mathfrak{X}$, an integral domain A with fraction field K, and a family of objects in $\mathfrak{X}$ over $\mathrm{Spec}(K)$, can one extend this to a family over some open subscheme of $\mathrm{Spec}(A)$? For example, can one always extend a principal G-bundle over $\mathrm{Spec}(K)$ over an open subscheme of $\mathrm{Spec}(A)$?

We will briefly survey some important "approximation theorems" used to address questions like this. First we discuss the notion of finite presentation and compactness, which plays a central role in Artin's criteria below. Then we discuss the "standard" relative approximation theorems of [G2, §. 8], and the absolute approximation theorems of [TT, App. C] and [R].

10.1   Stacks of finite presentation

Let R be a ring, and let Alg_R denote the category of R-algebras.

Exercise 10.1. *Show that if $\{B_i\}_{i \in I}$ is a diagram in Alg_R indexed by a filtered category I, then $\mathrm{colim}_{i \in I} B_i$ exists and is given by the colimit of underlying sets, along with its induced R-algebra structure.*

In any category $\mathcal{C}$ that admits filtered colimits, one defines an object $X \in \mathcal{C}$ to be *compact* if $\mathrm{Map}_{\mathcal{C}}(X, -)$ commutes with filtered colimits. Our first observation is that the condition of finite presentation for an R-algebra coincides with the categorical condition of compactness in Alg_R. (For this

reason, in certain categories the terms compact and finitely presented are used synonymously.)

Lemma 10.1. *$A \in \mathrm{Alg}_R$ is of finite presentation over R if and only if for any filtered colimit $B = \mathrm{colim}_{i \in I} B_i$, the canonical map gives a bijection*

$$\mathrm{colim}_{i \in I} \mathrm{Map}_{\mathrm{Alg}_R}(A, B_i) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, B). \tag{10.1}$$

Proof. First assume A is finitely presented, i.e., $A \cong R[a_1, \ldots, a_n]/(r_1, \ldots, r_m)$. Injectivity of (10.1) means that two maps $f_0, f_1 : A \to B_i$ induce the same map to B if and only if there is some $\phi : i \to j$ such that $\phi_* \circ f_0 = \phi_* \circ f_1 : A \to B_j$. This is the case because the map $A \to B$ is determined uniquely by where it takes a generating set $a_1, \ldots a_n$ of A, and the underlying set of B is a colimit of the sets B_i.

Surjectivity of (10.1) means that any map $\phi : A \to B$ factors through some $B_i \to B$. Because the underlying set of B is the colimit of the B_i, we can find an i large enough such that $\phi(a_1), \ldots, \phi(a_n)$ lift to B_i, and the resulting map $\phi_i : R[a_1, \ldots, a_n] \to B_i$ annihilates $r_1, \ldots, r_m$, and thus ϕ_i factors uniquely through a map $A \to B_i$ whose composition with $B_i \to B$ is ϕ.

Conversely, we assume that (10.1) is bijective for any filtered system $\{B_i\}$, and we wish to show that A is finitely presented. We first write A as a filtered union of its finitely generated subalgebras $B_i \subset A$, so $A = B = \mathrm{colim}_i B_i$. (10.1) implies that $\mathrm{id} : A \to A$ factors through B_i for some i, and hence $A = B_i$ is finitely generated. Thus we have an isomorphism $A \cong R[a_1, \ldots, a_n]/I$ for some ideal I.

Now we must show that I is finitely generated. Write $I = \bigcup_i I_i$ as the filtered union of its finitely generated $R[a_1, \ldots, a_n]$-submodules. This defines a filtered system $B_i = R[a_1, \ldots, a_n]/I_i$ in which any map $B_i \to B_j$ for $I_i \subset I_j$ is surjective. Once again $A = \mathrm{colim}_i B_i$, and the fact that $\mathrm{id} : A \to A$ factors through some B_i implies that $\pi : B_i \to A$ admits an R-algebra map $s : A \to B_i$ with $\pi s = \mathrm{id}_A$ for some i. It is not hard to show that if B is finitely generated when regarded as an A-algebra via s, with generators $y_1, \ldots, y_k \in B$, then $y_j - s(\pi(y_j))$ for $j = 1, \ldots, k$ generate $\ker(\pi)$. But it is clear that the image of $a_1, \ldots, a_n$ generate B as an A-algebra, and hence $\pi : B_i \to A$ is finitely presented. This implies that A is finitely presented over R. $\qquad\square$

Remark 10.2. To conclude that A was finitely presented, we only needed (10.1) to be surjective.

We now translate this into the language of schemes. If $B = \text{colim}_{i \in I} B_i$ is a filtered colimit in Alg_R, then $\text{Spec}(B) = \varprojlim_{i \in I^{\text{op}}} \text{Spec}(B_i)$ is a limit in R-schemes, because $\text{Spec} : \text{Alg}_R^{op} \to \text{Sch}_R$ is a right adjoint. We say that the category I^{op} is *cofiltered* because its opposite category is filtered, and refer to $\text{Spec}(B)$ as a cofiltered limit of the $\text{Spec}(B_i)$. If $F(-) = \text{Map}_{\text{Sch}_{/R}}(-, \text{Spec}(A))$, then the previous lemma says that A is finitely presented if and only if $F(-)$ maps cofiltered limits of affine schemes to filtered colimits of sets. This is the natural extension of the notion of compactness to the category of presheaves.

D:limit_preserving

Definition 10.3. A stack in groupoids $\mathcal{X}$ over $\text{Sch}_{/S}$ is *limit-preserving* if for any cofiltered limit $T = \varprojlim_i T_i$ of affine schemes over S, the canonical functor

$$\text{colim}_i \mathcal{X}(T_i) \to \mathcal{X}(T)$$

is an equivalence of categories. A morphism of stacks in groupoids $\mathcal{X} \to \mathcal{Y}$ is *limit-preserving* if for any cofiltered limit $T = \varprojlim_i T_i$ of affine schemes over S, the canonical commutative square

{E:limit_preserving_diagram}

$$\begin{array}{ccc} \text{colim}_i \mathcal{X}(T_i) & \longrightarrow & \mathcal{X}(T) \\ \downarrow f & & \downarrow f \\ \text{colim}_i \mathcal{Y}(T_i) & \longrightarrow & \mathcal{Y}(T) \end{array} \qquad (10.2)$$

is cartesian.

Remark 10.4. The filtered colimit of categories requires some explanation: given a filtered system $\{\mathcal{C}_i\}_{i \in I}$ of categories and functors $\phi_{i \to j} : \mathcal{C}_i \to \mathcal{C}_j$, the set of objects of $\mathcal{C} := \text{colim}_i \mathcal{C}_i$ is the disjoint union of the objects in all $\mathcal{C}_i$. To define maps between $X \in \mathcal{C}_i$ and $Y \in \mathcal{C}_j$, choose maps $i \to k$ and $j \to k$ for some $k \in I$, and let $X', Y' \in \mathcal{C}_k$ denote the image of X and Y respectively. Then

$$\text{Map}_{\mathcal{C}}(X, Y) := \text{colim}_{l \in (k/I)} \text{Map}_{\mathcal{C}_l}(\phi_{k \to l}(X'), \phi_{k \to l}(Y')).$$

Recall that because the condition of a morphism of schemes being locally of finite presentation is smooth local on the source and target, we define a morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ to be locally of finite presentation if for any smooth morphism $\text{Spec}(A) \to \mathcal{Y}$, the base change $\mathcal{X}_A$ admits a smooth surjective morphism from a scheme that is locally finitely presented over $\text{Spec}(A)$.

Proposition 10.5. [S4, Tag 0CMY] P:fp_limit_preserving *[A morphism of algebraic stacks $\mathfrak{X} \to \mathcal{Y}$ is locally of finite presentation if and only if it is limit preserving.*

Remark 10.6. In fact, a weaker condition called "limit preserving on objects" [S4, Tag 06CT] suffices to conclude that $\mathfrak{X} \to \mathcal{Y}$ is locally of finite presentation. This condition states that rather than the commutative square (10.2) being cartesian, the canonical functor

$$\mathrm{colim}_i \, \mathfrak{X}(T_i) \to \mathfrak{X}(T) \times_{\mathcal{Y}(T)} \mathrm{colim}_i \, \mathcal{Y}(T_i)$$

is essentially surjective.

We see therefore that "limit preserving" is a reasonable generalization of "locally of finite presentation" for a stack which we don't know to be algebraic.

In fact, maps to a finitely presented stack preserve other kinds of limits. Let $\mathcal{Z}$ be an algebraic stack, and let $\{\mathfrak{X}_i \to \mathcal{Z}\}_{i \in I}$ be a cofiltered system of algebraic stacks over $\mathcal{Z}$ such that for any $i \to j$ in I, the associated "bonding map" $\mathfrak{X}_i \to \mathfrak{X}_j$ is representable and affine.

Lemma 10.7. *The inverse limit $\varprojlim_i \mathfrak{X}_i$ exists, and the morphism $\mathfrak{X} \to \mathfrak{X}_i$ is affine for all i.*

Proof. Without loss of generality, you can assume that I has an initial object 0, because for any i in a cofiltered category, $(i/I^{\mathrm{op}}) \to I^{\mathrm{op}}$ is cofinal. In this case, every stack is affine over $\mathfrak{X}_0$, so $\mathfrak{X}_i = \mathrm{Spec}_{\mathfrak{X}_0}(\mathcal{A}_i)$ for some $\mathcal{A}_i \in \mathrm{Alg}(\mathrm{QCoh}(\mathfrak{X}_0))$. Then the limit is just $\mathrm{Spec}_{X_0}(\mathrm{colim}_i \mathcal{A}_i)$ as in the case of affine schemes (see Lemma 8.7). $\qquad\square$

P:finite_presentation_1

Proposition 10.8. *Let $\{\mathfrak{X}_i \to \mathcal{Z}\}_{i \in I}$ be a cofiltered system of algebraic stacks over an algebraic stack $\mathcal{Z}$ such that all bonding maps $\mathfrak{X}_i \to \mathfrak{X}_j$ are affine and every $\mathfrak{X}_i$ is qc.qs.. Then if $\mathfrak{X} := \varprojlim_i \mathfrak{X}_i$ and $\mathcal{Y} \to \mathcal{Z}$ is a morphism of algebraic stacks that is locally of finite presentation, the canonical functor is an equivalence*

$$\mathrm{colim}_i \, \mathrm{Map}_{\mathcal{Z}}(\mathfrak{X}_i, \mathcal{Y}) \to \mathrm{Map}_{\mathcal{Z}}(\mathfrak{X}, \mathcal{Y}).$$

Proof idea. $\mathfrak{X}$ lies in the smallest full subcategory of stacks over $\mathcal{Z}$ that contains affine schemes over $\mathcal{Z}$ and is closed under finite colimits. The result holds for affine schemes by definition. The result is also closed under the formation of finite colimits in the category of algebraic stacks over $\mathcal{Z}$, because filtered colimits commute with finite limits. $\qquad\square$

Example 10.9 (Suggested by Andres Fernandez Herrero). Let C be a smooth curve over a field k, and let $\mathcal{E}$ be a locally free sheaf of rank n on C. We say that a *filtration* of $\mathcal{E}$ is a filtration of $\mathcal{E}$ as a coherent sheaf $\mathcal{E} = \mathcal{E}_0 \supsetneq \mathcal{E}_1 \supsetneq \cdots \supsetneq \mathcal{E}_p \supsetneq 0$ such that $\mathrm{gr}_p(\mathcal{E}_\bullet) = \mathcal{E}_p/\mathcal{E}_{p+1}$ is also locally free. How does one study filtrations of $\mathcal{E}$? Any filtration of $\mathcal{E}$ induces a filtration of the restriction to the generic point $\mathcal{E}_{k(C)}$, which is just a vector space over the function field $k(C)$. We can use Proposition 10.8 to show that this gives a bijection between filtrations of $\mathcal{E}$ and filtrations of the $k(C)$-vector space $\mathcal{E}_{k(C)}$.

Proof. Recall that $\mathcal{E}$ corresponds to the principal GL_n-bundle $P := \mathrm{Isom}_C(\mathcal{O}^n, \mathcal{E})$ over C. If $n_p := \mathrm{rank}(\mathrm{gr}_p(\mathcal{E}_\bullet))$, then we let $P \subset G$ be the parabloic subgroup of block upper triangular matrices with block sizes $n_0 \times n_0, \ldots, n_p \times n_p$ along the diagonal. The data of a filtration of $\mathcal{E}$ is equivalent to a reduction of structure group of P from GL_n to P, which is the same as a section of the associated bundle $\pi : P \times_{\mathrm{GL}_n} (\mathrm{GL}_n /P) \to C$. By separatedness, a section is determined by its restriction to the generic point, i.e., the corresponding filtration of $\mathcal{E}_{k(C)}$. On the other hand,

$$k(C) = \mathrm{colim}_{\{p_1,\ldots,p_m\} \subset C} \mathcal{O}_{C \setminus \{p_1,\ldots,p_m\}}$$

where the colimit is taken over the filtered system of all proper closed subsets of C. By Proposition 10.8 that means that any section over $\mathrm{Spec}(k(C))$ extends to a section over $C \setminus \{p_1, \ldots, p_m\}$ for some proper finite subset of C. Then, because $P \times^{\mathrm{GL}_n} (\mathrm{GL}_n /P) \to C$ is proper, the valuative criterion for properness allows one to extend this section uniquely to a section over C. $\qquad\square$

Exercise 10.2. *We will see below that the moduli functor of Example 4.4 is an algebraic stack locally of finite presentation over the base S, but for the moment take this for granted. Let k be a field, let G be a smooth k-group-scheme, and let X be a flat projective G-scheme over $k[\![x]\!]$ with a G-equivariant ample invertible sheaf $\mathcal{L}$. Show that there is a finitely generated subalgebra $R \subset k[\![x]\!]$ and a flat projective G-scheme Y over R along with G equivariant invertible sheaf $\mathcal{M}$ such that there is a G-equivariant morphism $\phi : X \to Y$ that induces an isomorphism $X \cong Y_{k[\![t]\!]}$ and there is an isomorphism $\mathcal{L} \cong \phi^*(\mathcal{M})$.*

Exercise 10.3. *Let G be a smooth S-group-scheme, let $X \to S$ be a flat proper S-scheme, and consider the moduli functor $\mathrm{Bun}_G(X)$ of Exercise 9.8. Show that $\mathrm{Bun}_G(X)$ is locally of finite presentation over S.*

10.2 Relative approximation theorems

Let $R = \mathrm{colim}_i R_i$ be a filtered colimit of rings, then for any $R_i \to R_j$ the base change functors $(-) \otimes_{R_i} R_j$ induce functors on finitely presented objects $\mathrm{Alg}^{\mathrm{fp}}_{R_i} \to \mathrm{Alg}^{\mathrm{fp}}_{R_j}$. The first example of a "relative approximation" theorem says that the functor $\mathcal{F}(R) = \mathrm{Alg}_R$ is itself locally of finite presentation, i.e., the canonical functor

$$\mathrm{colim}_i \mathrm{Alg}^{\mathrm{fp}}_{R_i} \to \mathrm{Alg}^{\mathrm{fp}}_R$$

is an equivalence of categories. More precisely, we have

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L:finitely_presented_algebras_2
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Lemma 10.10. *Any finitely presentable $A \in \mathrm{Alg}_R$ is of the form $A_i \otimes_{R_i} R$ for some i and $A_i \in \mathrm{Alg}^{\mathrm{fp}}_{R_i}$, and for any $A_i, B_i \in \mathrm{Alg}_{R_i}$ with A_i finitely presentable, the canonical map is a bijection*

$$\mathrm{colim}_{j \in (i/I)} \mathrm{Map}_{R_j}(A_i \otimes_{R_i} R_j, B_i \otimes_{R_i} R_j) \to \mathrm{Map}_R(A_i \otimes_{R_i} R, B_i \otimes_{R_i} R).$$

The analogous claim holds for the category of finitely presented modules, i.e., any finitely presented $M \in R\text{-Mod}$ is of the form $M_i \otimes_{R_i} R$ for some finitely presented $M_i \in R_i\text{-Mod}$, and $\mathrm{Hom}_R(M_i \otimes_{R_i} R, N_i \otimes_{R_i} R) \cong \mathrm{colim}_i \mathrm{Hom}_{R_j}(M_i \otimes_{R_i} R_j, N_i \otimes_{R_i} R_j)$ for any other $N_i \in R_i\text{-Mod}$.

Proof. The description of the set or R algebra maps follows by the adjunction $\mathrm{Map}_{R_j}(A_i \otimes_{R_i} R_j, B_i \otimes_{R_i} R_j) \cong \mathrm{Map}_{R_i}(A_i, B_i \otimes_{R_i} R_j)$ and $\mathrm{Map}_R(A_i \otimes_{R_i} R, B_i \otimes_{R_i} R) \cong \mathrm{Map}_{R_i}(A_i, B_i \otimes_{R_i} R)$, the fact that $B_i \otimes_{R_i} (-)$ commutes with filtered colimits, and the compactness of A_i. To show that any finitely presented $A \in \mathrm{Alg}_R$ is the base change of a finitely presented R_i algebra for some i, just choose a presentation $A \cong R[x_1, \ldots, x_n]/(r_1, \ldots, r_m)$, then choose an i large enough so that all of the coefficients of the polynomials $r_1, \ldots, r_m$ lift to R_i. The proof of the statement for modules is similar. $\square$

Relative approximation theorems extend this key lemma from affine schemes and modules to more general stacks. First let us extend the statement about modules to the claim:

Proposition 10.11 (Approximating coherent sheaves)**.** *Let $\mathcal{X} = \varprojlim_i \mathcal{X}_i$ be a cofiltered (e.g., inverse) limit with affine bonding maps, and assume that $\mathcal{X}$ is qc.qs.. Then the pullback functor induces an equivalence*

$$\mathrm{colim}_i \mathrm{Coh}(\mathcal{X}_i) \to \mathrm{Coh}(\mathcal{X}).$$

Proof. The idea is to choose an atlas $X_0 = \mathrm{Spec}(A) \to \mathfrak{X}$ and deduce this from the corresponding claim for schemes. For simplicity, assume $\mathfrak{X}$ has affine diagonal, in which case $X_1 = X_0 \times_{\mathfrak{X}} X_0 = \mathrm{Spec}(B)$ is also affine. Likewise $\mathfrak{X}_i$ has a presentation $\mathrm{Spec}(B_i) \rightrightarrows \mathrm{Spec}(A_i)$, where $\mathrm{Spec}(A_i) = \mathfrak{X}_i \otimes_{\mathfrak{X}} \mathrm{Spec}(A)$, and likewise for B_i. Then $\mathrm{Coh}(\mathfrak{X})$ is equivalent to the category of finitely presented A-modules M with cocycle $\varphi : B \otimes_{s,A} M \to B \otimes_{t,A} M$. Lemma 10.10 implies that any finitely presented M, and homomorphism φ satisfying a cocycle equation arise via base change from some A_i. The same is true for homomorphisms between two objects, which are just homomorphisms of A-modules that commute with the given cocycles. $\qquad\square$

Another variant is that given a filtered colimit of rings $R = \mathrm{colim}_i R_i$ with initial index $i = 0$ and a map of finitely presentable R_0-algebras $A \to B$, many properties of the base change $R \otimes_{R_0} A \to R \otimes_{R_0} B$ must already hold for the intermediate base change $R_i \otimes_{R_0} A \to R_i \otimes_{R_0} B$ for sufficiently large i. The full formulation of this for stacks is:

Proposition 10.12 (Approximating properties of morphisms)**.** *Let $\mathfrak{Z} = \varprojlim_i \mathfrak{Z}_i$ be a cofiltered (e.g., inverse) limit of algebraic stacks with affine bonding maps, and assume that I has a terminal object 0. Let $\mathfrak{X}, \mathfrak{Y} \to \mathfrak{Z}_0$ be stacks of finite presentation over $\mathfrak{Z}_0$, and let $f_0 : \mathfrak{X} \to \mathfrak{Y}$ be a morphism over $\mathfrak{Z}_0$. Let P be one of the properties in the list:*

> *representable, a monomorphism, a closed immersion, an open immersion, surjective, flat, smooth, étale, separated, proper, affine, quasi-affine, quasi-finite.*

Then if the base change $f : \mathfrak{X}_{\mathfrak{Z}} \to \mathfrak{Y}_{\mathfrak{Z}}$ (resp. Δ_f) of f_0 satisfies a property P then so does $f_i : \mathfrak{X}_{\mathfrak{Z}_i} \to \mathfrak{Y}_{\mathfrak{Z}_i}$ (resp. Δ_{f_i}) for all i sufficiently large.

Note that all the properties in Proposition 10.12 are stable under base change and smooth-local on the base and target. The statement of Proposition 10.12 only lists properties that have come up in this course, but we refer the reader to [R, Prop.B3] for a proof and a more complete list.

Example 10.13. Let us prove Proposition 10.12 for the property $P = $"smooth." Because this property is smooth-local on the base, the target, and the source, one can reduce to the case where all are affine. So we have a map of finitely presentable R-algebras $A \to B$, which implies B admits a finite presentation as an A-algebra $\alpha : A[x_1, \ldots, x_n] \to B$ with finitely generated kernel $I = \ker(\alpha)$. Smoothness means that the naive cotangent complex $\mathbb{NL}_\alpha = (I/I^2 \to \Omega_{A[x_1,\ldots,x_n]/A})$ has homology that vanishes in degree -1

and is projective of finite rank in degree 0 (see **??**), and we must show that this holds for the R_i-algebra maps $A_i \to B_i$ for i sufficiently large. Lemma 10.10 implies that presentation α and the complex $\mathbb{NL}_\alpha$ are the base change from a presentation of $\alpha_i : A_i[x_1, \ldots, x_n] \to B_i$ for some sufficiently large i, and we let α_j denote the base change to R_j for any larger j. Now $H^0(\mathbb{NL}_\alpha) \cong H^0(\mathbb{NL}_{\alpha_i}) \otimes_{B_i} B$ being projective of finite rank means it is a summand of a finite free module, and Lemma 10.10 implies that the same must hold for $H^0(\mathbb{NL}_{\alpha_j})$ for j sufficiently large. Once $H^0(\mathbb{NL}_{\alpha_j})$ is finite projective, this implies that $H^{-1}(\mathbb{NL}_\alpha) \cong H^{-1}(\mathbb{NL}_{\alpha_j}) \otimes_{R_j} R$, so we use Lemma 10.10 once again to deduce that vanishing of H^{-1} over R implies vanishing over R_j for j sufficiently large.

Finally, we can extend Lemma 10.10 from algebras to stacks:

<div align="right"><code>P:finite_presentation_2</code></div>

Proposition 10.14 (Approximating finitely presented stacks)**.** *Let* $\mathcal{Z} \cong \varprojlim_i \mathcal{Z}_i$ *be a cofiltered (e.g., inverse) limit algebraic stacks with affine bonding maps, and assume without loss of generality that* $0 \in I$ *is a terminal object. Consider morphisms* $\mathcal{X}_0 \to \mathcal{Z}_0$ *with* $\mathcal{X}_0$ *qc.qs., and* $\mathcal{Y}_0 \to \mathcal{Z}_0$ *locally of finite presentation. Then*

$$\mathrm{colim}_i \, \mathrm{Map}_{\mathcal{Z}_i}((\mathcal{X}_0)_{\mathcal{Z}_i}, (\mathcal{Y}_0)_{\mathcal{Z}_i}) \to \mathrm{Map}_{\mathcal{Z}}((\mathcal{X}_0)_{\mathcal{Z}}, (\mathcal{Y}_0)_{\mathcal{Z}})$$

is an equivalence of categories. Furthermore, any algebraic stack of finite presentation over $\mathcal{Z}$ *is isomorphic to* $(\mathcal{Y}_i) \times_{\mathcal{Z}_i} \mathcal{Z}$ *for some finitely presented morphism* $\mathcal{Y}_i \to \mathcal{Z}_i$.

Proof idea. The equivalence on categories of morphisms follows from Proposition 10.8, and is similar to the proof in the affine case (Lemma 10.10). To show that any algebraic stack of finite presentation over $\mathcal{Z}$ arises via base change from some $\mathcal{Z}_i$, the idea is to choose a groupoid presenting $\mathcal{X}$, and then to approximate the spaces and structure maps of the groupoid. For example, if $\mathcal{X}$ is quasi-compact with affine diagonal and $\mathcal{Z} = \mathrm{Spec}(R)$ is affine, for any atlas $X_0 : \mathrm{Spec}(A) \to \mathcal{X}$, the fiber product $X_0 \times_{\mathcal{X}} X_0 = \mathrm{Spec}(B)$ is affine as well. Then Lemma 10.10 says that there is some i sufficiently large such that both A and B are the base change of finitely presented algebras A_i, B_i over R_i, then increasing i if necessary, the source, target, identity, and multiplication maps arise from maps between $\mathrm{Spec}(A_i), \mathrm{Spec}(B_i)$, and $\mathrm{Spec}(B_i \otimes_{A_i} B_i)$. Increasing i one more time, these maps will eventually satisfy the groupoid identities, and s and t will be smooth by Proposition 10.12 and Example 10.13. The proof in the general case is basically the same, but it uses a "bootstrapping" argument to deduced the claim first for separated schemes, then all schemes, etc... $\square$

Exercise 10.4. *Prove the following claim, which was used in the proof of Proposition 8.21: Let $\mathcal{Z} = \lim_i \mathcal{Z}_i$ be a cofiltered limit of algebraic stacks with affine bonding maps, and let 0 be the terminal index. Let $f_0 : \mathcal{X} \to \mathcal{Y}$ be a morphism between two stacks that are finitely presented over $\mathcal{Z}_0$. If the base change $f : \mathcal{X}_{\mathcal{Z}} \to \mathcal{Y}_{\mathcal{Z}}$ is quasi-projective, then so is $f_i : \mathcal{X}_{\mathcal{Z}_i} \to \mathcal{Y}_{\mathcal{Z}_i}$ for all i sufficiently large.*

10.3   Absolute approximation theorems

Another basic observation is that any R-algebra can be approximated by a compact R-algebra.

Lemma 10.15. *Every $B \in \mathrm{Alg}_R$ is a filtered colimit of finitely presented R-algebras.*

Proof. Let I be the category of all finitely presentable R-algebras equipped with a map $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to A$. The idea is to show that the category I is filtered. We leave the details to the reader. Once this is established, the canonical map

$$\mathrm{colim}_{i \in I} B_i \to A$$

is clearly surjective. Injectivity follows from the observation that if $b \in B_i$ maps to 0, then b is annihilated by the first map in the factorization $B_i \to B_i/(b) \to A$, and $B_i/(b)$ is also finitely presented. $\square$

This is most commonly used when R is a noetherian ring, such as $\mathbb{Z}$. It says that any affine scheme can be written as an inverse limit (in the category of schemes) of affine schemes of finite presentation over R. (Finite type and finite presentation are equivalent in the noetherian case). There are many situations in which relative and absolute approximation can be used to remove noetherian and finite presentation hypotheses from the statements of foundational theorems.

Example 10.16. Say one would like to prove a theorem about an arbitrary morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks of finite presentation over another algebraic stack $\mathcal{Z}$, and assume that the claim for f can be verified smooth-locally on the target $\mathcal{Y}$. Then one can base change along a smooth morphism $\mathrm{Spec}(A) \to \mathcal{Z}$ to reduce to the case where the base is affine, write $A = \mathrm{colim}_i A_i$ as a colimit of noetherian rings, and use relative approximation to reduce to the case of a map of finitely presented stacks over a noetherian base ring, a much easier context to work in.

For questions that are not smooth local on the base, however, one needs absolute approximation [TT].

Example 10.17. We already discussed [R, Thm. B] (see Theorem 8.19), which uses absolute approximation to show that any stack with quasi-finite and separated diagonal admits a finite and finitely presented cover by a scheme, generalizing the case where $\mathcal{X}$ is noetherian and Deligne-Mumford from [LMB, Thm. 16.6]. Another nice application of absolute approximation is that one can remove the hypothesis that $f : \mathcal{X} \to \mathcal{Z}$ is finitely presented in Proposition 10.12, as long as one restricts the property P to be "affine," "quasi-affine," or a long list of properties on the diagonal of f [R, Thm. C].

Let $\mathcal{Z}$ be a noetherian stack, or more generally a *pseudo-noetherian* algebraic stack, meaning that the conclusion of Proposition 7.12 holds for any stack that is finitely presented over $\mathcal{Z}$. The key idea behind absolute approximation of a morphism $f : \mathcal{X} \to \mathcal{Z}$ is to show that it admits a factorization as an affine morphism $\mathcal{X} \to \mathcal{X}_0$ followed by a morphism of finite presentation $\mathcal{X}_0 \to \mathcal{Z}$, and we call such a factorization an *approximation*. It is not too hard to show that if f admits an approximation, then: 1) the conclusion of Proposition 7.12 holds for $\mathrm{QCoh}(\mathcal{X})$, and 2) $\mathcal{X}$ can be written as a cofiltered limit, with affine bonding maps, of stacks of finite presentation over $\mathcal{Z}$. The main conceptual idea of [R] is to show that

- a composition of two morphism that admit approximations also admits and approximation, and

- if $\mathcal{X}' \to \mathcal{X}$ is a surjective, representable by algebraic spaces, and finitely presented étale morphism, and if $\mathcal{X}' \to \mathcal{Z}$ admits an approximation, then so does $\mathcal{X} \to \mathcal{Z}$.

This shows that if $\mathcal{X} \to \mathcal{Z}$ admits an approximation, then $\mathcal{X}$ is also pseudo-noetherian. It also leads to a large class of stacks over $\mathcal{Z}$ that admit approximations, the most important of which is probably the following:

D:global_type

Definition 10.18. We say that an algebraic stack $\mathcal{X}$ is of global type if it admits a representable (by algebraic spaces), surjective, finitely presented étale morphism from a basic quotient stack (Definition 8.14) $U/\mathrm{GL}_n \to \mathcal{X}$.

Many stacks are of global type, including qc.qs. schemes, algebraic spaces, DeligneMumford stacks and algebraic stacks with quasi-finite (and locally separated) diagonals. It is not hard to show that any stack of global type admits an approximation over $\mathrm{Spec}(\mathbb{Z})$, so any stack of global type is pseudo-noetherian. In addition any map between algebraic stacks of global type admits an approximation.

Lecture 11

Tannaka duality

References: [SR, DM2, L1, HR3] Date: 4/07/2020
Exercises: 2

11.1   Classical Tannaka duality

The original version of Tannaka duality applies to compact topological groups. It was extended to the algebraic setting by Saavedra Rivano [SR] and simplified by Deligne and Milne in [DM2].

The idea is that an affine group scheme over a field k can be recovered from the symmetric monoidal category $\mathcal{C} = \mathrm{Rep}_k(G)^{fin}$ of finite representations, along the forgetful functor $\omega : \mathrm{Rep}_k(G)^{fin} \to \mathrm{Vec}_k$. The k-points of G can be recovered as the group $\mathrm{Aut}^{\otimes}(\omega)$ of symmetric monoidal natural automorphisms of ω. This extends naturally to a functor on k-algebras $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Proposition 11.1. [DM2, Prop. 2.8] *There is a natural isomorphism of functors* $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Definition 11.2. Let $\mathcal{C}$ be a symmetric monoidal abelian category. An object $X \in \mathcal{C}$ is dualizable if there is another object $X^{\vee}\mathcal{C}$ such that the functor $X \otimes (-) : \mathcal{C} \to \mathcal{C}$ is left adjoint to $X^{\vee} \otimes (-)$. A symmetric monoidal abelian category is *rigid* if every object is dualizable.

Note that this means that for any $X, Y \in \mathcal{C}$, the inner Hom object $\underline{\mathrm{Hom}}(X, Y)$ exists and is isomorphic to $X^{\vee} \otimes Y$.

Theorem 11.3 ([DM2, Thm. 2.11]). *Let $\mathcal{C}$ be a rigid abelian tensor category such that $k \cong \mathrm{End}(\mathbf{1})$, and let $\omega : \mathcal{C} \to \mathrm{Vec}_k$ be an exact faithful k-linear tensor functor. Then,*

1. *The functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme G; and*

2. *The functor $\mathcal{C} \to \mathrm{Rep}_k(G)^{fin}$ defined by ω is an equivalence of tensor categories.*

The triple $(\mathcal{C}, \otimes, \omega)$ is known as a *neutral Tannakian category*. G is algebraic if and only if there is a tensor generator $X \in \mathcal{C}$.

Let us consider the stacky perspective, where G is a smooth group scheme. The fiber functor ω corresponds to the pullback $\mathrm{QCoh}(BG) \to \mathrm{QCoh}(\mathrm{Spec}(k))$ along the morphism $\mathrm{Spec}(k) \to BG$. However, if one only wants to recover the stack BG, then one can do that without the fiber functor, using the following:

<div align="right">P:tannaka_BG</div>

Proposition 11.4. *For any ring R, the canonical functor*

$$\mathrm{Map}_k(\mathrm{Spec}(R), BG) \to \mathrm{Fun}_k^{\otimes}(\mathrm{Rep}_k(G)^{fin}, \mathrm{Vec}_R)$$

that takes a map $f : \mathrm{Spec}(R) \to BG$ to the pullback functor f^* is an equivalence of categories.

This formulation has the disadvantage that it does not address the question of which categories are of the form $\mathrm{Rep}_k(G)^{fin}$ for some affine group scheme G, but it turns out this formulation extends to more general stacks.

11.2 Tannaka duality for algebraic stacks

<div align="right">T:tannaka_duality</div>

Theorem 11.5. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with quasi-affine diagonal, and let $\mathcal{Y}$ be a noetherian algebraic stack. The assignment of any map $f : \mathcal{Y} \to \mathcal{X}$ to $f^* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ defines a functor*

$$\mathrm{Map}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Fun}^{\otimes, \mathrm{cocont}}(\mathrm{QCoh}(\mathcal{X}), \mathrm{QCoh}(\mathcal{Y})), \qquad \overset{\{\text{E:tannaka}\}}{(11.1)}$$

where the right hand side denotes the category of symmetric monoidal functors that commute with all colimits. This functor is fully faithful, and its essential image consists of those f^* whose right adjoint $f_* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ commutes with filtered colimits.

Remark 11.6 (Simplification for pseudo-noetherian stacks). If $f^* : \mathcal{C} \to \mathcal{D}$ is a functor between categories that admits a right adjoint $f_* : \mathcal{D} \to \mathcal{C}$, then a simple formal argument shows that if f_* commutes with filtered colimits, then f^* preserves compact objects, and the converse holds if $\mathcal{C}$ is compactly generated, meaning that every object is a filtered colimit of compact objects. Therefore, with the additional hypothesis in Theorem 11.5 that $\mathrm{QCoh}(\mathcal{X})$ is compactly generated – for instance, if $\mathcal{X}$ is noetherian, of global type, or more generally pseudo-noetherian – the essential image of (11.1) can alternatively be described as

$$\{\text{cocontinuous } f^* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y}) \text{ that preserve } \mathrm{Coh}\}$$

this category admits a restriction functor to the category of additive, right-exact,[1] symmetric monoidal functors $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{Y})$ which is again an equivalence because $\mathrm{QCoh}(\mathcal{X})$ is compactly generated and $\mathrm{QCoh}(\mathcal{Y})$ admits filtered colimits. Combining these observations gives the following:

T:tannaka_coherent

Theorem 11.7 (Coherent Tannaka duality). $\mathcal{Y}$ *be a locally noetherian algebraic stack, and let* $\mathcal{X}$ *be a quasi-compact algebraic stack with quasi-affine diagonal such that* $\mathrm{QCoh}(\mathcal{X})$ *is compactly generated – for instance,* $\mathcal{X}$ *could be noetherian or global type. Then the assignment* $f \mapsto f^*$ *defines an equivalence of categories*

$$\mathrm{Map}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Fun}^{\otimes, \mathrm{rex}}(\mathrm{Coh}(\mathcal{X}), \mathrm{Coh}(\mathcal{Y})),$$

where the latter denotes the category of additive, right-exact symmetric monoidal functors.

Remark 11.8. The formulation of Theorem 11.5 is due to Lurie [L1], who also proved a version in the derived setting. These original versions gave a different description of the essential image of (11.1) that included the hypothesis that f^* preserve faithfully flat sheaves, which is hard to check in practice. Several improvements were made to both the derived and classical version of the theorem [BC, HR3, S2, S1, FI, BHL, B2]. For instance, [BC], shows that the functor (11.1) is an equivalence for arbitrary qc.qs. schemes. [HR3] deals with algebraic stacks that have affine automorphism groups, at the expense of a slightly weaker conclusion that is nevertheless good enough for all applications. The derived analog of Theorem 11.7, [?BhattHL], does not require the source $\mathcal{Y}$ to be locally noetherian.

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[1]For a non-noetherian stack, $\mathrm{Coh}(\mathcal{X})$ is not an abelian category, but it is closed under finite colimits, so by "additive and right exact," we mean "preserves finite colimits." This is equivalent to preserving direct sums and cokernels for maps between coherent sheaves.

Remark 11.9. ¡¡TODO: comment on Gabriel-Rosenberg reconstruction theorem, and Balmer spectrum¿¿

The key to Theorem 11.5 is the following simple special case.

Example 11.10. If $\mathcal{X} = \mathrm{Spec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(R)$, one can show the conclusion of Theorem 11.5 directly: Any cocontinuous symmetric monoidal functor $F : A\text{-Mod} \to R\text{-Mod}$ gives a map of rings

$$f : A = \mathrm{End}(\mathbf{1}_{A\text{-Mod}}) \to \mathrm{End}(F(\mathbf{1}_{A\text{-Mod}})) \cong \mathrm{End}(\mathbf{1}_{R\text{-Mod}}) = R.$$

We claim that $F \cong f^*$, and this isomorphism is uniquely determined by the tautological identification when restricted to full subcategory of A-Mod consisting of the single object $A = \mathbf{1}_{\mathrm{Coh}(A)}$. Indeed, because both functors commute with direct sums, they agree on the full subcategory $\mathcal{C} \subset A$-Mod whose objects are direct sums of copies of A. Finally, any $M \in A$-Mod admits a presentation

$$A^{\oplus I} \to A^{\oplus J} \to M \to 0,$$

so the fact that F and f^* are right exact implies that the previous isomorphism of functors $\mathcal{C} \to R$-Mod extends uniquely to an isomorphism of functors $F \cong f^*$. The same argument works when $\mathcal{Y}$ is an arbitrary algebraic stack, because a map $\mathcal{Y} \to \mathrm{Spec}(A)$ is the same as a map of rings $A \to \mathrm{End}(\mathcal{O}_\mathcal{Y})$. Note that in this example, you do not need the symmetric monoidal structure, and you do not need to stipulate that the right adjoint to commutes with filtered colimits.

We will prove Theorem 11.5 after establishing some preliminary results. Using smooth descent for $\mathrm{QCoh}(\mathcal{Y})$ and maps $\mathcal{Y} \to \mathcal{X}$, one can reduce the claim for an arbitrary $\mathcal{Y}$ to the special case where $\mathcal{Y} = \mathrm{Spec}(R)$ is affine. We omit the details of this bootstrapping procedure.

Lemma 11.11. *If $\mathcal{X}$ is a quasi-compact algebraic stack with quasi-affine diagonal, and $\mathcal{Y}$ is an algebraic stack, then the functor* (11.1) *is fully faithful.*

Proof. Consider two maps $f, g : \mathcal{Y} \to \mathcal{X}$. First note that if $[Y_1 \rightrightarrows Y_0]$ is a groupoid space presenting $\mathcal{Y}$, then smooth descent for maps of stacks implies that the set of 2-isomorphisms $f \Rightarrow g$ is the set of 2-isomorphisms $f|_{Y_0} \Rightarrow g|_{Y_0}$ that commute with the cocycles after restriction to Y_1. Likewise smooth descent for quasi-coherent sheaves implies the analogous fact for natural transformations $f^* \Rightarrow g^*$. Therefore if the claim holds for Y_0 and Y_1, it holds for $\mathcal{Y}$. Thus it suffices for us to establish the case where $\mathcal{Y} = \mathrm{Spec}(R)$ – from

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this one can use a bootstrapping procedure to extend this to the case where $\mathcal{Y}$ is a separated scheme, then a scheme, then a space, then a stack. So for the remainder of the proof we assume $\mathcal{Y} = \mathrm{Spec}(R)$ is affine.

Faithfulness of (11.1):

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism, and consider dotted arrows making the following diagram 2-commutative

$$\begin{array}{ccc} & & \mathcal{Y} \\ & {\scriptstyle\alpha}\nearrow & \downarrow {\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X} \end{array}\quad.$$

Let $\mathcal{A} = f_*(\mathcal{O}_\mathcal{Y})$. Because $\mathcal{X}$ has quasi-affine, the morphism f is quasi-affine. Thus the canonical morphism $\mathcal{Y} \to \mathrm{Spec}_\mathcal{X}(\mathcal{A})$ is an open embedding, hence a monomorphism. This means that the section α, along with the 2-isomorphism $f \circ \alpha \cong f$ is uniquely determined by the corresponding filling of

$$\begin{array}{ccc} & & \mathrm{Spec}_\mathcal{X}(\mathcal{A}) \\ & {\scriptstyle\alpha}\nearrow & \downarrow {\scriptstyle\pi} \\ \mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X} \end{array}\quad.$$

As the morphism π is affine, the set of such sections is in bijection with the set of $\mathcal{O}_\mathcal{Y}$-algebra maps $f^*(\mathcal{A}) \to \mathcal{O}_\mathcal{Y}$.

The identity section, i.e., $\alpha = \mathrm{id}_\mathcal{Y}$ with tautological 2-isomorphism $\pi \circ \alpha \cong f$, corresponds to the counit of adjunction $c : f^*(f_*(\mathcal{O}_\mathcal{Y})) \to \mathcal{O}_\mathcal{Y}$. On the other hand, the lift for $\alpha = \mathrm{id}$ and a possibly non-trivial 2-isomorphism $\eta : f \Rightarrow f$ corresponds to the composition $c \circ \eta^*_{f_*(\mathcal{O}_\mathcal{Y})} : f^*(f_*(\mathcal{O}_\mathcal{Y})) \to \mathcal{O}_\mathcal{Y}$, where $\eta^* : f^* \Rightarrow f^*$ is the symmetric monoidal natural transformation induced by η. It follows that if η^* is the identity, then $c \circ \eta^*_{f_*(\mathcal{O}_\mathcal{Y})} = c$, and thus $\eta = \mathrm{id} : f \Rightarrow f$.

Fullness of (11.1):

For simplicity, we prove this when $\mathcal{X}$ has affine diagonal and refer the reader to [HR3, Prop. 4.8.iib] for the extension of this argument to the case of quasi-affine diagonal.

Consider two maps $f, g : \mathcal{Y} \to \mathcal{X}$ and a natural transformation of symmetric monoidal functors $\gamma : f^* \Rightarrow g^*$. Let $\alpha_f : \mathcal{Y} \to \mathrm{Spec}_\mathcal{X}(f_*(\mathcal{O}_\mathcal{X}))$ and $\alpha_g : \mathcal{Y} \to \mathrm{Spec}_\mathcal{X}(g_*(\mathcal{O}_\mathcal{X}))$ denote the canonical morphisms, which are isomorphisms because $\mathcal{X}$ has affine diagonal. γ induces a natural transformation of right-lax symmetric monoidal functors $\gamma^\vee : g_* \Rightarrow f_*$. This gives a map

of $\mathcal{O}_{\mathcal{X}}$-algebras $g_*(\mathcal{O}_{\mathcal{X}}) \to f_*(\mathcal{O}_{\mathcal{X}})$ and hence a map $\beta : \mathrm{Spec}_{\mathcal{X}}(f_*(\mathcal{O}_{\mathcal{X}})) \to$ $\mathrm{Spec}_{\mathcal{X}}(g_*(\mathcal{O}_{\mathcal{X}}))$ over $\mathcal{X}$. Furthermore, $\gamma^\vee$ induces a natural transformation of right-lax symmetric monoidal functors $(\alpha_g)_* \Rightarrow \beta_*(\alpha_f)_*$ and hence a natural transformation $\alpha_f^* \beta^* \Rightarrow \alpha_g^*$. Example 11.10 now shows that $\beta \circ \alpha_f = \alpha_g$, because all three stacks are affine schemes. Identifying f with the composition $\mathcal{Y} \to \mathrm{Spec}_{\mathcal{X}}(f_*(\mathcal{O}_{\mathcal{Y}})) \to \mathcal{X}$ and likewise for g, this gives a 2-isomorphism $\eta : f \Rightarrow g$ such that $\eta^* = \gamma$.

$\square$

It therefore suffices to show that (11.1) is essentially surjective.

Definition 11.12. We refer to a cocontinuous symmetric monoidal functor $F : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ with whose right adjoint commutes with filtered colimits as a *virtual map*, and denote it $\mathcal{Y} \rightsquigarrow \mathcal{X}$. We say that F is *algebraic* if it is isomorphic to f^* for some morphism $f : \mathcal{Y} \to \mathcal{X}$. We say that $\mathcal{X}$ is *tensorial* if any virtual map $\mathcal{Y} \rightsquigarrow \mathcal{X}$ from a noetherian stack is algebraic.

We say that a diagram of quasi-compact algebraic stacks with affine diagonal that involves both algebraic and virtual maps is (2-)commutative if the corresponding diagram of symmetric monoidal categories (with reversed arrows) is (2-)commutative. Lemma 11.11 implies that this introduces no ambiguity, i.e., it agrees with the usual notion for a diagram involving only algebraic maps. Given a virtual map $f : \mathcal{Y} \rightsquigarrow \mathcal{X}$, we will use the slight abuse of notation $f^* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ to denote the corresponding functor.

L:tannaka_cover

Lemma 11.13. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with quasi-affine diagonal, and let $f : T \rightsquigarrow \mathcal{X}$ be a virtual map. If $p : T' \to T$ is a smooth cover such that the composition $f \circ p$ is algebraic, then so is f.*

Proof. This follows from smooth descent: Maps $T \to \mathcal{X}$ correspond to maps $T' \to \mathcal{X}$ along with an isomorphism of the two restrictions to $T'' :=$ $T' \times_T T'$ that satisfies a cocycle condition. On the other hand, smooth descent for $\mathrm{QCoh}(T)$ implies that cocontinuous symmetric monoidal functors $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(T)$ are symmetric monoidal functors $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(T')$ along with a natural isomorphism between the compositions with the two pullback functors $\mathrm{QCoh}(T') \to \mathrm{QCoh}(T'')$ that satisfies the corresponding cocycle condition. Lemma 11.11 guarantees that as long as the symmetric monoidal functor $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(T')$ is algebraic, the data of a cocycle for this functor is the same as the data of a cocycle for the corresponding map $T' \to \mathcal{X}$. $\square$

The key to using the criterion in Lemma 11.13 is the following:

Lemma 11.14. [HR3, Lem. 6.1] L:quasi-affine *Any algebraic stack $\mathfrak{X}$ that is a finitely nilpotent thickening of a basic quotient stack is tensorial.*

We will not give the proof, which is a bit technical. It is not known if a nilpotent thickening of a basic quotient stack is again a basic quotient stack – even for the classifying stack for an affine group scheme over a field – so the hard part of Lemma 11.14 is showing that the nilpotent thickening is tensorial.

Exercise 11.1. *Show that a basic quotient stack $\mathfrak{X} = U/\operatorname{GL}_{n,\mathbb{Z}}$, where U is quasi-affine, is tensorial by following these steps:*

1. *Show that $B\operatorname{GL}_{n,\mathbb{Z}}$ is tensorial directly;*

2. *Use this to show that $\operatorname{Spec}(A)/\operatorname{GL}_{n,\mathbb{Z}}$ is tensorial using Lemma 8.7;*

3. *Use this to show that $U/\operatorname{GL}_{n,\mathbb{Z}}$ is tensorial for a quasi-compact equivariant open $U \subset \operatorname{Spec}(A)$.*

L:tannaka_lifting

Lemma 11.15. *Let $\mathfrak{X}$ be an algebraic stack, let $\mathcal{A} \in \operatorname{Alg}(\operatorname{QCoh}(\mathfrak{X}))$, and let $f : T \rightsquigarrow \mathfrak{X}$ be a virtual map. Then the category of virtual maps $T \rightsquigarrow \operatorname{Spec}_{\mathfrak{X}}(\mathcal{A})$ over $\mathfrak{X}$ is equivalent to the category of $\mathcal{O}_T$-algebra maps $f^*(\mathcal{A}) \to \mathcal{O}_T$.*

Exercise 11.2. *Prove this lemma.*

Now in the context of Theorem 11.5, choose a smooth surjective morphism from an affine scheme $\pi : U \to \mathfrak{X}$, and let $\mathcal{A} := \pi_*(\mathcal{O}_U)$. Because the diagonal of $\mathfrak{X}$ is quasi-affine, the canonical morphism $j : U \to \operatorname{Spec}_{\mathfrak{X}}(\mathcal{A})$ is a quasi-compact open immersion. Let $J \subset \mathcal{A}$ be the ideal defining the complement of U with its reduced structure. Because for any algebraic stack QCoh is "locally finitely generated," we can write J as a union of finitely generated sub-ideals. Because $U \to \operatorname{Spec}_{\mathfrak{X}}(\mathcal{A})$ is quasi-compact, one of these sub-ideals $I \subset J$ will define a finitely presented (because I is finitely generated) closed immersion $\mathcal{Z} \hookrightarrow \operatorname{Spec}_{\mathfrak{X}}(\mathcal{A})$ whose complement is also U.

For any virtual map from an affine scheme $f : T \rightsquigarrow \mathfrak{X}$, let $Z_T \hookrightarrow \operatorname{Spec}(f^*(\mathcal{A}))$ denote the closed subscheme defined by the image of $f^*(I) \to f^*(\mathcal{A})$, and let $j_T : U_T \to \operatorname{Spec}(f^*(\mathcal{A}))$ denote its complement. [HR3, Thm. 5.1] implies that the composition

$$j_T^* \circ f^* : \mathcal{A}\text{-Mod}(\operatorname{QCoh}(\mathfrak{X})) \to \operatorname{QCoh}(U_T)$$

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factors uniquely as a composition $g^* j^*$ for some virtual map $g : U_T \rightsquigarrow U$. However, because U is affine, this virtual map must be algebraic and hence so is the composition $U_T \to U \to \mathfrak{X}$.

Lemma 11.16. $U_T \to T$ *is finitely presented.*

Proof. For any affine scheme $p : S \to T$, we have a canonical bijection

$$\mathrm{Map}_T(S, U_T) \cong \begin{cases} \mathrm{Map}_{\mathfrak{X}}(S, U), & \text{if } f \circ p \text{ is algebraic,} \\ \emptyset, & \text{otherwise} \end{cases}.$$

Now consider a cofiltered system of affine schemes S_i over T, and consider the diagram

$$
\begin{array}{ccc}
\mathrm{colim}_i \mathrm{Map}_T(S_i, U_T) & \longrightarrow & \mathrm{Map}_T(\varprojlim_i S_i, U_T) \\
\downarrow & & \downarrow \\
\mathrm{colim}_i \mathrm{Map}_{\mathfrak{X}}(S_i, U) & \longrightarrow & \mathrm{Map}_{\mathfrak{X}}(\varprojlim_i S_i, U)
\end{array}.
$$

We wish to show that the top horizontal arrow is a bijection. If any of the sets $\mathrm{Map}_T(S_i, U_T)$ are non-empty, then all of the sets in this diagram are non-empty and both vertical arrows are bijective, so the claim follows from the fact that $U \to \mathfrak{X}$ is of finite presentation (see [Proposition 10.5](#)). On the other hand, if the set $\mathrm{Map}_{\mathfrak{X}}(\varprojlim_i S_i, U)$ is empty, then all of the sets are empty, so that claim holds trivially. This reduces the proof to showing the following:

Claim: If $\mathrm{Map}_{\mathfrak{X}}(\varprojlim_i S_i, U)$ *is non-empty, then* $\mathrm{Map}_T(S_i, U_T)$ *is non-empty for some i.*

Consider a map $g : \varprojlim_i S_i \to U$ over $\mathfrak{X}$, and consider the following commutative diagram:

$$
\begin{array}{ccc}
& V \longrightarrow U & , \\
\nearrow \uparrow\uparrow \qquad \uparrow\uparrow & \\
\varprojlim_i S_i \xrightarrow{\;j_\infty\;} \mathrm{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{\varprojlim_i S_i})) \longrightarrow \mathrm{Spec}_{\mathfrak{X}}(\mathcal{A}) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \pi \\
S_i \rightsquigarrow_{j_i} \mathrm{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{S_i})) \longrightarrow \mathfrak{X}
\end{array}
$$

where V is defined to be the preimage of U, and the map $\varprojlim_i S_i \to V$ is induced by g. Because f_* commutes with filtered colimits,

$$\mathrm{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{\varprojlim_i S_i})) \cong \varprojlim_i \mathrm{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{S_i})).$$

Then because $V \subset \operatorname{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{\varprojlim_i S_i}))$ is a quasi-compact open immersion, it is the preimage of a quasi-compact open subset $V_i \subset \operatorname{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{S_i}))$ for some i (and hence all $j > i$). By construction the composition

$$\varprojlim_i S_i \longrightarrow S_i \rightsquigarrow \operatorname{Spec}_{\mathfrak{X}}(f_*(\mathcal{O}_{S_i}))$$

factors through V_i, and thus if $\mathcal{Z}_i$ is a finitely presented closed immersion complimentary to V_i, then $j_{i'}^*(\mathcal{O}_{\mathcal{Z}_{i'}}) = 0$ for some $i' > i$. Then [HR3, Thm. 5.1] implies that the virtual map $j_{i'}$ factors through V_i as well. Thus by only considering indexes above i', we can replace the diagram above with the following:

$$
\begin{array}{ccccc}
\varprojlim_i S_i & \xrightarrow{\ j_\infty\ } & \varprojlim_i V_i & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\
S_i & \underset{j_i}{\rightsquigarrow} & V_i & \longrightarrow & \mathfrak{X}
\end{array}
$$

Thus because $U \to \mathfrak{X}$ is finitely presented, there is a lift $V_i \to U$ for some i sufficiently large. Because U is affine, the corresponding virtual map $S_i \rightsquigarrow U$ must be algebraic by Example 11.10, and thus the map $S_i \to \mathfrak{X}$ is algebraic, and the map $S_i \to U$ over $\mathfrak{X}$ induces a map $S_i \to U_T$ over T. $\square$

Lemma 11.17. *Let $\mathfrak{X}$ be a quasi-compact algebraic stack with quasi-affine diagonal, and let $i : \mathcal{Y} \hookrightarrow \mathfrak{X}$ be a finitely presented closed immersion, and assume that $\mathcal{Y}$ has affine diagonal. If $\mathcal{Y}$ and $\mathfrak{X} \setminus \mathcal{Y}$ are tensorial with respect to virtual maps from locally noetherian schemes, then so is $\mathfrak{X}$.*

Proof. We have shown that (11.1) is fully faithful in Lemma 11.11, so it suffices to show that for any noetherian affine scheme T, and virtual map $f : T \rightsquigarrow \mathfrak{X}$ is algebraic. By Lemma 11.13, it suffices to show that the map $U_T \to T$ constructed above is smooth.

If $\mathcal{Y}$ is defined by the ideal $I \subset \mathcal{O}_{\mathfrak{X}}$, then let $Y \hookrightarrow T$ denote the closed subscheme defined by the image of $f^*(I) \to \mathcal{O}_T$. If $\mathcal{Y}^{(n)} = \operatorname{Spec}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}/I^n)$ is the nth infinitesimal neighborhood of $\mathcal{Y}$, then $f^*(\mathcal{O}_{\mathcal{Y}^{(n)}}) = \mathcal{O}_{Y^{(n)}}$ so the virtual map $Y^{(n)} \rightsquigarrow \mathfrak{X}$ factors through $\mathcal{Y}^{(n)}$.

Each virtual map $Y^{(n)} \rightsquigarrow \mathcal{Y}^{(n)}$ is algebraic by Lemma 11.14, and by hypothesis the unique virtual map $T \setminus Y \rightsquigarrow \mathfrak{X} \setminus \mathcal{Y}$ induced by [HR3, Thm. 5.1] is also algebraic. Thus we have a finitely presented morphism $U_T \to T$, by Lemma 11.16, whose restriction to $Y^{(n)} \hookrightarrow T$ is smooth for all n, and whose restriction to $T \setminus Y$ is also smooth. It follows, because T is noetherian, that $U_T \to T$ is smooth [S4, Tag 0A43]. $\square$

Proof of Theorem 11.5. By **??**, it suffices by induction to produce a finitely presented stratification of $\mathcal{X}$ whose strata are basic quotient stacks. To complete the proof, we thus invoke Proposition 8.21. $\qquad\square$

Exercise 11.3. *Let k be a field, and let $\mathcal{X}$ be a quasi-compact algebraic stack with quasi-affine diagonal. Show that any morphism*

$$f : \mathbb{A}_k^1/\mathbb{G}_m \to \mathcal{X}$$

that induces a trivial group homomorphism of k-groups $\mathbb{G}_m = \mathrm{Aut}_{\mathbb{A}_k^1/\mathbb{G}_m}(0) \to \mathrm{Aut}_{\mathcal{X}}(f(0))$ factors uniquely through the projection $\mathbb{A}_k^1/\mathbb{G}_m \to \mathrm{Spec}(k)$.

Exercise 11.4. *Let $\pi : Y \to X$ be a flat proper morphism of reduced noetherian schemes that satisfies the following: for every point $x \in X$ with residue field $k(x)$, if $\pi_x : Y_x \to \mathrm{Spec}(k(x))$ denotes the fiber, then $R(\pi_x)_*(\mathcal{O}_{Y_x}) \cong k(x)$. Show that for any quasi-compact algebraic stack with quasi-affine diagonal $\mathcal{X}$, the functor induced by composition with π*

$$\mathrm{Map}(X, \mathcal{X}) \to \mathrm{Map}(Y, \mathcal{X}),$$

is fully faithful, and its essential image consists of morphisms $Y \to \mathcal{X}$ whose restriction to Y_x factors through π_x for all $x \in X$. (Hint: first show that the canonical map $F \to \pi_*(\pi^*(F))$ is an isomorphism for any $F \in \mathrm{Coh}(X)$.)

Remark 11.18. Using the derived version of Tannaka duality [**?HLBhatt**], one can show that the conclusion of the previous exercise holds without the hypothesis that π is flat, but with the fibers Y_x over $x \in X$ replaced by the derived fibers.

Lecture 12

Artin's criteria

References: [A4], [H2], [S4, Tag 07SZ]
Date: 4/14/2020
Exercises: 3

12.1 The representability theorem

Artin's representability theorem is a list of conditions on a fibered category that imply it is an algebraic stack.

For simplicity, let us assume that the base scheme $S = \operatorname{Spec}(A)$, where A is an excellent ring. This means that it is noetherian, a G ring , a $J - 2$ ring, and universally catenary [**????**]. Examples include fields, $\mathbb{Z}$, complete local noetherian rings, and any localization of a finitely generated ring over one of these examples.

Before stating the theorem, we need to introduce some properties of stacks. Let us fix a category fibered in groupoids over S, denoted $\mathfrak{X}$.

Rim-Schlessinger condition

Consider a surjection of S-algebras $A' \twoheadrightarrow A$ with kernel I satisfying $I^n = 0$ for some $n \geq 0$, and consider another map of S-algebras $B \to A$. Then the diagram

$$\begin{array}{ccc} \operatorname{Spec}(A) & \hookrightarrow & \operatorname{Spec}(A') \\ \downarrow & & \downarrow \\ \operatorname{Spec}(B) & \hookrightarrow & \operatorname{Spec}(A' \times_A B) \end{array}$$

{E:RS_pushout}
(12.1)

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is a pushout diagram in the category $\mathrm{Sch}_{/S}$. This means that for any scheme X over S, the map $X(B \times_A A') \to X(B) \times_{X(A)} X(A')$ is a bijection, where as usual we are using the shorthand notation $X(A) := X(\mathrm{Spec}(A))$.

Definition 12.1. We say that a category fibered in groupoids $\mathfrak{X}$ over S satisfies the *strong Rim-Schlessinger condition* if for any diagram as in (12.1) with all rings finite type over S, the canonical functor

$$\mathfrak{X}(B \times_A A') \to \mathfrak{X}(A') \times_{\mathfrak{X}(A)} \mathfrak{X}(B)$$

is an equivalence of groupoids. We say that it satisfies the *Rim-Schlessinger condition* if it satisfies this condition only with respect to diagrams (12.1) in which A, A', and B are local artinian algebras of finite type over S.

Effectiveness of formal points

Let R be a complete local noetherian S-algebra, meaning that if $\mathfrak{m} \subset R$ is the maximal ideal then the canonical map $R \mapsto \hat{R} := \varprojlim_n R/\mathfrak{m}^n$ is an isomorphism. A *formal R point* of $\mathfrak{X}$ is a collection of $\xi_n \in \mathfrak{X}(R/\mathfrak{m}^n)$ along with isomorphisms $\xi_{n+1}|_{R/\mathfrak{m}^n} \cong \xi_n$ for all n.

Definition 12.2. We say that *formal points of $\mathfrak{X}$ are effective* if for any complete local S-algebra R, the canonical functor

$$\mathfrak{X}(R) \to \varprojlim_n \mathfrak{X}(R/\mathfrak{m}^n)$$

is an equivalence of groupoids.

There is a slightly more conceptual way to think of formal points: Let $\mathrm{Spf}(R) := \mathrm{colim}_n \mathrm{Spec}(R_n)$ denote the colimit of fibered categories over $\mathrm{Sch}_{/S}$, which is just the fibered category associated to the colimit or presheaves represented by $\mathrm{Spec}(R_n)$. Then by definition, this means that

$$\mathrm{Map}_S(\mathrm{Spf}(R), \mathfrak{X}) = \varprojlim \mathrm{Map}_S(\mathrm{Spec}(R_n), \mathfrak{X})$$

is the groupoid of formal R-points of $\mathfrak{X}$. Because $(\mathrm{colim}\, F_i)(T) = \mathrm{colim}\, F_i(T)$ for presheaves, we have

$$\mathrm{Map}_S(T, \mathrm{Spf}(R)) = \mathrm{colim}_n \mathrm{Map}_S(T, \mathrm{Spec}(R_n))$$

for any $T \in \mathrm{Sch}_{/S}$. In particular, one can describe $\mathrm{Spf}(R) \to \mathrm{Spec}(R)$ more intrinsically as the subfunctor corresponding to maps $T \to \mathrm{Spec}(R)$ whose set-theoretic image lies in $\mathrm{Spec}(R_0) \hookrightarrow \mathrm{Spec}(R)$.

This shows that any inverse system of surjections $R \to \cdots \to R_2 \to R_1 \to R_0$ will define the same functor as long as every closed subscheme set theoretically supported on $\mathrm{Spec}(R_0)$ is eventually contained in some $\mathrm{Spec}(R_n)$. This is equivalent to $R \to \varprojlim R_n$ being an isomorphism.

Openness of versality

Consider a scheme U of finite type over S and a morphism $\xi : U \to \mathcal{X}$.

Definition 12.3. We say that ξ is *versal at a finite type point* $u_0 \in U$ if for any surjection of finite type local artinian S-algebras $A' \twoheadrightarrow A$, any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathcal{X} \end{array} \qquad \text{\{E:versality\}} \quad (12.2)$$

in which the top horizontal arrow maps the unique point of $\mathrm{Spec}(A)$ to u_0, admits a dotted arrow that makes the whole diagram commute.

Note that although we are not assuming that $\mathcal{X}$ is algebraic, if it *were* algebraic and locally finite type over S, then the map $U \to \mathcal{X}$ would be representable by algebraic spaces. In this case the infinitesimal lifting criterion (12.2) is equivalent to ξ being smooth at the point u_0 [**????**].

D:openness_of_versality

Definition 12.4. We say that $\mathcal{X}$ satisfies *openness of versality* if for any finite type S-scheme U and a morphism $\xi : U \to \mathcal{X}$ that is versal at a finite type point $u_0 \in U$, there is an open subscheme $U' \subset U$ such that ξ is versal at every finite type point of U'.

Some formal deformation theory

Now consider an S-scheme $T = \mathrm{Spec}(A)$, and a morphism $\xi : T \to \mathcal{X}$. For any square-zero thickening $i : T \hookrightarrow T'$, i.e., a closed immersion defined by an ideal I such that $I^2 = 0$, we define the groupoid of *infinitesimal deformations of* ξ to be the groupoid

$$\begin{aligned} \mathrm{Def}(\xi, T') &:= \mathcal{X}(T') \times_{\mathcal{X}(T)} \{\xi\} \\ &= \left\{ \text{commutative diagrams} \quad \begin{array}{ccc} T & \xrightarrow{\xi} & \mathcal{X} \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \xi'} & \\ T' & & \end{array} \right\} \qquad \text{\{E:liftings\}} \quad (12.3) \end{aligned}$$

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i.e., objects are maps $\xi' : T' \to \mathfrak{X}$ along with an isomorphism $\xi' \circ i \cong \xi$. We will let $\pi_0 \operatorname{Def}(\xi, T')$ denote the set of isomorphism classes.

If $\mathfrak{X}$ satisfies the Rim-Schlessinger condition and $T = \operatorname{Spec}(k)$, then for any $M \in k$-Mod, the set of isomorphism classes $\pi_0 \operatorname{Def}(\xi, T[M])$ canonically has the structure of a k vector space. Similarly, the group of automorphisms of any $\xi' \in \operatorname{Def}(\xi, T')$ canonically has the structure of an k vector space, denoted $\operatorname{Aut}(\xi'/\xi)$.

More generally, if $\mathfrak{X}$ satisfies the strong Rim-Schlessinger condition, then for any $T = \operatorname{Spec}(A)$ and $M \in A$-Mod, $\operatorname{Aut}_{\operatorname{Def}(\xi, T[M])}(\xi_{\mathrm{canon}})$ and $\pi_0 \operatorname{Def}(\xi, T[M])$ canonically have the structure of A-modules, where $T[M] = \operatorname{Spec}(A \oplus M)$ denotes the trivial square-zero extension and ξ_{canon} denotes the canonical lift given by composing ξ with the canonical projection $T[M] \to T$.

D:tangent_space

Definition 12.5. We define the *tangent functor* of $\mathfrak{X}$ at $\xi : \operatorname{Spec}(A) \to \mathfrak{X}$ to be $\mathbb{T}_\xi(M) := \pi_0 \operatorname{Def}(\xi, T[M]) \in A$-Mod, and we define the *infinitesimal automorphism functor* to be $\operatorname{Inf}_\xi(M) = \operatorname{Aut}_{\operatorname{Def}(\xi, T[M])}(\xi_{\mathrm{canon}}) \in A$-Mod.

For a non-split square-zero thickening $T \hookrightarrow T'$ defined by an ideal $I \subset A$, $\pi_0 \operatorname{Def}(\xi, T')$ has a free and transitive action of $\mathbb{T}_\xi(I)$ as an abelian group, if it is non-empty. Similarly, the group of automorphisms of any $\xi' \in \operatorname{Def}(\xi, T')$ can be canonically identified with $\operatorname{Inf}_\xi(I)$. See [S4, Tag 07Y6] for more discussion and a proof of these claims.

A preliminary statement

We begin with a version of the representability theorem that is not the one people usually use, but whose proof is somewhat more straightforward.

P:artin_mild

Proposition 12.6. *Let S be an excellent affine scheme, and let $\mathfrak{X}$ be a category fibered in groupoids over $\operatorname{Sch}_{/S}$. Then $\mathfrak{X}$ is an algebraic stack locally of finite presentation over S if and only if it satisfies the following conditions:*

1. *$\mathfrak{X}$ is a stack for the étale topology;*

2. *The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable by algebraic spaces;*

3. *$\mathfrak{X}$ is limit-preserving (Definition 10.3);*

4. *$\mathfrak{X}$ satisfies the Rim-Schlessinger condition (Definition 12.1);*

5. *For any $\xi : \operatorname{Spec}(k) \to \mathfrak{X}$, with k finite type over S, $T_\xi(k)$ and $\operatorname{Inf}_\xi(k)$ are finite dimensional k vector spaces;*

6. Formal points of $\mathfrak{X}$ are effective ([Definition 12.2](#));

7. $\mathfrak{X}$ satisfies openness of versality ([Definition 12.4](#)).

In practice, openness of versality and effectiveness of formal points are the hardest to verify.

Proof idea. For simplicity, we will assume that S is Spec of an algebraically closed field. By (1) and (2), any morphism from an S-scheme $U \to \mathfrak{X}$ will be representable. All we need is to construct such a morphism that is smooth and surjective.

Consider a point $\xi_0 : \mathrm{Spec}(k) \to \mathfrak{X}$, with k finite type over S. Schlessinger's theorem [**????**] says that (4) and (5) imply that there is an inverse system of artinian local rings $\cdots \twoheadrightarrow R_2 \twoheadrightarrow R_1 \twoheadrightarrow R_0 = k$ with inverse limit a complete local neotherian ring R, and a formal point $\{\xi_n \in \mathfrak{X}(R_n)\}$: $\mathrm{Spf}(R) \to \mathfrak{X}$ that is "formally versal" in the sense that for any commutative diagram of fibered categories (the solid arrows)

$$
\begin{array}{ccc}
\mathrm{Spec}(A) & \longrightarrow & \mathrm{Spf}(R) \\
\cap \downarrow & \nearrow & \downarrow \\
\mathrm{Spec}(A') & \longrightarrow & \mathfrak{X}
\end{array}
\qquad \text{\{E:formally\_versal\}} \quad (12.4)
$$

in which $A' \twoheadrightarrow A$ is a surjection of finite type artinian local S-algebras whose unique point maps to ξ, one can fill in the dotted arrow so that the diagram still commutes.

Condition (6) implies that $\mathfrak{X}(\mathrm{Spf}(R)) \cong \mathfrak{X}(\mathrm{Spec}(R))$, so $\{\xi_n\}$ is the restriction of an actual map $\xi : \mathrm{Spec}(R) \to \mathfrak{X}$. Any map from a finite type artin local S-algebra $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ that maps the unique point of $\mathrm{Spec}(A)$ to the unique closed point of $\mathrm{Spec}(R)$ factors uniquely through $\mathrm{Spf}(R) \to \mathrm{Spec}(R)$. It follows that the lifting condition ([12.4](#)) implies the same "versality" condition with the formal point $\{\xi_n\} : \mathrm{Spf}(R) \to \mathfrak{X}$ replaced by $\xi : \mathrm{Spec}(R) \to \mathfrak{X}$.

Now, the technical heart of the argument is the following result of Artin [**????**]: Condition (3) above implies that exists a finite type S-scheme U, a closed point $u_0 \in U$ with residue field k, a map $\xi' : U \to \mathfrak{X}$ such that $\xi'(u_0) \cong \xi_0$, and an isomorphism $\widehat{\mathcal{O}}_{U,u_0} \cong R$ that identifies the two formal points

$$
\xi'|_{\mathrm{Spf}(\widehat{\mathcal{O}}_{U,u_0})} \cong \xi|_{\mathrm{Spf}(R)} = \{\xi_n\}.
$$

This gives you a map $\xi' : U \to \mathfrak{X}$ that is locally of finite presentation and versal at u_0.

Finally, (7) allows us to replace U with an open subscheme $U' \subset U$ such that $\xi' : U' \to \mathcal{X}$ is versal at *every* finite type point of U'. The same will be true after base change along any map from a finite type S-scheme $V \to \mathcal{X}$, and this implies that $\xi' : U' \to \mathcal{X}$ is smooth. If one repeats this process at *every* point of $\mathcal{X}$ over a finite type S-field, one gets a surjective smooth cover of $\mathcal{X}$ (assuming the cardinality of the set of such points of $\mathcal{X}$ is not too large). $\qquad\square$

12.1.1 Final statement, and openness of versality

Another important idea of Artin is that one can deduce openness of versality from the existence of an additional piece of data, called an "obstruction theory." Different versions of Artin's criteria involve slightly different formulations of what this means. For our purposes, we will introduce the following notion, which is not the minimal amount of data needed, but is easy to state and check in practice.

D:control_deformations

Definition 12.7. We say that a complex $E \in D^- \operatorname{Coh}(T)$ *controls infinitesimal deformations* if there is a canonical class $\operatorname{ob}(\xi, T') \in \operatorname{Ext}^1(E, I)$, functorial with respect to maps of square zero extensions, i.e., maps $T' \to T''$ commuting with the maps from T, and satisfying the following properties:

1. $\operatorname{Def}(\xi, T')$ is nonempty if and only if $\operatorname{ob}(\xi, T') = 0$;

2. If $\operatorname{Def}(\xi, T') \neq \emptyset$, there is an equivalence of groupoids

$$\operatorname{Def}(\xi, T') \cong \tau^{\leq 0}(\operatorname{RHom}(E, I))$$

   that is natural with respect to maps of square-zero extensions;

3. Given maps of square zero extensions $T[I_1] \to T'_2 \to T'_3$ that induces a short exact sequence on defining ideals $0 \to I_3 \to I_2 \to I_1 \to 0$, and given $\xi'_2 \in \operatorname{Def}(\xi, T'_2)$, then the connecting homomorphism δ in the long exact cohomology sequence

$$\cdots \to \operatorname{Hom}(E, I_2) \to \operatorname{Hom}(E, I_1) \xrightarrow{\delta} \operatorname{Ext}^1(E, I_3) \to \cdots$$

   maps the class in $\operatorname{Hom}(E, I_1)$ corresponding to $\xi'_2|_{T[I_1]}$ to the class $\operatorname{ob}(\xi, T'_3)$.

The complex E is meant to be an approximation of the cotangent complex of $\mathcal{X}$. Indeed, if $\mathcal{X}$ were algebraic and locally almost of finite presentation, then for any morphism $\xi : T \to \mathcal{X}$, Theorem 9.5 says precisely that $\xi^*(\mathbb{L}_{\mathcal{X}/S}) \in$

$D^- \operatorname{Coh}(T)$ controls infinitesimal deformations in the sense of Definition 12.7. In fact, that theorem says that $\mathbb{L}_{\mathcal{X}/S}$ controls infinitesimal defomations for maps from arbitrary algebraic stacks, not just affine S-schemes. Note, however, that Definition 12.7 only depends on the truncation $\tau^{\geq -1}(E)$, so E is not uniquely determined.

Remark 12.8. The lack of uniqueness of E in Definition 12.7 makes this a "construction" rather than a "condition." This is remedied somewhat in the derived setting, where $\xi^*(\mathbb{L}_{\mathcal{X}/S})$ is completely determined by deformation theory, see Section 9.2.3. It is also partially remedied in other formulations of Artin's criteria [HR2], which identify further conditions on the moduli functor which imply the existence of such a complex E.

With this notion, we can formulate a more useful version of Artin's criteria:

Theorem 12.9 (Artin's representability criteria). [H2] *Let S be an excellent affine scheme, and let $\mathcal{X}$ be a category fibered in groupoids over S. Then $\mathcal{X}$ is an algebraic stack locally of finite presentation over S if and only if the following criteria hold:*

1. *$\mathcal{X}$ is a stack for the étale topology;*

2. *$\mathcal{X}$ is limit-preserving (Definition 10.3);*

3. *$\mathcal{X}$ satisfies the* strong *Rim-Schlessinger condition (Definition 12.1);*

4. *Formal points of $\mathcal{X}$ are effective (Definition 12.2);*

5. *For any affine scheme T of finite type over S and any $\xi : T \to \mathcal{X}$ there is an $E \in D^- \operatorname{Coh}(T)$ that controls infinitesimal deformations (Definition 12.7).*

One interesting feature is that you no longer need any hypotheses on the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$. This is because the criteria are formulated in such a way that they imply the analogous relative version of the criteria for $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, so one can first use the criteria to show that $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, then show that $\mathcal{X}$ is algebraic using Proposition 12.6.

The fact that (5) implies openness of versality is [S4, Tag 07YZ]. The rough idea is the following: $\pi : U \to \mathcal{X}$ is versal at a closed point $u \in U$ if and only if for any map from a finite type artinian local S-algebra $\xi : \operatorname{Spec}(A) \to U$ that maps the unique point of $\operatorname{Spec}(A)$ to u, the canonical map $\operatorname{Def}_U(\xi, A') \to \operatorname{Def}_{\mathcal{X}}(\pi \circ \xi, A')$ is essentially surjective for any square-zero extension $A' \twoheadrightarrow A$. Using the properties of Definition 12.7 and strong

Rim-Schlessinger, one can construct a map $E \to \tau^{\geq -1}(\mathbb{L}_{U/S})$ such that π is versal at a closed point $u \in U$ if and only if

$$H^{-1}\left(k(u) \otimes^L_{\mathcal{O}_U} \mathrm{Cone}(E \to \tau^{\geq -1}(\mathbb{L}_{U/S}))\right) = 0.$$

This is an open condition by the semicontinuity of fiber dimension for the cohomology of coherent complexes.

12.2 Example: the stack of G-bundles on a proper, flat scheme

Our goal for this section is to prove:

<div align="right"><code>T:bung_algebraic</code></div>

Theorem 12.10. *Let k be a field, let $\pi : X \to S$ be a flat and finitely presented proper morphism of k-schemes, and let G be a finite type affine k-group scheme. Then the stack $\mathrm{Bun}_G(X/S)$ on $\mathrm{Sch}_{/S}$ defined by*

$$\mathrm{Bun}_G(X/S)(T) := \{principal\ G\text{-}bundles\ on\ X \times_S T\}$$

is an algebraic stack, locally of finite presentation and with affine diagonal over S.

We already know that $\mathrm{Bun}_G(X/S)$ is a stack for the étale topology **??**. The algebraicity of this stack can be checked smooth-locally, so we may assume S if affine. Furthermore, writing $S = \varprojlim S_i$ as a cofiltered inverse limit affine schemes of finite type over $\mathrm{Spec}(k)$, we may use relative approximation **??** to realize X as the base change of a proper map $X_i \to S_i$ for some S_i. Because we have $\mathrm{Bun}_{G_i}(X_i/S_i) \times_{S_i} S \cong \mathrm{Bun}_G(X/S)$, it suffices to replace $X \to S$ with $X_i \to S_i$ and prove algebraicity under the additional hypothesis that S is finite type over $\mathrm{Spec}(k)$.

We now verify the conditions of Theorem 12.9 one-by-one:

Affine diagonal

We must show that for any two morphisms $\xi_1, \xi_2 : T \to \mathrm{Bun}_G(X/S)$, corresponding to two G-bundles P_1 and P_2 on X_T, there is an affine morphism $Y \to T$ that represents the functor $\mathrm{Isom}_{BG}(P_1|_{X_{T'}}, P_2|_{X_{T'}})$ on schemes T' over T. Because affine morphisms satisfy smooth descent **??**, it suffices to show this when T and T' are affine.

The stack BG has affine diagonal, so there is a quasi-coherent $\mathcal{O}_{X_T}$-algebra $\mathcal{B}$ that satisfies

$$
\begin{aligned}
\operatorname{Isom}_{BG}(P_1|_{X_{T'}}, P_2|_{X_{T'}}) &\cong \operatorname{Map}_{\operatorname{Alg}(\operatorname{QCoh}(X_T))}(\mathcal{A}, \mathcal{O}_{X_{T'}}) \\
&\cong \operatorname{Map}_{\operatorname{Alg}(\operatorname{QCoh}(X_T))}(\mathcal{A}, \pi^*(\mathcal{O}_{T'})).
\end{aligned}
$$

The key fact is that, assuming T admits a dualizing complex (as it does here, because T is finite type over $\operatorname{Spec}(\mathbb{Z})$), the functor $\pi^* : \operatorname{QCoh}(S) \to \operatorname{QCoh}(X)$ admits a left adjoint $\pi_+ : \operatorname{QCoh}(X) \to \operatorname{QCoh}(S)$ that preserves coherent sheaves. This is actually proved first by establishing the derived version:

Lemma 12.11 ([H1, Prop. 3.1] or [HLP, Prop. 5.1.6]). *If S is a noetherian scheme that admits a dualizing complex, and $\pi : X \to S$ is a flat and proper morphism of schemes, then $\pi^* : D(S) \to D(X)$ admits a left adjoint $L\pi_+ : D(X) \to D(S)$.*

Because π is flat, π^* is exact, and thus $L\pi_+$ is right exact. It follows that for $E \in \operatorname{QCoh}(X)$ and $F \in \operatorname{QCoh}(S)$,

$$
\begin{aligned}
\operatorname{Hom}_X(E, \pi^*(F)) &\cong \operatorname{RHom}_X(E, \pi^*(F)) \cong \operatorname{RHom}_S(L\pi_+(E), F) \\
&\cong \operatorname{RHom}_S(H^0(L\pi_+(E)), F) \cong \operatorname{Hom}_S(H^0(L\pi_+(E)), F),
\end{aligned}
$$

so $\pi_+ := H^0(L\pi_+(-)) : \operatorname{QCoh}(X) \to \operatorname{QCoh}(S)$ is our left adjoint. Because π^* commutes with filtered colimits, π_+ must preserve compact objects, i.e., coherent sheaves.

Using the existence of π_+ one can construct a functor

$$
\pi_+^{alg} : \operatorname{Alg}(\operatorname{QCoh}(X)) \to \operatorname{Alg}(\operatorname{QCoh}(S))
$$

that is left adjoint to the pullback of algebras functor π^*. For $F \in \operatorname{QCoh}(\mathfrak{X})$, we have $\pi_+^{alg}(\operatorname{Sym}_{\mathcal{O}_X}(F)) = \operatorname{Sym}_{\mathcal{O}_S}(\pi_+(F))$, and in general one can compute π_+^{alg} by realizing any algebra as a coequalizer of two maps between algebras of the form $\operatorname{Sym}_{\mathcal{O}_X}(F)$. We thus have

$$
\operatorname{Map}_{\operatorname{Alg}(\operatorname{QCoh}(X_T))}(\mathcal{B}, \pi^*(\mathcal{O}_{T'})) \cong \operatorname{Map}_{\operatorname{Alg}(\operatorname{QCoh}(T))}(\pi_+^{alg}(\mathcal{B}), \mathcal{O}_{T'}),
$$

so $Y = \operatorname{Spec}_T(\pi_+^{alg}(\mathcal{B}))$ represents the isomorphism functor. For general T, which might not admit a dualizing complex, one can construct Y using relative approximation.

Limit preserving

Given a cofiltered inverse system of affines $T = \varprojlim_i T_i$ over S, we have $X_T = \varprojlim_i X_{T_i}$. Because BG itself is limit preserving, we have $\mathrm{Map}(X_T, BG) = \mathrm{colim}_i \mathrm{Map}(X_{T_i}, BG)$ by **??**.

Strong Rim-Schlessinger

The key fact here is the following:

Lemma 12.12. [**????**] *Let*

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

be a pushout diagram of schemes, in which the vertical morphisms are affine, and the horizontal morphisms are finite order nilpotent thickenings. Then the canonical restriction map is an equivalence of categories of flat *quasi-coherent sheaves*

$$
\mathrm{QCoh}(X')^{\mathrm{flat}} \to \mathrm{QCoh}(X)^{\mathrm{flat}} \times_{\mathrm{QCoh}(Y)^{\mathrm{flat}}} \mathrm{QCoh}(Y')^{\mathrm{flat}},
$$

Proof. By Zariski descent we may reduce to the case where all the schemes are affine.... $\square$

In particular, this implies the same for the category of vector bundles $\mathrm{Vec}_{X'} \cong \mathrm{Vec}_X \times_{\mathrm{Vec}_Y} \mathrm{Vec}_{Y'}$. This is a symmetric monoidal equivalence, so Proposition 11.4 implies that

$$
\begin{aligned}
\mathrm{Map}(X', BG) &\cong \mathrm{Fun}^{\otimes}(\mathrm{Vec}_{BG}, \mathrm{Vec}_{X'}) \\
&\cong \mathrm{Fun}^{\otimes}(\mathrm{Vec}_{BG}, \mathrm{Vec}_X) \times_{\mathrm{Fun}^{\otimes}(\mathrm{Vec}_{BG}, \mathrm{Vec}_Y)} \mathrm{Fun}^{\otimes}(\mathrm{Vec}_{BG}, \mathrm{Vec}_{Y'}) \\
&\cong \mathrm{Map}(X, BG) \times_{\mathrm{Map}(Y, BG)} \mathrm{Map}(Y', BG).
\end{aligned}
$$

In fact, this strong Rim-Schlessinger condition holds with BG replaced by any algebraic stack $\mathfrak{X}$ [S4, Tag 07WN].

Effectivity

Let R be a noetherian ring that is complete along an ideal I, and let $R_n := R/I^n$. Completeness means that $R \to \varprojlim_n R_n$ is an isomorphism. A key fact is the following

Lemma 12.13. *The restriction functor* $\text{Coh}(R) \to \varprojlim_n \text{Coh}(R_n)$ *taking* $M \mapsto \{M \otimes_R R_n\}$ *is an equivalence of symmetric monoidal categories.*

Proof. ¡¡TODO:¿¿ □

As a consequence, if $\mathcal{X}$ is a noetherian stack with quasi-affine diagonal, **??** implies that

$$\begin{aligned} \mathcal{X}(R) &\cong \text{Fun}^{\otimes, rex}(\text{Coh}(\mathcal{X}), \text{Coh}(R)) \\ &\cong \varprojlim_n \text{Fun}^{\otimes, rex}(\text{Coh}(\mathcal{X}), \text{Coh}(R_n)) \\ &\cong \varprojlim_n \mathcal{X}(R_n). \end{aligned}$$

In fact, let us consider the more general notion:

D:coherent_completeness

Definition 12.14. Let $\mathcal{Y}$ be a noetherian algebraic stack, let $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ be a closed substack, and let $\mathcal{Y}_n$ denote the nth infinitesimal neighborhood of $\mathcal{Y}_0$. We say that $\mathcal{Y}$ is *coherently complete* along $\mathcal{Y}_0$ if the canonical restriction functor defines an equivalence of categories

$$\text{Coh}(\mathcal{Y}) \xrightarrow{\cong} \varprojlim_n \text{Coh}(\mathcal{Y}_n).$$

Then by the same reasoning as above we have

L:artin_integrability

Lemma 12.15. *Let $\mathcal{Y}$ be a noetherian algebraic stack that is coherently complete along a closed substack $\mathcal{Y}_0$. Then for any (pesudo-)noetherian stack $\mathcal{X}$ with quasi-affine diagonal, the canonical functor is an equivalence of categories*

$$\text{Map}(\mathcal{Y}, \mathcal{X}) \xrightarrow{\cong} \varprojlim_n \text{Map}(\mathcal{Y}_n, \mathcal{X}).$$

Then the *Grothendieck existence theorem*, also called "Formal GAGA" says that the canonical restriction functor defines an equivalence of categories

$$\text{Coh}(X_R) \xrightarrow{\cong} \varprojlim_n \text{Coh}(X_{R_n}). \qquad \text{\{E:grothendieck\_existence\}} \tag{12.5}$$

The case where $X_R = \text{Spec}(R)$ is the Artin-Reese lemma, which says that $\text{Coh}(R) \cong \varprojlim_n \text{Coh}(R_n)$. Properness of $X \to S$ is essential here, and in fact, the derived

Controlling infinitesimal deformations

This follows from Theorem 9.5, which says that given a square-zero extension $A' \twoheadrightarrow A$ with kernel I and a map $f : X_A \to BG$, and extension of f to a map $f' : X_{A'} \to BG$ exists if and only if a certain obstruction class in $\text{Ext}^1_{X_A}(Lf^*(\mathbb{L}_{BG}), \pi_A^*(I))$ vanishes, where $\pi_A : X_A \to \text{Spec}(A)$ is the projection, and if the obstruction class vanishes, then

$$\text{Map}(X_{A'}, BG) \times_{\text{Map}(X_A, BG)} \{f\} \cong \tau^{\leq 0} \text{RHom}_X(Lf^*(\mathbb{L}_{BG}), \pi_A^*(I)).$$

We have discussed above that $\pi^* : D(S) \to D(X)$ admits a left adoint $L\pi_+$, and this left adjoint preserves $D^- \text{Coh}$, so it is clear that $L\pi_+(Lf^*(BG)) \in D - \text{Coh}(A)$ controls infinitesimal deformations in the sense of Definition 12.7.

Exercise 12.1. *Show that the properness of $X \to S$ is necessary for Theorem 12.10 by showing that the moduli funtor $\text{Bun}_{\mathbb{G}_m}(\mathbb{A}^1_k / \text{Spec}(k))$ does not satisfy Artin's criteria.*

Exercise 12.2. *Let $X \to S$ be a flat proper morphism of k-schemes, with S excellent, let G be a finitely presented k-group scheme, and let Y be an affine G-scheme. Use Theorem 12.10 to show that the moduli functor on $\text{Sch}_{/S}$*

$$\text{Map}(X, Y/G) : T \mapsto \text{Map}_k(T \times_S X, Y/G) \cong \text{Map}_T(X_T, (Y/G) \times_k T)$$

is an algebraic stack locally of finite type over S and with affine diagonal.
Exercise 12.3. *Show that the stack of flat families of projective schemes along with a relatively ample invertible sheaf, introduced in Example 4.4 is an algebraic stack.*

Remark 12.16. When X is projective and $G = \text{GL}_n$, there is a direct way to construct this moduli stack. (For brevity, we omit the verification that the diagonal is affine.) There is a projective S-scheme $\text{Quot}(\mathcal{O}_X(-n)^{\oplus m})_P$, the quot scheme, parameterizing flat families of quotients of $\mathcal{O}_X(-n)^{\oplus m}$, as a coherent sheaf, with Hilbert polynomial P. One can show that there is an open subscheme $U_{m,n,P} \subset \text{Quot}(\mathcal{O}_X(-n)^{\oplus m})$ parameterizing quotients $\mathcal{O}_X(-n)^{\oplus m} \to F$ with kernel K for which F is locally free and the canonical map

$$\text{Hom}(F, K) \to \text{Ext}^1(F, F),$$

which is the canonical map from the tangent space of Quot to the tangent space of the stack $\text{Bun}_{\text{GL}_n}(X/S)$ at the point F, is surjective. The canonical morphism

$$\phi_{m,n,P} : U_{m,n,P} \to \text{Bun}_{\text{GL}_n}(X/S)$$

that maps $\mathcal{O}_X(-n)^{\oplus m} \twoheadrightarrow F$ to F is smooth, and we claim that any map from an affine scheme $T \to \operatorname{Bun}_{\operatorname{GL}_n}(X/S)$ factors through one of the morphisms $\phi_{m,n,P}$. Indeed, if E is a vector bundle on X_T classified by this morphism, then for $n \gg 0$, $E(n)$ is globally generated and has vanishing H^1 on every fiber over T. This gives a surjection $\mathcal{O}_X(-n)^{\oplus m} \to E$ for some n, and guarantees that $\operatorname{Ext}^1(\mathcal{O}_X(-n)^{\oplus m}, E) = 0$ on every fiber over T, which implies the surjectivity by the long exact cohomology sequence for Ext groups.

When X/S is proper but not projective, the argument using Artin's criteria is the only one I am aware of for proving Theorem 12.10. Most applications only use the projective case, but we have taken the approach via Artin's criteria as a basic illustration of how these methods are used.

Lecture 13

Local structure theorems

References: [AHR1] [AHR2]
Date: 4/16/2020
Exercises: 0

We have seen in Proposition 8.21 that any stack admits a stratification by basic quotient stacks. Although this is usefull (See **??**), there are many local questions for which one would like an *étale* cover by basic quotient stacks. These are the local structure theorems.

13.1 Statement of structure theorem

Say $\mathcal{X}$ is a stack locally of finite type over an algebraically closed field k. Let $x \in \mathcal{X}(k)$ be a point, and let $G_x := \mathrm{Aut}_{\mathcal{X}}(x)$ denote its automorphism group. By definition $G_x = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)$.

Now let $U \to \mathcal{X}$ be an atlas. Because k is algebraically closed, the map $x : \mathrm{Spec}(k) \to \mathcal{X}$ admits a lift to U, and this induces a map of groupoid spaces

$$[G_r \rightrightarrows \mathrm{Spec}(k)] \to [U \times_{\mathcal{X}} U \rightrightarrows X],$$

and hence a morphism of stacks $BG_x \to \mathcal{X}$.[1] This map is a locally closed immersion, because k is algebraically closed. So we have a description of the point as a quotient stack, and the question is when is it possible to extend this to identify an (étale) neighborhood of x with a quotient stack.

---

[1] Note that G_x might not be reduced, so $[G_x \rightrightarrows \mathrm{Spec}(k)]$ might only be an fppf groupoid, but this still defines an algebraic stack, as discussed in **??**.

Theorem 13.1. [AHR1, Thm. 1.2] *Let $\mathfrak{X}$ be a quasi-separated algebraic stack
with affine automorphism groups, locally of finite type over an algebraically
closed field k. Let $x \in \mathfrak{X}(k)$ be a point and $H \subset G_x$ be a subgroup scheme
that is linearly reductive and such that G_x/H is smooth (resp. étale). Then
there exists an affine scheme $\mathrm{Spec}(A)$ with an action of H and a smooth
(resp. étale) morphism*

$$f : (\mathrm{Spec}(A)/H, w) \to (\mathfrak{X}, x)$$

*such that $B\mathrm{Aut}(w) = BH \cong f^{-1}(BG_x)$. If $\mathfrak{X}$ has affine diagonal, it can be
arranged that f is affine.*

In particular, if G_x is linearly reductive, then the map $BG_x \to \mathfrak{X}$ can be
extended to an étale cover of a neighborhood of x by a stack of the form
$\mathrm{Spec}(A)/G_x$.

For a general qc.qs. algebraic stack, a point $x \in \mathfrak{X}(k)$ still defines a map
$BG_x \to \mathfrak{X}$, but it is not a locally closed immersion. Instead, one has an
embedding $\mathcal{G}_x \hookrightarrow \mathfrak{X}$, where $\mathcal{G}_x \to \mathrm{Spec}(k')$ is a gerbe (see **??**), called the
residual gerbe, and k' is called the *residue field*. By construction $\mathcal{G}_x$ has a k
point, but it might not have any k'-points and thus is not of the form BG
for some k'-group G. But one can still formulate a version of Theorem 13.1
in this setting.

The crucial hypothesis in Theorem 13.1 is that H is linearly reductive,
which means that the pushforward functor $\mathrm{QCoh}(BH) \to k\text{-Mod}$ is exact.
So to generalize Theorem 13.1, we introduce the following:

Definition 13.2. We say that an algebraic stack $\mathfrak{X}$ is *cohomologically affine*
if the global section functor $\Gamma(\mathfrak{X}, -) : \mathrm{QCoh}(\mathfrak{X}) \to \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})\text{-Mod}$ is exact.

Example 13.3. The main example over a field k is a stack of the form
$\mathrm{Spec}(A)/G$, where G is linearly reductive.

Example 13.4. If a basic quotient stack $\mathfrak{X} = U/\mathrm{GL}_n$ is cohomologically affine,
then U is affine. To see this, observe that the morphism $U \to U/\mathrm{GL}_n$ is affine,
so if the latter is cohomologically affine then $\Gamma(U, -)$ is exact on $\mathrm{QCoh}(U)$,
which implies U is affine by Serre's criterion. The converse only holds in
characteristic 0, because $B\mathrm{GL}_n$ is not linearly reductive in characteristic p.

Example 13.5. If $\mathfrak{X}$ has affine automorphism groups and $x \in \mathfrak{X}(k)$ is such
that G_x is affine and linearly reductive, then the residual gerbe $\mathcal{G}_x \hookrightarrow \mathfrak{X}$ is
cohomologically affine.

Theorem 13.6. [AHR2, Thm. 1.1]*,* [**?AHHLR**] *Let $\mathfrak{X}$ be an algebraic stack
locally of finite presentation and quasi-separated over an excellent algebraic*

space S and with affine automorphism groups. Let $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ be a locally closed substack, and let $f_0 : \mathcal{W}_0 \to \mathfrak{X}_0$ be a smooth (respectively étale) morphism such that $\mathcal{W}_0$ is a cohomologically affine basic quotient stack. Then there is a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{W}_0 & \xrightarrow{\ f_0\ } & \mathfrak{X}_0 \\[2pt]
\big\downarrow & & \big\downarrow \\[2pt]
\mathcal{W} & \xrightarrow{\ f\ } & \mathfrak{X}
\end{array}
$$

such that f is smooth (respectively étale) and $\mathcal{W} \cong \operatorname{Spec}(A)/\operatorname{GL}_n$ for some GL_n-algebra A. Furthermore, if $\mathfrak{X}$ has separated (respectively affine) diagonal it can be arranged that f is representable (respectively affine).

Remark 13.7. If $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ is a field, then one can use approximation techniques to remove the hypothesis that S is excellent. All one needs is that S is a quasi-separated algebraic space.

We will summarize the proof after discussing some applications.

13.2   Application: Sumihiro's theorem

Let us work over a field k. Consider an action of $\mathbb{G}_m$ on projective space $\mathbb{P}^n$ coming from a linear action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ (in fact, any action arises from a linear action). Let us choose homogeneous coordinates $z_0, \ldots, z_n$ that diagonalize the $\mathbb{G}_m$-action, say $t \cdot z_i = t^{\alpha_i} z_i$ for some $\alpha_i \in \mathbb{Z}$. Then if the α_i are distinct for different i, the fixed points of this action are precisely the points p_i with $z_j = 0$ for all $j \neq i$.

For each p_i, the map $\operatorname{Spec}(k[z_0/z_i, \ldots, z_n/z_i]) \to \mathbb{P}^n$ is an equivariant neighborhood of p_i, where the coordinate z_j/z_i has weight $\alpha_j - \alpha_i$. Letting i vary, we see that any point in $\mathbb{P}^n$ has an equivariant affine open neighborhood. This implies the same for any $\mathbb{G}_m$-equivariant closed subscheme $X \hookrightarrow \mathbb{P}^n$. The same argument works for any action of a split torus on a projective variety with a linearizable T action. This result is known as Sumihiro's theorem.

For an action of a group G on an algebraic space X, the best one could hope for is equivariant étale neighborhoods, and one consequence of Theorem 13.1 is the following:

Proposition 13.8. *Let k be an algebraically closed field, and let G be a finite type affine k-group. Let X be a quasi-separated algebraic space with an action of G, and let $x \in X(k)$ be a point whose stabilizer $G_x \subset G$ is linearly*

reductive. Then there exists a G-equivariant affine étale neighborhood of x, $\mathrm{Spec}(A) \to X$.

13.3 The construction of neighborhoods

Remark 13.9. The proof of Theorem 13.6 below is very similar to the proof of Artin's criteria Proposition 12.6, and indeed one can think of the construction of a smooth cover in that proof as the special case of Theorem 13.6 where $\mathcal{W}_0 = \mathrm{Spec}(k)$ for some field of finite type over the base.

Consider the morphism $f : \mathcal{W}_0 = \mathrm{Spec}(A_0)/\mathrm{GL}_n \to \mathcal{X}$, and let $I \subset \mathcal{O}_{\mathcal{X}}$ be the ideal defining $\mathcal{X}_0$, and let $\mathcal{X}_n$ denote the nth infinitesimal thickening. Each $\mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ is a square-zero embedding with ideal I^n/I^{n+1}. We first inductively construct a sequence of 2-cartesian diagrams

$$
\begin{array}{ccc}
\mathcal{W}_n & \longrightarrow & \mathcal{W}_{n+1} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_n} \\
\mathcal{X}_n & \longrightarrow & \mathcal{X}_{n+1}
\end{array}
$$

in which the vertical maps are representable and smooth. This is essentially a deformation theory problem: the obstruction to extending at the nth stage lies in

$$\mathrm{Ext}^2_{\mathcal{W}_n}(\mathbb{L}_{\mathcal{W}_n/\mathcal{X}_n}, f_n^*(I^n/I^{n+1})),$$

which vanishes because f_n is smooth, hence $\mathbb{L}_{\mathcal{W}_n/\mathcal{X}_n}$ only has homology in degree zero, and $H^0(\mathbb{L}_{\mathcal{W}_n/\mathcal{X}_n}) = \Omega_{\mathcal{W}_n/\mathcal{X}_n}$ is locally free.

The directed system $\mathcal{W}_0 \hookrightarrow \mathcal{W}_1 \hookrightarrow \cdots$ is analogous to the system of artin local algebras in the proof of Proposition 12.6, $\mathrm{Spec}(R_0) \hookrightarrow \mathrm{Spec}(R_1) \hookrightarrow \cdots$ produced by Schlessinger's theorem. We would like a stack $\mathcal{W}$ analogous to $\mathrm{Spec}(R)$ of the completion:

Theorem 13.10. [AHR2, Thm. 1.10] *There is a basic quotient stack $\widehat{\mathcal{W}}$ that is cohomologically affine and a compatible sequence of closed immersions $\cdots \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1} \hookrightarrow \cdots \hookrightarrow \widehat{\mathcal{W}}$ such that $\widehat{\mathcal{W}}$ is coherently complete (Definition 12.14) along $\mathcal{W}_0$.*

By Lemma 12.15, Tannaka duality and coherent completeness imply that the compatible sequence of maps $\mathcal{W}_n \to \mathcal{X}$ extend uniquely to a map $\hat{f} : \widehat{\mathcal{W}} \to \mathcal{X}$.[2]

---

[2]In the case when $\mathcal{X}$ does not have quasi-affine diagonal, this requires the more refined versions of Tannaka duality in **??**.

The map $\hat{f}$ is formally smooth (see Exercise 9.3), but not locally of finite presentation. Thus the last step in the proof involves approximating $\widehat{\mathcal{W}}$ by a stack that is smooth over $\mathcal{X}$. This follows from the following stronger form of Artin approximation:

Theorem 13.11. [**?AHHLR**, Thm. 2.3] *Let S be an excellent affine scheme, and let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite type over S. Let $\widehat{\mathcal{W}}$ be a noetherian basic quotient stack that is cohomologically affine and coherently complete along a closed substack $\mathcal{W}_0$. Let $\hat{f} : \widehat{\mathcal{W}} \to \mathcal{X}$ be a morphism mapping $\mathcal{W}_0$ to a closed substack $\mathcal{X}_0$ such that the induced map of infinitesimal neighborhoods $\mathcal{W}_n \to \mathcal{X}_n$ is smooth for all $n \geq 0$.*

Then there exists a stack $\mathcal{W}$ of the form $\mathrm{Spec}(A)/\mathrm{GL}_n$, a closed substack $\mathcal{W}'_0$, and a commutative triangle

$$
\begin{array}{ccc}
\widehat{\mathcal{W}} & \xrightarrow{\;\varphi\;} & \mathcal{W} \\
& {\hat{f}} \searrow & \downarrow {f} \\
& & \mathcal{X}
\end{array}
$$

such that f is finite type, φ maps $\mathcal{W}_0$ to $\mathcal{W}'_0$, and the induced map on infinitesimal neighborhoods $\varphi_n : \mathcal{W}_n \to \mathcal{W}'_n$ is an isomorphism for all $n \geq 0$.

The last condition implies that $\widehat{\mathcal{W}} \to \mathcal{W}$ is formally étale along $\mathcal{W}_0$, and thus $\mathcal{W} \to \mathcal{X}$ is formally smooth along $\mathcal{W}'_0 = f^{-1}(\mathcal{X}_0)$. Hence f is smooth in a neighborhood $\mathcal{U} = U/\mathrm{GL}_n \subset \mathcal{W}$ that $\mathcal{W}_0$.

Finally, we must shrink U so that it is affine. Consider the closed complement $Z = \mathrm{Spec}(A) \setminus U$, with its reduced scheme structure, and let $W_0 \hookrightarrow \mathrm{Spec}(A)$ be the closed subscheme corresponding to $\mathcal{W}_0$. It is a fundamental theorem for the action of reductive groups on affine schemes that given two G-equivariant closed subsets of $\mathrm{Spec}(A)$, there is an element $f \in A^{\mathrm{GL}_n}$ such that f vanishes on Z but not on W_0. Hence $\mathrm{Spec}(A_f) \subset U$ is a G-equivariant open subset containing W_0, and $\mathrm{Spec}(A_f)/\mathrm{GL}_n$ is the desired neighborhood of $\mathcal{W}_0$.

Lecture 14

Good moduli spaces

References: [A2], [C]
Date: 4/21/2020
Exercises: 3

14.1   Moduli spaces and the Keel-Mori theorem

One of the classical goals of moduli theory was to construct spaces (ideally quasi-projective schemes) whose points are in bijection with isomorphism classes of the kind of object one is interested in studying.

Example 14.1. In Brill-Noether theory, one translates questions about special divisors on a smooth curve C, i.e., divisors with fewer than expected global sections, into geometric questions about the moduli spaces of such objects, which are subvarieties of the Jacobian $\mathrm{Jac}(C)$.

There is a sense in which this question has a tautological answer: if an algebraic stack has trivial automorphism groups, then it *is* an algebraic space. However, there are many examples of algebraic stacks that are close to being represented by an algebraic space in the following sense:

Definition 14.2. Let $\mathfrak{X}$ be an algebraic stack over a scheme S. A *coarse moduli space* is a morphism $\mathfrak{X} \to X$ to an algebraic space over S such that

1. Any map $\mathfrak{X} \to Y$, where Y is an algebraic space over S, factors uniquely through $\mathfrak{X} \to X$; and

2. For any algebraically closed field k over S, the map $\pi_0\mathfrak{X}(k) \to X(k)$ is a bijection, where $\pi_0(-)$ denotes the set of isomorphism classes.

Note that the first condition implies that $\mathcal{X} \to X$ is unique, as it co-represents the functor $\mathrm{Map}(\mathcal{X}, -)$ on algebraic spaces.

Example 14.3. Given a gerbe (see Definition 8.22) $\mathcal{G}$ over an algebraic space X, the structure map $\mathcal{G} \to X$ is a coarse moduli space. We have seen in the proof of Proposition 8.21 (see [S4, Tag 06RC]) that any qc.qs. algebraic stack admits a stratification by gerbes, so even if a moduli problem does not admit a coarse moduli space, it can always be broken into locally closed pieces that do. This tends not to be a very useful fact in practice, unless one can say more about the stratification or the resulting moduli spaces.

A more substantial result is the following:

T:keel_mori

Theorem 14.4. [KM, C] *[Keel-Mori Theorem] Let S be a scheme, and let $\mathcal{X}$ be an algebraic stack that is locally of finite presentation over S and such that the inertia stack $I_\mathcal{X} \to \mathcal{X}$ is finite over $\mathcal{X}$. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$. The map π is proper and quasi-finite, and if S is locally noetherian then X is locally of finite type over S.*

Example 14.5. Consider the stack $\mathcal{M}_g \to \mathrm{Sch}_{/S}$ parameterizing smooth families of curves of fixed genus $g \geq 2$ discussed in Example 4.5. One can use Exercise 12.3 to show that this is an algebraic stack locally of finite presentation over S. The automorphism group of any curve is finite, and in fact the morphism $I_{\mathcal{M}_g} \to \mathcal{M}_g$ is finite as well. It follows that there is a coarse moduli space $\mathcal{M}_g \to M_g$.

Example 14.6. A generalization of the previous examples is the moduli of stable maps. Let X be a proper scheme over a noetherian base scheme S, with a relatively ample bundle L. We introduce the stack $\overline{\mathcal{M}}_{g,n}(X)$ of *stable maps to X*. A family of stable maps to X over an S-scheme T consist of: 1) a flat proper morphism $C \to T$ whose geometric fibers are connected nodal curves of arithmetic genus g, 2) n sections $s_1, \ldots, s_n : T \to C$ that avoid the all of the nodes in the fibers, and 3) a map $C \to X$ over S. This data must satisfy a "stability" condition: at every geometric point $\mathrm{Spec}(k) \to T$, if C_k denotes the fiber and $\tilde{C}_k$ denotes its normalization, then each component $Z \subset \tilde{C}_k$ satisfies one of the following:

1. The map $Z \to X$ does not contract Z to a point,

2. Z has genus ≥ 2,

3. Z has genus 1 and contains at least one special point, i.e., point over a node in $\tilde{C}_k$ or one of the marked points s_i.

4. Z has genus 0 and contains at least three special points.

One can show, using Theorem 12.9, that $\overline{\mathcal{M}}_{g,n}(X)$ is a Deligne-Mumford stack locally of finite presentation over S (hence locally noetherian). The open and closed substack of maps of a fixed degree with respect to L is proper over S. In particular, the diagonal of this stack is propert and quasi-finite, hence finite (using noetherian hypotheses). This implies that the inertia is finite, and thus $\overline{\mathcal{M}}_{g,n}(X)$ admits a coarse moduli space by Theorem 14.4.

Theorem 14.4 gives a more-or-less complete solution to the problem of constructing moduli spaces in the presence of finite automorphism groups. Note however, that it offers no insight into whether the moduli space X is quasi-projective – this question is mostly handled on a case-by-case basis for different moduli problems (see [K] for one method).

14.2   Good moduli spaces

We will see that many interesting examples of stacks do not admit coarse moduli spaces:

Exercise 14.1. *Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scaling. Explain why a map $T \to \mathbb{A}^n/\mathbb{G}_m$ consists of an invertible sheaf $\mathcal{L}$ on T along with n sections $\sigma_1, \ldots, \sigma_n \in \Gamma(T, \mathcal{L})$. Show that $\mathbb{A}^n/\mathbb{G}_m$ does not admit a coarse moduli space.*

To deal with examples such as this, with positive dimensional affine automorphism groups whose dimension jumps as the point varies, we will study a slightly more general notion of moduli space. In particular, this will be necessary for studying the stack of principal G-bundles on a smooth curve.

`D:good_moduli_space`

Definition 14.7. Let $\mathcal{X}$ be an algebraic stack. A *good moduli space* is a quasi-compact morphism to an algebraic space $q : \mathcal{X} \to X$ such that

1. $q_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is exact, and

2. the canonical map $\mathcal{O}_X \to q_*(\mathcal{O}_\mathcal{X})$ is an isomorphism.

`EX:affine_gms`

Example 14.8. Let G be a linearly reductive algerbaic group over a field k, let $\mathrm{Spec}(A)$ be an affine G-scheme. Then the morphism

$$q : \mathrm{Spec}(A)/G \to \mathrm{Spec}(A^G)$$

is a good moduli space morphism. To see this, observe that

$$q_* : \mathrm{QCoh}(\mathrm{Spec}(A)/G) \to A^G\text{-Mod}$$

is the functor that takes a G-equivariant A-module M to the group M^G equipped with its canonical structure as an A^G-module. Theferfore, exactness of q_* follows from the exactness of $(-)^G$, i.e., the linear reductivity of G.

Exercise 14.2. *Let $\mathfrak{X}$ be a stack with a good moduli space. If $x \in \mathfrak{X}(k)$ correspond to a closed point, so that the residual gerbe $\mathcal{G}_x \hookrightarrow \mathfrak{X}$ is a closed embedding. Show that the automorphism group G_x is linearly reductive in two steps: first show that the global section functor on $\mathrm{QCoh}(\mathcal{G}_x)$ is exact, then show that this implies the same for BG_x.*

Exercise 14.3. *Show that if $q : \mathfrak{X} \to X$ is a good moduli space and $\mathcal{A} \in \mathrm{Alg}(\mathrm{QCoh}(\mathfrak{X}))$, then $\mathrm{Spec}_X(q_*(\mathcal{A}))$ is a good moduli space for $\mathrm{Spec}_{\mathfrak{X}}(\mathcal{A})$.*

14.2.1 Local structure theory and basic properties

It might be unclear, at first, why Definition 14.7 is a reasonable notion of a moduli space. It turns out, however that good moduli spaces have many useful properties:

Theorem 14.9 (Properties of good moduli spaces [A2])**.** *Let $q : \mathfrak{X} \to X$ be a good moduli space, with $\mathfrak{X}$ defined over a base scheme S. Then*

1. *q is surjective and universally closed.*

2. *For any morphism of algebraic spaces $X' \to X$, the base change $X' \times_X \mathfrak{X} \to X'$ is a good moduli space.*

3. *$q : \mathfrak{X} \to X$ is universal for maps to algebraic spaces.*

4. *Two geometric points $x_1, x_2 \in \mathfrak{X}(k)$ are identified in X if and only if their closures in $\mathfrak{X} \times_S \mathrm{Spec}(k)$ intersect.*

5. *The pullback functor $q^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathfrak{X})$ is fully faithful.*

Furthermore, if $\mathfrak{X}$ is finitely presented over a quasi-separated algebraic space S, then:

6. *X is finitely presented over S.*

7. *$\pi : \mathfrak{X} \to X$ has affine diagonal.*

8. *The pushforward $q_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$ preserves coherent sheaves.*

9. $F \in \mathrm{Coh}(\mathfrak{X})$ *lies in the essential image of* q^* *if and only if for any closed point in* $\mathfrak{X}$, *represented by a map* $x : \mathrm{Spec}(k) \to \mathfrak{X}$, *the automorphism group* $G_x = \mathrm{Aut}_{\mathfrak{X}}(x)$ *acts trivially on* $H^i(Lx^*(F))$ *for* $i = 0, 1$, *where* Lx^* *denotes the derived pullback.*

Property (4) implies that for every geometric point of the base $s \in S(k)$, every fiber of the map $\mathfrak{X}_k \to X_k$ contains a unique closed point. So q induces a bijection between geometric points of X and geometric points of $\mathfrak{X}$ that are closed in their fiber over S, which are sometimes called the *polystable points of* $\mathfrak{X}$. So properties (3) and (4) are analogous to the defining properties of a coarse moduli space.

Theorem 14.9 is proved in [A2] directly from Definition 14.7, although properties (3) and (6)-(9) had additional noetherian hypotheses which were subsequently removed in [AHR2]. (9) is stated only for vector bundles in [A2], but we will deduce the more general statement below.

We will take a somewhat ahistorical approach to Theorem 14.9. By restricting our attention to $\mathfrak{X}$ of finite presentation and with affine stabilizers and separated diagonal over a qc.qs. algebraic space S, we will see that the local structure theorem Theorem 13.6 can be used to reduce Theorem 14.9 to the basic case where $\mathfrak{X} = \mathrm{Spec}(A)/\mathrm{GL}_n$.

Theorem 14.10. [AHR2, Thm. 13.1] *Let $\mathfrak{X}$ be an algebraic stack that is locally of finite presentation with separated diagonal and affine stabilizers over a qc.qs. algebraic space S, and let $\mathfrak{X} \to X$ be a good moduli space. Then there is a surjective étale morphism* $\mathrm{Spec}(A) \to X$ *such that*

$$\mathfrak{X} \times_X \mathrm{Spec}(A) \cong \mathrm{Spec}(B)/\mathrm{GL}_n$$

for some GL_n*-algebra* B, *and* $\mathrm{Spec}(B)/\mathrm{GL}_n \to \mathrm{Spec}(A)$ *is a good moduli space, i.e.,* $A = B^{\mathrm{GL}_n}$ *and* $\mathrm{Spec}(B)/\mathrm{GL}_n$ *is cohomologically affine.*

Furthermore, when $S = \mathrm{Spec}(k)$ *for an algebraically closed field* k, *one can arrange that* $\mathfrak{X} \times_X \mathrm{Spec}(A) \cong \mathrm{Spec}(B)/G$ *for a linearly reductive* k*-group* G.

Remark 14.11. In fact, Theorem 14.10 has a refinement in which the map $\mathrm{Spec}(A) \to X$ is a "Nisnevich" cover, meaning that it is étale and surjective on k-points for any k.

Proof. Exercise 14.2 shows that closed points have linearly reductive stabilizer groups, and it follows that $\mathcal{G}_x$ is a cohomologically affine basic quotient stack. We can therefore apply Theorem 13.6 with $\mathcal{W}_0 = \mathcal{G}_x$ to obtain a representable étale map $f : \mathcal{W} = \mathrm{Spec}(B)/\mathrm{GL}_n \to \mathfrak{X}$ with a closed point

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$w \in \mathcal{W}$ inducing an isomorphism of residual gerbes $\mathcal{G}_w \to f^{-1}(\mathcal{G}_x)$ and such that $\mathcal{W}$ is cohomologically affine. Let $\pi_{\mathcal{W}} : \mathcal{W} \to W = \mathrm{Spec}(B^{\mathrm{GL}_n})$ be the good moduli space of $\mathcal{W}$. Then Luna's fundamental lemma [A1, Thm. 6.10] (see **??** below), implies that there is an affine open subscheme $\mathrm{Spec}(A) \subset W$ containing the image of w such that if $\mathcal{U} = \pi_{\mathcal{W}}^{-1}(\mathrm{Spec}(A))$, then $\mathcal{U} \to \mathrm{Spec}(A) \times_X \mathcal{X}$ is an isomorphism. $\qquad\square$

We now discuss the proof of Theorem 14.9. All of the properties are étale local over the good moduli space X, so Theorem 14.10, combined with the fact that a morphism $\mathcal{X} \to X$ is a good moduli space if and only if its base change along an étale cover is so [A2, Prop. 4.7], implies that it suffices to consider the case where $\mathcal{X} = \mathrm{Spec}(A)/G$ and $X = \mathrm{Spec}(A^G)$ with $G = \mathrm{GL}_n$. We will also assume for simplicity that we are working over a noetherian base $S = \mathrm{Spec}(R)$. We refer the reader to [A2, AHR2] for a more complete discussion, including the proof of property (3).

Remark 14.12. Even though the functor $(-)^G$ is not exact on $\mathrm{Rep}(G)$ in this case, because we are not necessarily in characteristic 0, we are assuming that $\mathcal{X} \to X$ is a good moduli space, so $(-)^G$ is exact on $A\text{-}\mathrm{Mod}(\mathrm{Rep}(G))$.

Projection formula and base change

We claim that for $M \in \mathrm{QCoh}(\mathcal{X})$ and $N \in A^G\text{-}\mathrm{Mod}$, then canonical homomorphism

$$q_*(M) \otimes N \to q_*(M \otimes q^*(N)) \qquad (14.1)$$

is an isomorphism. This is clearly true when $N = A^G$, and both the left side and the right side are right-exact functors of N (using the fact that q_* is exact). So, the general case follows by applying both functors to a presentation $(A^G)^{\oplus I} \to (A^G)^{\oplus J} \to N \to 0$ of N.

If we apply this to a map of R-algebras $A^G \to B$, it implies that $(A \otimes_{A^G} B)^G$, so good moduli spaces are stable under affine base change. This implies property (2) of Theorem 14.9.

Pullbacks and pushforwards of sheaves

The fully faithfulness of $q^* : A^G\text{-}\mathrm{Mod} \to \mathrm{QCoh}(\mathrm{Spec}(A)/G)$ is equivalent, via the adjunction

$$\mathrm{Hom}_{\mathcal{X}}(q^*(M), q^*(N)) \cong \mathrm{Hom}_X(M, q_*(q^*(N))),$$

to the claim that the canonical map $N \to q_*(q^*(N))$ is an isomorphism for any $N \in A^G$-Mod, which holds by the projection formula (14.1) with $M = \mathcal{O}_{\mathcal{X}}$.

For preservation of coherence, let $M \in \mathrm{Coh}(\mathcal{X})$ be a coherent sheaf. It suffices to show that for some coherent R-submodule $V \subset M^G$, the induced map $A^G \otimes_R V \to M^G$ is surjective.[1] Note that if $A \cdot M^G \subset M$ is the A-submodule generated by M^G, then $(A \cdot M^G)^G = M^G$, so we replace M with $A \cdot M^G$, which is coherent because A is noetherian. We will therefore assume that $A \cdot M^G = M$.

Write $M^G = \bigcup_\alpha V_\alpha$ as a filtered union of coherent R submodules. Because A is noetherian and M is coherent, the homomorphism of A-modules

$$A \otimes_R V_\alpha \to M,$$

must be surjective for some α. It follows that $q_*(A \otimes_R V_\alpha) = A^G \otimes_R V_\alpha \to M^G$ is surjective, by the projection formula and exactness of q_*.

Universally closed

The key fact here is the following

Lemma 14.13. *If $I_1, I_2 \subset A$ are two G-equivariant ideals, then*

$$(I_1 + I_2) \cap A^G = I_1 \cap A^G + I_2 \cap A^G.$$

Proof. (See [A2, Lem. 4.9].) Observe that $A^G \cap I = I^G$, so we must show $\pi_*(I_1 + I_2) = \pi_*(I_1) + \pi_*(I_2)$. Consider the short exact sequence $0 \to I_1 \to I_1 + I_2 \to I_2/(I_1 \cap I_2) \to 0$. Then we have the short exact sequence

$$0 \to \pi_*(I_1) \to \pi_*(I_1 + I_2) \to \pi_*(I_2/I_1 \cap I_2) \to 0,$$

and the composition $\pi_*(I_2) \to \pi_*(I_1 + I_2) \to \pi_*(I_2/I_1 \cap I_2)$ is surjective, and the claim follows. $\qquad\square$

The main consequence of this is:

Corollary 14.14. *For any two disjoint G-equivariant closed subsets $Z_1, Z_0 \hookrightarrow \mathrm{Spec}(A)$, there is a function $f \in A^G$ which is 1 on Z_1 and 0 on Z_0.*

Proof. Applying the lemma, if $1 \in I_1 + I_2$, then $1 \in \pi_*(I_1) + \pi_*(I_2)$. $\qquad\square$

---

[1] We are not assuming A^G is noetherian here. Once we have finite generation of M, we let $K \in \mathrm{Coh}(\mathcal{X})$ be the kernel of $A \otimes_R V \to q^*(M^G)$. Then we apply the same claim to show that $q_*(K)$, which is the kernel of $A^G \otimes_R V \to M^G$, will be finitely generated as well.

Now to see that π is closed imagine that $\pi(Z_1) \subset \mathrm{Spec}(A^G)$ is not closed, and let $p \in \mathrm{Spec}(A^G)$ be a point that does not lie in the set theoretic image of Z_1. By base change to a local ring at p, we may assume that p is a closed point. Let $I_2 \subset A$ be the ideal defining the closed subscheme $\pi^{-1}(p)$. Then by the previous observation we may find an $f \in A^G$ that vanishes on Z_1 but $f(p) = 1$, which shows that p does not lie in the scheme theoretic image of Z_1 either.

Finite generation of A^G

This is a more subtle question, so we will just discuss the relatively simple argument when G is linearly reductive over a field k, and refer the reader to [S3, Thm. 2] and the more general [A3, Thm. 6.3.3] for a more thorough discussion.

First observe that if $I \subset A^G$ is an ideal, then $(A \cdot I)^G \subset A^G$ is the image of $I = (A \otimes_{A^G} I)^G \to A^G$ because G is linearly reductive. Therefore $(A \cdot I)^G = I$. Now consider an ascending chain of ideals $I_0 \subset I_1 \subset \cdots \subset A^G$. The ideals $A \cdot I_0 \subset A \cdot I_1 \subset \cdots A$ must stabilize because A is noetherian, but applying $(-)^G$ shows that the original chain $I_0 \subset I_1 \subset \cdots$ must stabilize, and hence A^G is noetherian.

To show that A^G is finitely generated, by [Proposition 7.12](#) we can choose a G-equivariant surjection $\mathrm{Sym}_k(V) \twoheadrightarrow A$, where V is a finite dimensional sub-representation of A. $\mathrm{Sym}_k(V)^G \to A^G$ will again be surjective because G is linearly reductive, so it suffices to prove finite generation of $\mathrm{Sym}(V)^G$.

$\mathrm{Sym}_k(V)^G$ is a positively graded k-algebra with k in degree 0. An inductive argument shows that in this situation, any set of generators for the ideal spanned by positively graded elements will also generate $\mathrm{Sym}_k(V)^G$ as an algebra. Because we know $\mathrm{Sym}_k(V)^G$ is noetherian, finitely many generators will suffice.

14.3 Application: finiteness of cohomology

An important fact about a scheme X that is proper over a noetherian affine scheme $S = \mathrm{Spec}(A)$ is that for any coherent sheaf $F \in \mathrm{Coh}(X)$, the cohomology groups $H^i(X, F)$ are finitely generated A-modules.

Lemma 14.15. *Let $\mathcal{X}$ be an algebraic stack that admits a good moduli space that is proper over a noetherian scheme $S = \mathrm{Spec}(A)$. Then there is a constant $N > 0$ such that for any $F \in \mathrm{Coh}(\mathcal{X})$, $H^i(\mathcal{X}, F) = 0$ for $i \geq N$ and $H^i(\mathcal{X}, F)$ is a finitely generated A-module for $i < N$.*

Proof. The cohomology is defined using the fact that $\mathrm{QCoh}(\mathfrak{X})$ is a Grothendieck abelian category (see **??**). Let $q : \mathfrak{X} \to X$ be the good moduli space morphism, and let $\pi : X \to S$ be the given morphism. Then $R\Gamma(\mathfrak{X}, -) \cong Rq_* \circ R\pi_*$ by **??**, and $Rq_* = q_*$ because q_* is exact. It follows that

$$H^i(\mathfrak{X}, F) = H^i(X, q_*(F))$$

for any $F \in \mathrm{QCoh}(\mathfrak{X})$ and i, so the claim reduces to the claim for algebraic spaces, which is [**????**]. $\square$

One application of this is the following:

14.3.1 Integration in K-theory

For an essentially small abelian category $\mathcal{A}$, we let $K_0(\mathcal{A})$ be the Grothendieck group, i.e. the free abelian generated by symbols $[F]$ for $F \in \mathcal{A}$ modulo the relation $[E_2] = [E_1] + [E_3]$ for any short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. This is equivalent to the free abelian group on *finite chain complexes* in $\mathcal{A}$ modulo the same relation for exact triangles, where a complex $\cdots E_n \to E_{n-1} \to \cdots$ is identified with $\sum_n (-1)^n H^n(E_\bullet)$.

If $\mathfrak{X}$ admits a proper good moduli space over a field k, then one can define an "index" homomorphism on K-theory

$$\mathrm{Ind} : K_0(\mathrm{Coh}(\mathfrak{X})) \to \mathbb{Z}, \ \text{taking}$$
$$[F] \mapsto \sum_i \dim_k H^i(\mathfrak{X}; F).$$

The right hand side is only well-defined because the cohomology groups are finite dimensional and are non-vanishing in only finitely many degrees.

$K_0(\mathrm{Coh}(-))$ behaves like a homology theory for schemes, and more generally for stacks. For instance, any closed substack $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ gives a fundamental class $[\mathcal{O}_{\mathfrak{Z}}] \in K_0(\mathrm{Coh}(\mathfrak{X}))$. Furthermore, two substacks $\mathfrak{Z}_0, \mathfrak{Z}_1$ have $[\mathcal{O}_{\mathfrak{Z}_0}] = [\mathcal{O}_{\mathfrak{Z}_1}]$ if they are isotopic in the following sense: there is an open subset $U \subset \mathbb{A}^1$ containing 0 and 1 and a closed substack $\tilde{Z} \hookrightarrow \mathfrak{X} \times U$ that is flat over $0, 1 \in U$ such that $\tilde{\mathfrak{Z}}|_{\mathfrak{X} \times \{i\}} = \mathfrak{Z}_i$ for $i = 0, 1$. From this perspective, we think of Ind as an "integration map" on homology.

14.4 Remarks on adequate moduli spaces

There is a better notion of a moduli space for stacks in mixed or positive characteristic, called an *adequate moduli space,* and introduced and developed in [A3]. The definition replaces the condition that q_* be exact in

Definition 14.7 with a weaker condition, and in particular a good moduli space is always an adequate moduli space. But, for a reductive group G, the map $q_* : \operatorname{Spec}(A)/G \to \operatorname{Spec}(A^G)$ is always an adequate moduli space, even though q_* will not be exact in general in positive or mixed characteristic.

While many of the properties of good moduli spaces are known to hold for adequate moduli spaces, the analog of Theorem 14.10 is not known, and there are fewer general results establishing the existence of adequate moduli spaces. For this reason, we will restrict our focus to good moduli spaces.

Remark 14.16. For an algebraic stack of finite presentation over a scheme, the coarse moduli space $q : \mathfrak{X} \to X$ is an adequate moduli space, but not necessarily a good moduli space. An algebraic stack with finite inertia is defined to be *tame* precisely if q_* is exact, i.e. the coarse moduli space is a good moduli space.

Lecture 15

Construction of good moduli spaces

References: [AHLH]
Date: 4/23/2020
Exercises: 2

15.1 Geometric invariant theory

Now let G be a linearly reductive k-group for some field k. Consider a G-scheme X along with a projective G-equivariant morphism $X \to \mathrm{Spec}(A)$ to some finite type affine G-scheme. Let $\mathcal{L} \in \mathrm{Pic}(X/G)$ be a G-linearized ample bundle, i.e., an invertible sheaf that is ample on X when you forget the G-equivariant structure.

Note that for any $\sigma \in \Gamma(\mathcal{X}, \mathcal{L})^G = \Gamma(X, \mathcal{L})^G$, the non-vanishing locus $X_\sigma := \{x \in X | \sigma_x \neq 0\}$ is G-equivariant and affine, because $\mathcal{L}$ is ample. It follows from Example 14.8 that

$$X_\sigma/G \to \mathrm{Spec}(\Gamma(X_\sigma, \mathcal{O}_{X_\sigma})^G)$$

is a good moduli space.

P:git_1

Proposition 15.1. *With the set up above, let us define the $\mathcal{L}$-semistable locus*

$$X^{\mathrm{ss}}(\mathcal{L}) := \bigcup_{n \geq 1, \sigma \in \Gamma(X, \mathcal{L}^n)^G} X_\sigma,$$

and let $A = \bigoplus_n A_n$ be the graded algebra with $A_n := \Gamma(X^{\mathrm{ss}}, \mathcal{L}^n)^G$. Then the induced morphism is a good moduli space

$$X^{\mathrm{ss}}(\mathcal{L})/G \to \mathrm{Proj}(A).[1]$$

Proof. First, let us recall the canonical map: $\mathrm{Proj}(A)$ is covered by standard affine open subset $D_+(\sigma)$ for $\sigma \in A_n = \Gamma(\mathcal{X}, \mathcal{L}^n)^G$ for $n \geq 1$. We have $D_+(\sigma) \cong \mathrm{Spec}(A[\sigma^{-1}]_0)$, where $(-)_0$ denotes the degee 0 part of the graded algebra. Then restriction of sections to X_σ defines a map of algebras

$$A[\sigma^{-1}]_0 \to (\bigoplus_{n \geq 0} \Gamma(X^{\mathrm{ss}}, \mathcal{L}^n)^G)[\sigma^{-1}]_0 \cong \Gamma(X^{\mathrm{ss}}, \mathcal{O}_{X^{\mathrm{ss}}})^G,$$

where the isomorphism is induced by the natural inclusion of the right-hand-side into the left hand side, which is an isomorphism because $\sigma|_{X^{\mathrm{ss}}}$ is non-vanishing. This defines a map $X_\sigma/G \to D_+(\sigma)$ for every homogeneous $\sigma \in A$. These maps are compatible with the identifications $X_\sigma \cap X_\gamma = X_{\sigma\gamma}$ and $D_+(\sigma) \cap D_+(\gamma) = D_+(\sigma\gamma)$ for two homogeneous elements σ, γ, and in fact $X_\sigma \subset X^{\mathrm{ss}}$ is the preimage of $D_+(\sigma)$ under the canonical map $X^{\mathrm{ss}} \to \mathrm{Proj}(A)$ (one can see this from the functor-of-points of Proj [S4, Tag 01N9]).

By the étale base change formula **??**, it suffices to check the each of the conditions of Definition 14.7 for the map $X_\sigma/G \to D_+(\sigma)$, i.e., we must show that

$$\Gamma(X_\sigma, \mathcal{O}_X)^G \cong A[\sigma^{-1}]_0.$$

This isomorphism even holds before taking G-invariants. For the graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$, the degree 0 piece of the localization with respect to $\sigma \in \Gamma(X, \mathcal{L}^k)$ is

$$\mathrm{colim}\left(\Gamma(X, \mathcal{O}_X) \xrightarrow{\sigma \cdot (-)} \Gamma(X, \mathcal{L}^k) \xrightarrow{\sigma \cdot (-)} \Gamma(X, \mathcal{L}^{2k}) \to \cdots\right),$$

which is the global sections of the sheaf $\mathrm{colim}(\mathcal{O}_X \to \mathcal{L}^k \to \mathcal{L}^{2k} \to \cdots)$. The later sheaf admits a canonical isomorphism with $j_* \mathcal{O}_{X_\sigma}$, where $j : X_\sigma \subset X$ is the inclusion. □

15.2 Strongly étale maps

To summarize the proof of Proposition 15.1: the good moduli space of X^{ss}/G is obtained by gluing together the good moduli spaces $X_\sigma/G \to$

---

[1]Note that this is projective over $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^G)$.

$\mathrm{Spec}(\Gamma(X_\sigma, \mathcal{O}_X)^G)$ along their intersections. Theorem 14.10 implies that for a general stack with a good moduli space, a similar construction must work as well, but gluing along étale maps rather than Zariski open immersions.

The key fact needed for the proof of Proposition 15.1 was that for homogeneous elements $\sigma, \gamma \in A$, the open immersion $X_{\sigma\gamma}/G \subset X_\sigma/G$ induces an open immersion on good moduli spaces, and furthermore the diagram

$$
\begin{array}{ccc}
X_{\sigma\gamma}/G & \longrightarrow & X_\sigma/G \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\Gamma(X_{\sigma\gamma}, \mathcal{O}_X)^G) & \longrightarrow & \mathrm{Spec}(\Gamma(X_\sigma, \mathcal{O}_X)^G)
\end{array}
$$

is cartesian. The latter condition was needed to ensure that X_σ is the preimage of $D_+(\sigma) \subset \mathrm{Proj}(A)$. This motivates the following:

Definition 15.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks with good moduli spaces X and Y respectively. We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is *strongly étale* if the induced map $X \to Y$ is étale, and the corresponding map $\mathcal{X} \to X \times_Y \mathcal{Y}$ is an isomorphism.

We say that a morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *inertia preserving* if for any $\xi \in \mathcal{X}(T)$ the induced group homomorphism $\mathrm{Aut}_{\mathcal{X}}(\xi) \to \mathrm{Aut}_{\mathcal{Y}}(f(\xi))$ is an isomorphism. We observe the following lemma, which will be our main tool for constructing good moduli spaces.

L:gms_construction

Lemma 15.3. *Let* $f : \mathcal{X}_0 \to \mathcal{X}$ *be a surjective inertia preserving étale morphism of algebraic stacks. If* $\mathcal{X}_0$ *and* $\mathcal{X}_1 := \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$ *admit good moduli spaces, and the two projections* $p_1, p_2 : \mathcal{X}_1 \to \mathcal{X}_0$ *are strongly étale, then* $\mathcal{X}$ *admits a good moduli space, and the morphism* $\mathcal{X}_0 \to \mathcal{X}$ *is strongly étale.*

Proof. The hypotheses imply the existence of a diagram

$$
\begin{array}{ccc}
\mathcal{X}_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{X}_0 & \xrightarrow{f} & \mathcal{X}\ , \\
\downarrow \qquad \downarrow & & \\
X_1 \underset{q_2}{\overset{q_1}{\rightrightarrows}} X_0 & &
\end{array}
$$

where both the square with p_1, q_1 and the square with p_2, q_2 are cartesian, and the vertical arrows are good moduli spaces. Because good moduli spaces are closed under base change, $X_1 \times_{X_0} X_1$ is the good moduli space for $\mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_1$, and this gives a composition map $X_1 \times_{X_0} X_1 \to X_1$ that makes $X_1 \rightrightarrows X_0$ an

étale groupoid. One can show that f being inertia preserving implies that $X_1 \to X_0 \times X_0$ is a monomorphism, hence $X_1 \rightrightarrows X_0$ is an étale equivalence relation and defines an algebraic space X. By étale descent, there is a map $\mathfrak{X} \to X$ such that $\mathfrak{X}_0 \to X_0 \times_X \mathfrak{X}$ is an isomorphism. It follows that $\mathfrak{X} \to X$ is a good moduli space. $\qquad\square$

15.3 Codimension-2 filling conditions

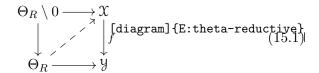Before stating our main theorem, we must introduce the relevant "valuative" criteria for algebraic stacks.

15.3.1 Θ-reductivity

Definition 15.4. We define the stack $\Theta := \mathbb{A}^1/\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t])/\mathbb{G}_m$, where the coordinate t on $\mathbb{A}^1$ has weight -1. For any other stack $\mathfrak{X}$, we denote $\Theta_{\mathfrak{X}} := \Theta \times_{\mathbb{Z}} \mathfrak{X}$. If R is a discrete valuation ring (DVR) with maximal ideal $(\pi) \subset R$, then we denote the codimension 2 closed point $\{t = \pi = 0\}$ in Θ_R by 0.

Definition 15.5. A morphism $f : \mathfrak{X} \to \mathcal{Y}$ of locally noetherian algebraic S-stacks is Θ-*reductive* if for every DVR R, any commutative diagram with solid arrows

$$\begin{array}{ccc} \Theta_R \setminus 0 & \longrightarrow & \mathfrak{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Theta_R & \longrightarrow & \mathcal{Y} \end{array} \qquad \texttt{[diagram]\{E:theta-reductive\}} \quad (15.1)$$

admits a unique dotted arrow making the diagram commute. We say that a stack $\mathfrak{X}$ is Θ-reductive if it is Θ-reductive over the base S.

The important example to keep in mind is the following:

Example 15.6. The stack $B\operatorname{GL}_n$ is Θ-reductive. When $\mathcal{Y} = S$ is the base, then Definition 15.8 amounts to the condition that any map $\Theta_R \setminus 0 \to \mathfrak{X}$ over S can be uniquely extended to Θ_R. What this means, concretely, is that any $\mathbb{G}_m$-equivariant locally free sheaf on $\operatorname{Spec}(R[t]) \setminus \{t = \pi = 0\}$ extends uniquely to an equivariant locally free sheaf. A result of Serre says that if X is a normal 2-dimensional scheme and $j : U \subset X$ is the open compliment of a closed point, then $j_* : \operatorname{QCoh}(U) \to \operatorname{QCoh}(X)$ maps locally free sheaves to locally free sheaves. Applying this to $j : \operatorname{Spec}(R[t]) \setminus \{t = \pi = 0\} \to \operatorname{Spec}(R[t])$

shows what we need, because j_* automatically takes equivariant sheaves to equivariant sheaves.

Exercise 15.1. *Show that any stack of the form* $\mathrm{Spec}(A)/\mathrm{GL}_n$ *is* Θ*-reductive in two steps:*

- *Show that a composition of* Θ*-reductive morphisms is* Θ*-reductive.*

- *Show that an affine morphism is* Θ*-reductive.*

Then apply Example 15.6.

15.3.2 $\mathbb{S}$-completeness

This condition is quite similar to Θ-reductivity, but has some nice implications.

Definition 15.7. For any DVR R with maximal ideal $(\pi) \subset R$, we define the stack
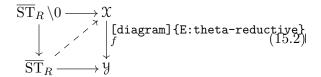$$\overline{\mathrm{ST}}_R := \mathrm{Spec}(R[s,t]/(st - \pi))/\mathbb{G}_m,$$
where the $\mathbb{G}_m$ action is termined by the grading of the algebra in which s has weight 1 and t has weight -1. We denote the point $\{s = t = 0\}$ by $0 \in \overline{\mathrm{ST}}_R$

Note that $R[s,t]/(st - \pi)$ is a regular noetherian scheme of dimension 2, and 0 is a closed point.

Definition 15.8. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of locally noetherian algebraic S-stacks is $\mathbb{S}$-*complete* if for every DVR R, any commutative diagram with solid arrows

$$
\begin{array}{ccc}
\overline{\mathrm{ST}}_R \backslash 0 & \longrightarrow & \mathfrak{X} \\
\downarrow & \nearrow & \downarrow {\scriptstyle f} \\
\overline{\mathrm{ST}}_R & \longrightarrow & \mathfrak{Y}
\end{array}
\qquad \text{[diagram]\{E:theta-reductive\}} \qquad (15.2)
$$

admits a unique dotted arrow making the diagram commute. We say that a stack $\mathfrak{X}$ is $\mathbb{S}$-complete if it is $\mathbb{S}$-complete over the base S.

As in the case of Θ-reductivity, we have:

Example 15.9. Any stack of the form $\mathrm{Spec}(A)/\mathrm{GL}_n$ is $\mathbb{S}$-complete. Indeed, the argument for showing $B\,\mathrm{GL}_n$ is $\mathbb{S}$-complete is identical to Example 15.6, and one extends this to $\mathrm{Spec}(A)/\mathrm{GL}_n$ exactly as in Exercise 15.1.

Remark 15.10. This shows that for any field k and reductive k-group G, the stack BG is $\mathbb{S}$-complete. In fact the converse holds as well [AHLH, Prop. 3.47], i.e., an affine algebraic k-group G is reductive if and only if BG is $\mathbb{S}$-complete.

Let R be a DVR and K its field of fractions. Then we have

$$\overline{\mathrm{ST}}_R \setminus \{t = 0\} \cong \mathrm{Spec}(R[t^{\pm}])/\mathbb{G}_m \cong \mathrm{Spec}(R), \text{ and}$$
$$\overline{\mathrm{ST}}_R \setminus \{s = 0\} \cong \mathrm{Spec}(R[s^{\pm}])/\mathbb{G}_m \cong \mathrm{Spec}(R).$$

So, $\overline{\mathrm{ST}}_R \setminus 0$ is isomorphic to two copies of $\mathrm{Spec}(R)$ glued along $\mathrm{Spec}(K)$. Note also that the fiber of the map $\overline{\mathrm{ST}}_R \to \mathrm{Spec}(R)$ over the special point $\{\pi = 0\}$ is $\mathrm{Spec}(k[s,t]/(st))/\mathbb{G}_m$, which is two copies of $\mathbb{A}^1$ with opposite $\mathbb{G}_m$-actions glued together at the origin.

Thus $\mathbb{S}$-completeness is something like a separatedness condition. It says that two maps $\mathrm{Spec}(R) \to \mathcal{X}$ that agree at the generic point do not necessarily agree everywhere (the stacks we are considering are rarely separated!), but the image of the special points x_1, x_2 are connected by a map $\mathbb{A}^1_k \cup_{\{0\}} \mathbb{A}^1_k/\mathbb{G}_m \to \mathcal{X}$ mapping the non-zero point on the left side to x_1 and the nonzero point on the right side to x_2.

The relationship with separatedness is made more precise by the following:

Proposition 15.11 (Sepatedness and Properness, see [AHLH, Prop. 3.45])**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type and with affine diagonal over a locally noetherian algebraic space S, and let $\mathcal{X} \to X$ be a good moduli space. Then $X \to S$ is separated if and only if $\mathcal{X}$ is $\mathbb{S}$-complete, and $X \to S$ is proper if and only if $\mathcal{X} \to S$ is quasi-compact and satisfies the "existence part" of the noetherian valuative criterion:*

For any DVR R with fraction field K, any commutative diagram of solid arrows admits an extension of DVR's $R \subset R'$ and a filling of dotted arrows making the diagram commute:

$$\begin{array}{ccc}
\mathrm{Spec}(K') \dashrightarrow \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\
\downarrow \qquad\qquad\qquad \downarrow & & \downarrow \\
\mathrm{Spec}(R') \dashrightarrow \mathrm{Spec}(R) & \longrightarrow & S
\end{array}$$

15.4 Existence criteria for good moduli spaces

Theorem 15.12. [AHLH, Thm. A] *Let $\mathcal{X}$ be an algebraic stack locally of finite type and with affine diagonal over a quasi-separated locally noetherian*

algebraic space S. Then $\mathfrak{X}$ admits a separated good moduli space if and only if:

1. *Every point of $\mathfrak{X}$ specializes to a closed point, and closed points have linearly reductive stabilizers;*

2. *$\mathfrak{X}$ is Θ-reductive relative to S; and*

3. *$\mathfrak{X}$ is $\mathbb{S}$-complete relative to S.*

If $\mathfrak{X} \to S$ is quasi-compact and $\mathfrak{X}$ is of characteristic 0, then (1) is implied by (3) (see Remark 15.10) and is thus not necessary.

Remark 15.13. In fact, in **??**, it suffices to check the filling conditions (2) and (3) for DVR's that are essentially of finite type over S. [AHLH] also provides slightly more refined criteria for good moduli spaces that are not necessarily separated.

In the rest of this section, we will explain the proof, which is a slight reorganization of the methods of [AHLH].

The necessity of (1) is Exercise 14.2. The necessity of (2) and (3) follow from the stability of these properties under representable étale base change [AHLH, Prop. 3.18, Prop. 3.41], Theorem 14.10, and the fact that the map $\mathrm{Spec}(A)/\mathrm{GL}_n \to \mathrm{Spec}(A^{\mathrm{GL}_n})$ for an affine GL_n-scheme is Θ-reductive and $\mathbb{S}$-complete (see Exercise 15.1 and Example 15.9).

We now turn to the sufficiency of these criteria.

Θ-surjective étale covers

Definition 15.14. For any field k, denote by $j : \mathrm{Spec}(k) \cong (\mathbb{A}^1_k \setminus \{0\})/\mathbb{G}_m \hookrightarrow \Theta_k$ the open immersion. Then a morphism of algebraic stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is Θ-surjective if and only if for any algebraically closed field k, any commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(k) & \longrightarrow & \mathfrak{X} \\
{\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
\Theta_k & \longrightarrow & \mathfrak{Y}
\end{array}
\qquad \text{[diagram]\{E:theta-surjective\}} \quad (15.3)
$$

of solid arrows can be filled in with a dotted arrow.

This condition is important because it allows one to descend good moduli spaces:

Proposition 15.15. *Let $\mathcal{X}$ and $\mathcal{X}_0$ be algebraic stacks locally of finite type and with affine diagonal over a noetherian ring R, and let $f : \mathcal{X}_0 \to \mathcal{X}$ be a surjective affine étale morphism. If $\mathcal{X}_0$ admits a good moduli space whose diagonal is affine, and f is inertia preserving and Θ-surjective, then $\mathcal{X}$ admits a good moduli space and f is strongly étale.*

We wish to apply Lemma 15.3 to construct a good moduli space for $\mathcal{X}$. So, we will need the following criterion for a map to be strongly étale, known as Luna's fundamental lemma:

Lemma 15.16 ([A1, Thm. 6.10]). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable étale morphism of algebraic stacks with good moduli spaces $\pi_{\mathcal{X}} : \mathcal{X} \to X$ and $\pi_{\mathcal{Y}} : \mathcal{Y} \to Y$. If $x \in \mathcal{X}$ is a closed point such that*

1. *$f(x) \in |\mathcal{Y}|$ is closed, and*

2. *f induces an isomorphism of automorphism groups at x,*

then there is an open subspace $U \subset X$ such that the induced map $\pi_{\mathcal{X}}^{-1}(U) \to \mathcal{Y}$ is strongly étale.

The key observation is that one can use Θ-surjectivity to show that a morphism preserves closed points.

Lemma 15.17. *Let R be a noetherian ring, let A be a finite type GL_n-equivariant R-algebra, and let $\mathcal{Y}$ be an algebraic stack that is locally of finite type over R. Then any Θ-surjective morphism $f : \mathcal{Y} \to \mathrm{Spec}(A)/\mathrm{GL}_n$ maps closed points of $\mathcal{Y}$ to closed points of $\mathrm{Spec}(A)/\mathrm{GL}_n$.*

Proof. Let $y \in \mathcal{Y}(k)$ be a representative of a closed point in $\mathcal{Y}$, and assume $f(y) \in \mathrm{Spec}(A)/\mathrm{GL}_n$ is not a closed point. The Hilbert-Mumford criterion, **??** below, says that after replacing k with a finite extension, we can find a map $\Theta_k \to \mathrm{Spec}(A)/\mathrm{GL}_n$ mapping $\{1\}$ to $f(y)$ and mapping $\{0\}$ to a closed point. But Θ-surjectivity implies that one can lift this to a map $\Theta_k \to \mathcal{Y}$ mapping $\{1\}$ to y and mapping $\{0\}$ to a distinct point in $|\mathcal{Y}|$. This contradicts that y was closed. $\square$

The previous lemma used the following important result of [MFK, Sect. 2.1].

Lemma 15.18 (Hilbert-Mumford criterion). *Let R be a noetherian ring, and let $\mathcal{X} \in \mathrm{Spec}(A)/\mathrm{GL}_n$ where A is a finite type R-algebra. Let k be a finite type R-field, and let $x \in \mathcal{X}(k)$ be a point that specializes to a closed point $y \in |\mathcal{X}|$. Then after passing to a finite extension $k \subset k'$ there is a morphism $f : \Theta_{k'} \to \mathcal{X}$ with $f(1) = x$ and $f(0) = y$ in $|\mathcal{X}|$.*

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Proof of Proposition 15.15. Let $q : \mathfrak{X}_0 \to X_0$ be a good moduli space. By Theorem 14.10 we may choose an affine étale cover $X_0' \to X_0$ such that $\mathfrak{X}_0' := \mathfrak{X}_0 \times_{X_0} X_0'$ is a disjoint union of cohomologically affine stacks of the for $\mathrm{Spec}(A)/\mathrm{GL}_n$. The induced map $\mathfrak{X}_0' \to \mathfrak{X}$ still satisfies the hypotheses of the proposition, so it suffices to assume that $\mathfrak{X}_0$ is a disjoint union of stacks of this form.

The projections $\mathfrak{X}_1 := \mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{X}_0 \to \mathfrak{X}_0$ are affine morphisms, so if $\mathfrak{X}_1 = \mathrm{Spec}_{\mathfrak{X}_0}(\mathcal{A})$ for some algebra $\mathcal{A} \in \mathrm{QCoh}(\mathfrak{X}_0)$, $\mathrm{Spec}_X(q_*(\mathcal{A}))$ is a good moduli space for $\mathfrak{X}_1$. The properties of being Θ-surjective and inertia preserving are stable under base change, so the projection maps $\mathfrak{X}_1 \to \mathfrak{X}_0$ are strongly étale by Lemma 15.16. Thus the conclusion follows from Lemma 15.3. □

Exercise 15.2. *The reason the hypothesis in Proposition 15.15 is "Θ-surjective" and not "maps closed points to closed points" is that the latter is not stable under base change:*

Let k be an algebraically closed field, and let $C \subset \mathbb{P}^2$ be the nodal cubic. Then the normalization $\tilde{C}$ is $\mathbb{P}^1$, and the $\mathbb{G}_m$-action on $\tilde{C}$ fixing the two points over the node descends to a $\mathbb{G}_m$-action on C. Let $X = \mathrm{Spec}(k[s,t]/(st))$ be the union of two copies of $\mathbb{A}^1$ at the origin, equipped with a $\mathbb{G}_m$-action in which t has weight -1 and s has weight 1. There is an étale $\mathbb{G}_m$-equivariant cover $X \to C$ mapping $0 \in X$ to the node in C. Let $\mathfrak{X} = C/\mathbb{G}_m$ and $\mathfrak{X}_0 = X/\mathbb{G}_m$, and show the following:

1. *$\mathfrak{X}$ does* not *admit a good moduli space;*

2. *The map $\mathfrak{X}_0 \to \mathfrak{X}$ is surjective, affine, étale, preserves closed points, is inertia preserving;*

3. *The projection (either one) $\mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{X}_0 \to \mathfrak{X}_0$ does not preserve closed points.*

The proof of Theorem 15.12

Proposition 15.15 reduces the existence of good moduli spaces to the ability to construct Θ-surjective and inertia preserving affine étale covers.

T:gms_shrink

Theorem 15.19. *Let R be a noetherian ring, let $\mathcal{W} = \mathrm{Spec}(A)/\mathrm{GL}_n$ for a finite type GL_n-equivariant R-algebra A, and let $\mathfrak{X}$ be an algebraic stack locally of finite type with affine diagonal over R. Let $f : \mathcal{W} \to \mathfrak{X}$ be an affine étale morphism, and let $w \in \mathcal{W}$ be a closed point mapping to a closed point $x \in \mathfrak{X}$ such that the map of residual gerbes $\mathcal{G}_w \to \mathcal{G}_x$ is an isomorphism. If $\mathfrak{X}$*

is Θ-reductive and $\mathbb{S}$-complete, then there is an invariant element $\sigma \in A^{\mathrm{GL}_n}$ that does not vanish at w and such that the restriction

$$f : \mathrm{Spec}(A[\sigma^{-1}])/\mathrm{GL}_n \to \mathcal{X}$$

is Θ-surjective and inertia preserving.

This is a combination of several results of [AHLH], whose proofs are somewhat involved. So, rather than provide the argument here, we will simply explain how to extract this statement from the results of [AHLH].

Proof. This is precisely the statement of [AHLH, Prop. 4.4], except that instead of the hypothesis that $\mathcal{X}$ is $\mathbb{S}$-complete, that proposition assumes $\mathcal{X}$ satisfies a different condition, called *unpunctured inertia*, which we have not discussed. The claim that $\mathbb{S}$-completeness and Θ-reductivity imply unpunctured inertia is essentially [AHLH, Thm. 5.4], although there it is stated with the additional hypothesis that closed points of $\mathcal{X}$ have linearly reductive stabilizers. The reductive stabilizer hypothesis is only used in the proof to produce a local quotient presentation $\mathrm{Spec}(A)/\mathrm{GL}_n \to \mathcal{X}$ around a given closed point in $\mathcal{X}$, but that data is already supplied by hypothesis in the statement of this theorem. $\qquad\square$

We can now complete the proof of Theorem 15.12 as follows:

One can use étale descent over S to reduce to the case where $S = \mathrm{Spec}(R)$ is affine. Then, around every closed point $x \in \mathcal{X}$, choose a local quotient presentation $f : (\mathrm{Spec}(A)/\mathrm{GL}_n, w) \to (\mathcal{X}, x)$, in which the morphism is affine and $\mathrm{Spec}(A)/\mathrm{GL}_n$ is cohomologically affine. Theorem 15.19 implies that after shrinking the source, we may assume that f is Θ-surjective and inertia preserving. If $\mathcal{X}$ is quasi-compacw, we can repeat this for finitely many closed points of $\mathcal{X}$ as needed, resulting in a morphism $f : \bigsqcup_i \mathrm{Spec}(A_i)/\mathrm{GL}_{n_i} \to \mathcal{X}$ that satisfies the conditions of Proposition 15.15, so we are done.

If $\mathcal{X}$ is not quasi-compact, then the map f will not be surjective, but will satisfy the conditions of Proposition 15.15 with respect to its open image $\mathcal{U} \subset \mathcal{X}$. Thus we can cover $\mathcal{X}$ by a family of open substacks such that $\mathcal{U}$ admits a good moduli space and the embedding $\mathcal{U} \subset \mathcal{X}$ is Θ-surjective. One can check that open substacks of this form is closed under unions, so we get our moduli space for $\mathcal{X}$.

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Lecture 16

Filtrations and stability I

References:
Date: 4/30/2020
Exercises: 5

Theorem 15.12 gives criteria for the existence of a good moduli space, but how does one find stacks satisfying these criteria in practice? Typically there is a naive moduli problem that is "large," in the sense that it defines a non quasi-compact algebraic stack. Then one introduces a stability condition on objects, and when everything works nicely, the moduli of semistable objects is bounded and admits a good moduli space. Over the next three lectures, we will discuss semistability, and the structures around it.

16.1 Example: the moduli of vector bundles on a curve

Let us first state what the results say in an interesting classical example, the stack of vector bundles on a smooth projective curve C over a field k, which we denote $\mathrm{Bun}(C)$. For any k-scheme T, a T-point of $\mathrm{Bun}(C)$ is a locally free sheaf on $T \times C$. We introduced this stack in Exercise 4.1, and it follows from Theorem 12.10 that it is algebraic, locally of finite type over k, and has affine diagonal.

Bun is not quasi-compact

We first break $\mathrm{Bun}(C)$ into pieces based on "topological invariants," i.e., quantities that are locally constant in families. Given a locally free sheaf E

on $T \times C$, we will let C_t denote the fiber of the projection $T \times C \to T$ over any point $t \in T$, and we will let E_t denote the restriction of E to C_t. We observe:

- The function $t \mapsto \operatorname{rank}(E_t)$ is locally constant in t.

- We define the degree of E_t, $\deg(E_t)$, to be the degree of the top exterior power of E_t. The function $t \mapsto \deg(E_t)$ is locally constant in t.

The fact that $\operatorname{rank}(E_t)$ and $\deg(E_t)$ are locally constant means that the substack $\operatorname{Bun}_{r,d}(C) \subset \operatorname{Bun}(C)$ parameterizing families of constant rank r and degree d is an open and closed substack. Thus $\operatorname{Bun}(C) = \bigsqcup_{r,d} \operatorname{Bun}_{r,d}(C)$ is a stack-theoretic (as opposed to set-theoretic) disjoint union, and each component is non-empty.

Remark 16.1. One important property of the rank and degree that we will use below is that they are *additive* in the sense that for a short exact sequence of locally free sheaves $0 \to E_1 \to E_2 \to E_3 \to 0$, one has $\operatorname{rank}(E_2) = \operatorname{rank}(E_1) + \operatorname{rank}(E_3)$ and $\deg(E_2) = \deg(E_1) + \deg(E_3)$. We can also extend the definition of rank and deg to any coherent sheaf on C. For any sheaf F whose support has dimension 0, we define $\deg(F) = \operatorname{length}(F)$. For a general coherent sheaf E, let $F \subset E$ denote the maximal subsheaf with zero-dimensional support, sometimes called the *torsion subsheaf* of E, and define $\deg(E) = \deg(F) + \deg(E/F)$. One can check that additivity still holds for rank and deg with these definitions.

One typically focuses on one component $\operatorname{Bun}_{r,d}(C)$ at a time, but even these stacks are not quasi-compact:

EXC:bung_not_qc

Exercise 16.1. *Fix an ample line bundle $\mathcal{O}_C(1)$ on C. Show that there is no finite type k-scheme T with a locally free sheaf E on $T \times C$ such that all of the bundles $\mathcal{O}_C(n) \oplus \mathcal{O}_C(-n)$ for $n \in \mathbb{Z}$ appear as fibers E_t for some $t \in T$. These bundles all have rank 2 and degree 0, so $\operatorname{Bun}_{2,0}(C)$ does not admit a surjection from a scheme T of finite type over k, and hence it is not quasi-compact. Generalize this to show that none of the stacks $\operatorname{Bun}_{r,d}(C)$ are quasi-compact for $r > 1$.*

One might suspect that the issue in Exercise 16.1 is that we have missed some additional locally constant functions, and can therefore break up $\operatorname{Bun}_{r,d}(C)$ into smaller locally closed substacks. In fact, the situation is worse than that. One can use Claim 16.2 below, and a refined version of Exercise 16.1 to show that: 1) no point of $\operatorname{Bun}_{r,d}(C)$ with $r > 1$ is closed, and 2) no connected component of $\operatorname{Bun}_{r,d}(C)$ with $r > 1$ is quasi-compact.

Claim 16.2. *Assuming k is algebraically closed, let E be a locally free sheaf on C of rank $r > 1$, and let $\mathcal{O}_C(1)$ be an ample line bundle on C. The for any $n \in \mathbb{Z}$ sufficiently large, one can find a family of locally free sheaves over $\mathbb{A}_k^1$ whose fiber at 1 is isomorphic to E and whose fiber at 0 is isomorphic to $\mathcal{O}_C(-n) \oplus Q$ for some locally free sheaf Q.*

Proof. We first show that for $n \gg 0$, one can find an embedding $\mathcal{O}_C(-n) \hookrightarrow E$ whose quotient $Q = E/\mathcal{O}_C(-n)$ is locally free. For all n sufficiently large, $E(n)$ is globally generated, which gives a short exact sequence

$$0 \to K \to \mathcal{O}_C^{\oplus m} \to E(n) \to 0$$

Taking the total space of these locally free sheaves gives a closed embedding of schemes $\operatorname{Tot}_C(K) \hookrightarrow \operatorname{Tot}_C(\mathcal{O}_C^{\oplus m}) \cong C \times \mathbb{A}^m$. We have $\dim \operatorname{Tot}_C(K) = \operatorname{rank}(K) + 1 = m - \operatorname{rank}(E) + 1 < m$, so the composition

$$\operatorname{Tot}_C(K) \to C \times \mathbb{A}_k^m \to \mathbb{A}_k^m$$

can not be surjective. Choosing a fiber of the map $C \times \mathbb{A}_k^m \to \mathbb{A}_k^m$ that does not meet $\operatorname{Tot}_C(K)$ corresponds to a section of $\mathcal{O}_C^{\oplus m}$ that does not lie in the fiber K_p for any $p \in C$. In particular the section s of $E(n)$ induced by $\mathcal{O}^{\oplus m} \to E(n)$ is nowhere vanishing, and hence the resulting map $s : \mathcal{O}_C \to E(n)$ is injective with locally free quotient. Twisting by $\mathcal{O}_C(-n)$ gives an embedding $\mathcal{O}_C(-n) \hookrightarrow E$ whose quotient Q is locally free.

Now consider the map $x : Q \to Q$ given by multiplication by a scalar in $x \in k$. We can pull back the extension (**??**) to define a new vector bundle

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & E_x & \longrightarrow & Q & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow {\scriptstyle x} & & \\
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0
\end{array}
$$

If x is a unit, then the vertical map is an isomorphism, but if $x = 0$, then the vertical map is not an isomorphism and $E_x \cong L \oplus Q$.

We can actually regard x as the coordinate on $\mathbb{A}_k^1$, so that multiplication by x is actually a map of vector bundles $p^*(Q) \to p^*(Q)$ on $\mathbb{A}^1 \times C$, where $p : \mathbb{A}^1 \times C \to C$ is the projection. Then the construction of E_x above actually defines a family of vector bundles over $\mathbb{A}_k^1$, whose fiber at any non-zero point is isomorphic to E, and whose fiber at 0 is isomorphic to $L \oplus Q$. $\qquad \square$

Exercise 16.2. *Every locally free sheaf on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_{\mathbb{P}^1}(n)$. Using the construction in the proof of*

Claim 16.2, *describe what vector bundles can appear as the fiber over* 0 *in a family over* $\mathbb{A}^1$ *whose restriction to* $\mathbb{A}^1 \setminus 0$ *is the constant family with fiber* $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$.

Semistability

For any locally free sheaf E on C, we define the *slope* of E as

$$\nu(E) := \deg(E)/\operatorname{rank}(E).$$

Then we introduce the following

Definition 16.3. A locally free sheaf E on C is *unstable* if there is a subobject $F \subset E$ with $\nu(F) > \nu(E)$. E is defined to be

- *semistable* if it is not unstable;

- *stable* if it is proper and there are no proper subobjects with $\nu(F) = \nu(E)$ either;

- *polystable* if it is a direct sum of stable objects of the same slope (which implies semistable as well).

One can show that the automorphism group of a stable object consists only of $\mathbb{G}_m$ acting by scaling. The main result is the following:

Theorem 16.4. *The substack* $\operatorname{Bun}(C)^{\mathrm{ss}} \subset \operatorname{Bun}(C)$ *parameterizing semistable bundles is an open substack, hence algebraic.* $\operatorname{Bun}_{r,d}(C)^{\mathrm{ss}}$ *is quasi-compact and admits a proper (in fact, projective) good moduli space* $q : \operatorname{Bun}_{r,d}(C)^{\mathrm{ss}} \to M_{r,d}(C)$. *Furthermore,*

1. *Every semistable E admits a filtration, called a* Jordan-Holder *filtration, whose associated graded locally free sheaf E' is polystable.*

2. *The isomorphism class of E' is uniquely determined by E, and two sheaves lies in the same fiber of q if and only if they lead to the same polystable sheaf.*

The classical proof of this fact uses geometric invariant theory [????]. We will ultimately prove this, in characteristic 0, and its generalization to G-bundles for general G, using Theorem 15.12.

Harder-Narasimhan theory

In passing to the semistable locus, Theorem 16.4 discards many locally free sheaves. It turns out that the unstable sheaves also have a nice structure. Given a filtration of a locally free sheaf E with locally free graded pieces

$$0 \subsetneq E_p \subsetneq E_{p-1} \subsetneq \cdots \subsetneq E_0 = E,$$

we refer to the *shape* of the filtration as the set of ordered pairs $\alpha = ((r_p, d_p), \ldots, (r_0, d_0))$, where $r_i = \mathrm{rank}(\mathrm{gr}_i(E))$ and $d_i = \deg(\mathrm{gr}_i(E))$. We can define the *Shatz polytope* $\mathrm{Pol}(\alpha) \subset \mathbb{R}^2$ of the shape α to be the convex hull of the points

$$(0,0), (r_p, d_p), (r_p + r_{p-1}, d_p + d_{p-1}), \cdots, (r_p + \cdots + r_0, d_p + \cdots + d_0).$$

We define a partial order on shapes by saying $\alpha \leq \beta$ if $\mathrm{Pol}(\alpha) \subset \mathrm{Pol}(\beta)$.

Theorem 16.5 (Harder-Narasimhan)**.** *For any locally free sheaf E on C, there is a* unique *filtration, called the* Harder-Narasimhan (HN) filtration, $0 \subsetneq E_p \subsetneq E_{p-1} \subsetneq \cdots \subsetneq E_0 = E$ *such that* $\mathrm{gr}_i(E)$ *is locally free and semistable for all i, and the slopes $\nu(\mathrm{gr}_i(E))$ are strictly increasing in i. Furthermore:*

1. *For any $\alpha = ((r_p, d_p), \ldots, (r_0, d_0))$, representing the shape of a HN filtration, the stack $\mathcal{S}_\alpha$ that assigns to T the groupoid of filtered vector bundles $0 \subsetneq E_p \subsetneq \cdots \subsetneq E_0$ on $T \times C$ whose restriction to every fiber over T is a HN filtration of shape α is algebraic, and the map $\mathcal{S}_\alpha \to \mathrm{Bun}(C)$ that forgets the filtration is a locally closed immersion.*

2. *There is map $\prod \mathrm{gr}_i : \mathcal{S}_\alpha \to \prod_{i=0}^{p} \mathrm{Bun}_{r_i, d_i}^{\mathrm{ss}}(C)$ is a map of algebraic stacks.*

3. *The closure of $\mathcal{S}_\alpha \subset \mathrm{Bun}(C)$ lies in the union of $\mathcal{S}_\beta$ for all $\beta \geq \alpha$ in the partial ordering on shapes described above.*

The classical way to show the existence and uniqueness of the Harder-Narasimhan filtration is to show that E has a maximal subsheaf $F \subset E$ of maximal slope, and that the quotient E/F is locally free. This forms the basis of an inductive construction: the HN filtration of E has first term F, and the remaining terms are the preimage of the HN filtration of E/F. We refer to [**????**] for the details.

While this argument is very nice and generalizes to many other examples of moduli functors parameterizing objects in an additive category, it does not generalize, for instance to the stack of G bundles on a curve for other reductive groups G. We will therefore take a different approach, which simultaneously gives the properties (1)-(3) above as well.

16.2 The stack of filtered objects

In order to generalize [Theorem 16.4](#) and [Theorem 16.5](#) to other moduli problems, we will need a notion of "filtration" in an arbitrary moduli problem.

We begin with a classical construction. Let E be a quasi-coherent sheaf on C, and consider a $\mathbb{Z}$-weighted filtration of E, by which we mean a sequence of subsheaves $\cdots E_{w+1} \subset E_w \subset \cdots \subset E$ indexed by $w \in \mathbb{Z}$ and such that $E = \bigcup_w E_w$.

CONST:rees

Construction 16.6 (Rees construction). Given a diagram of sheaves on C indexed by $w \in \mathbb{Z}$,

$$\cdots \to E_{w+1} \to E_w \to \cdots \quad \text{\{E:filtration\_diagram\}} \quad (16.1)$$

we define the graded $\mathcal{O}_C[t]$-module, where t has weight -1,

$$R(E_\bullet) := \bigoplus_{w \in \mathbb{Z}} E_w$$

multiplication by t is the given map $E_{w+1} \to E_w$. This establishes an equivalence between the category of diagrams of the form (16.1) and graded quasi-coherent $\mathcal{O}_C[t]$-modules. This equivalence identifies $\mathbb{Z}$-*weighted filtrations*, i.e., diagrams in which the maps $E_{w+1} \to E_w$ are injective, with graded $\mathcal{O}_C[t]$-modules that are flat over $k[t]$. This equivalence of categories also holds with C replaced by any other scheme or algebraic stack.

We also know that graded quasi-coherent $\mathcal{O}_C[t]$-modules correspond to $\mathbb{G}_m$-equivariant quasi-coherent sheaves on $\mathbb{A}^1 \times C$, or equivalently quasi-coherent sheaves on the quotient stack

$$\mathbb{A}^1 \times C / \mathbb{G}_m \cong \Theta \times C,$$

where $\Theta = \mathbb{A}^1/\mathbb{G}_m$ as in [Definition 15.4](#). This essentially follows from the identification of $\mathrm{QCoh}((B\mathbb{G}_m)_C)$ with the category of graded objects in $\mathrm{QCoh}(C)$, and using the fact that $\Theta_C \to (B\mathbb{G}_m)_C$ is an affine morphism.

EXC:filtrations

Exercise 16.3. *Show that a diagram of the form* (16.1) *corresponds to a locally free sheaf on $\Theta \times C$ if and only all of the maps $E_{w+1} \to E_w$ are injective, $\mathrm{gr}_w := E_w/E_{w+1}$ is locally free, $E_w = 0$ for $w \gg 0$, and E_w stabilizes $w \ll 0$.*

[Exercise 16.3](#) shows that a $\mathbb{Z}$-weighted filtered locally free sheaf on C corresponds to a map $\Theta \to \mathrm{Bun}(C)$, which motivates the following:

Definition 16.7. Let $\mathcal{X}$ be a stack over a scheme S. Then the stack of ($\mathbb{Z}$-weighted) *filtrations in* $\mathcal{X}$, denoted $\mathrm{Filt}(\mathcal{X})$, assigns

$$\mathrm{Filt}(\mathcal{X}) : T \mapsto \mathrm{Map}_S(\Theta_T, \mathcal{X}).$$

The stack of ($\mathbb{Z}$-weighted) graded points of $\mathcal{X}$, denoted $\mathrm{Grad}(\mathcal{X})$, assigns

$$\mathrm{Grad}(\mathcal{X}) : T \mapsto \mathrm{Map}_S((B\mathbb{G}_m)_T, \mathcal{X}).$$

When we wish to emphasize the base, we will write $\mathrm{Filt}_S(\mathcal{X})$ and $\mathrm{Grad}_S(\mathcal{X})$.

Observe that there are several operations one can do with filtrations that have purely geometric interpretations:

- Restricting a map $f : \Theta_k \to \mathrm{Bun}(C)$ to the point $\{1\} \in \Theta_k$ corresponds to taking $(\cdots E_{w+1} \to E_w \to \cdots) \mapsto \bigcup_w E_w$. Thus in general we regard a map $f : \Theta_k \to \mathcal{X}$ as a filtration of the point $f(1) \in \mathcal{X}(k)$. This defines a map of stacks

  $$\mathrm{ev}_1 : \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}.$$

- Restricting $f : \Theta_k \to \mathrm{Bun}(C)$ to $\{0\}/\mathbb{G}_m$ corresponds to taking $(\cdots \to E_{w+1} \to E_w \to \cdots) \mapsto \bigoplus_w E_w/E_{w+1}$ as a graded vector bundle. So in general we define the associated graded object of a filtration $f : \Theta_k \to \mathcal{X}$ to be the restriction $f|_{\{0\}} : B\mathbb{G}_m \to \mathcal{X}$, and this defines a map of stacks

  $$\mathrm{ev}_0 : \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X}).$$

- Given a graded object $g : (B\mathbb{G}_m)_k \to \mathcal{X}$, one can compose with the projection $\Theta_k \to (B\mathbb{G}_m)_k$ to get a filtration $s(g) : \Theta_k \to \mathcal{X}$. In the case of $\mathrm{Bun}(C)$ this takes a graded locally free sheaf $\bigoplus_w F_w$ to the filtered locally free sheaf with $E_w = \bigoplus_{w' \geq w} F_{w'}$, so in general we think of $s(g)$ as the "split filtration" associated to the graded object g. This defines a map of stacks
  $$s : \mathrm{Grad}(\mathcal{X}) \to \mathrm{Filt}(\mathcal{X}).$$

An important property of these stacks is the following:

<div align="center">T:stack_of_filtered_objects</div>

Theorem 16.8. *Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation and with affine automorphism groups relative to an algebraic space S. Then $\mathrm{Filt}(\mathcal{X})$ is an algebraic stack locally of finite presentation and with affine automorphism groups over S. If $\mathcal{X}$ has affine diagonal, then so does $\mathrm{Filt}(\mathcal{X})$.*

This is proved in [HLP] (when $\mathfrak{X}$ has quasi-affine diagonal) and [HR3] using a version of Artin's criteria Theorem 12.9. The proof of Theorem 16.8 is very similar to the proof of Theorem 12.10, except that $\Theta_S \to S$ is not proper.

Exercise 16.4. *Follow the proof of Theorem 12.10 to prove Theorem 16.8 in the special case where $S = \mathrm{Spec}(k)$ is a field, and $\mathfrak{X} = BG$ for an affine k-group G. The key technical steps are:*

1. *Use Construction 16.6 to show that the pullback functor $\pi^* : k\text{-Mod} \to \mathrm{QCoh}(\Theta_k)$ admits a left adjoint $\pi_+ : \mathrm{QCoh}(\Theta_k) \to k\text{-Mod}$.*

2. *Use ?? to show that the stack $\mathrm{Filt}(\mathfrak{X})$ is effective.*

Now that we have a general notion of filtration with which to define stability, let us study what this definition gives us in the most concrete examples.

Example 16.9. If X is an qc.qs. algebraic space with a $\mathbb{G}_m$-action, then we can define the functor of fixed points

$$X_0(T) = \{\mathbb{G}_m\text{-equivariant maps } T \to X\},$$

where $\mathbb{G}_m$ acts trivially on T, and we define the functor of the attracting locus to be

$$X_+(T) = \{\mathbb{G}_m\text{-equivariant maps } T \times \mathbb{A}^1 \to X\},$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling, i.e., the coordinate function on $\mathbb{A}^1$ has weight -1. Now, there is a canonical point $\mathrm{pt} \to \mathrm{Grad}(B\mathbb{G}_m)$ classifying the identity map $B\mathbb{G}_m \to B\mathbb{G}_m$, and a canonical point $\mathrm{pt} \to \mathrm{Filt}(B\mathbb{G}_m)$ classifying the projection $\Theta = \mathbb{A}^1/\mathbb{G}_m \to B\mathbb{G}_m$. Then one can check that

$$X_+ \cong \mathrm{pt} \times_{\mathrm{Filt}(B\mathbb{G}_m)} \times \mathrm{Filt}(X/\mathbb{G}_m)$$
$$X_0 \cong \mathrm{pt} \times_{\mathrm{Grad}(B\mathbb{G}_m)} \mathrm{Grad}(X/\mathbb{G}_m),$$

so Theorem 16.8 implies that X_+ and X_0 are representable by algebraic spaces.

The notation reflects the fact that if $X = \mathrm{Spec}(A)$ and one chooses an equivariant embedding into a linear representation $X \hookrightarrow \mathbb{A}^n$, then X_+ and X_0 are the intersections of X with the linear subspace spanned by vectors of positive weight and weight 0 respectively. If I_+ and I_- denote the ideals in A generated by homogeneous elements of weight > 0 and < 0 respectively, then $X_+ = \mathrm{Spec}(A/I_+)$ and $X_0 = \mathrm{Spec}(A/(I_+ + I_-))$.

Example 16.10. For any cocharacter $\lambda : \mathbb{G}_m \to B\,\mathrm{GL}_n$, one can construct a map $E_\lambda : \Theta \to B\,\mathrm{GL}_n$ that corresponds to the $\mathbb{G}_m$-equivariant GL_n bundle on $\mathbb{A}^1$ given by $\mathbb{A}^1 \times \mathrm{GL}_n \to \mathbb{A}^1$ with $\mathbb{G}_m$ action given by $z \cdot (t, g) = (zt, \lambda(z)g)$. In terms of locally free sheaves, for any integer w there is a unique invertible sheaf $\mathcal{O}_\Theta(w)$ whose fiber at $\{0\}$ has weight w. A cocharacter $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$ corresponds to grading on the free $\mathcal{O}_S$ module of rank n, $\mathcal{O}_S^n = \bigoplus_w V_w$, and in this case $E_\lambda \cong \bigoplus_w \mathcal{O}(w) \otimes V_w$ as a locally free sheaf on Θ.

One can compute

$$P_\lambda := \mathrm{Aut}_{B\,\mathrm{GL}_n}(E_\lambda) = (\mathrm{GL}_n)_{\lambda+},$$

where $G_{\lambda+}$ denotes the construction of Example 16.9 for the action of $\mathbb{G}_m$ on GL_n given by $z \cdot g = \lambda(z)g\lambda(z)^{-1}$. More concretely, if $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$ corresponds to the grading $\mathcal{O}_S^n \cong \bigoplus_w V_w$, then P_λ is the group of automorphisms ϕ of $\mathcal{O}_S^n$ that are "upper triangular" in the sense that $\phi(V_w) \subset \sum_{w' \geq w} V_{w'}$. Choosing a basis for the V_w identifies P_λ with a group of block-upper-triangular matrices. This defines a map

$$\bigsqcup_{\lambda : \mathbb{G}_m \to G/\text{conjugation}} BP_\lambda \to \mathrm{Filt}(B\,\mathrm{GL}_n), \qquad \begin{matrix}\text{\{E:filt\_BGLn\}}\\(16.2)\end{matrix}$$

and one can show this is an isomorphism.

Exercise 16.5. *Show that the map* (16.2) *is an isomorphism of algebraic stacks.*

Combining the previous two examples, we have

Example 16.11. If $\mathcal{X} = X/\mathrm{GL}_n$ for an algebraic space X, then we can consider the preimage of the connected component $BP_\lambda \subset \mathrm{Filt}(B\,\mathrm{GL}_n)$ under the canonical map $\mathrm{Filt}(X/\mathrm{GL}_n) \to \mathrm{Filt}(B\,\mathrm{GL}_n)$. Note that the $(-)_+$ construction of Example 16.9 commutes with products of $\mathbb{G}_m$-schemes. Using this one can show that $X_{\lambda+}$ inherits a canonical action of $P_\lambda = (\mathrm{GL}_n)_{\lambda,+}$ for any cocharacter λ. We claim that we have a cartesian diagram

$$\begin{array}{ccc} X_{\lambda+}/P_\lambda & \longrightarrow & \mathrm{Filt}(X/\mathrm{GL}_n) \,, \\ \downarrow & & \downarrow \\ BP_\lambda & \longrightarrow & \mathrm{Filt}(B\,\mathrm{GL}_n) \end{array}$$

and hence

$$\mathrm{Filt}(X/\mathrm{GL}_n) \cong \bigsqcup_{\lambda : \mathbb{G}_m \to \mathrm{GL}_n\,/\text{conjugation}} X_{\lambda,+}/P_\lambda. \qquad \begin{matrix}\text{\{E:filt\_quotient\_stack\}}\\(16.3)\end{matrix}$$

Indeed this amounts to considering the base change along the map $E_\lambda : \mathrm{pt} \to$ $\mathrm{Filt}(B\,\mathrm{GL}_n)$, along with its induced action of P_λ. With a little massaging of the definitions, the fiber of $\mathrm{Filt}(X/\,\mathrm{GL}_n) \to \mathrm{Filt}(B\,\mathrm{GL}_n)$ corresponds to the functor

$$T \mapsto \{\mathbb{G}_m \times \mathrm{GL}_n \text{ equivariant maps } T \times \mathbb{A}^1 \times \mathrm{GL}_n \to X\}$$

where the left factor of $\mathbb{G}_m$ acts trivially on X and acts by $z\cdot(t,g) = (zt, \lambda(t)g)$ on $\mathbb{A}^1 \times \mathrm{GL}_n$. Because the GL_n action by right multiplication is free and transitive on $\mathbb{A}^1 \times \mathrm{GL}_n$, we can identify this with maps $\mathbb{A}^1 \to X$ that are equivariant in the sense that $f(zt) = \lambda(z)f(t)$. Thus for the point classified by E_λ, we have $\mathrm{pt} \times_{\mathrm{Filt}(B\,\mathrm{GL}_n)} \mathrm{Filt}(X/\,\mathrm{GL}_n) \cong X_{\lambda+}$.

Remark 16.12. If G is a linear algebraic k group with a split maximal torus, then one has a similar identification for any qc.qs. algebraic k-space X

$$\mathrm{Filt}(X/G) = \bigsqcup_{\lambda:\mathbb{G}_m \to G/\text{conjugation}} X_{\lambda+}/P_\lambda.$$

Lecture 17

Filtrations and stability II

References:
Date:
Exercises:

17.1   Θ-stratifications

Definition 17.1. A Θ-stratum is an open and closed substack $\mathcal{S} \subset \mathrm{Filt}(\mathcal{X})$ such that $\mathrm{ev}_1 : \mathcal{S} \to \mathcal{X}$ is a closed immersion.

Definition 17.2. A Θ-stratification of $\mathcal{X}$ indexed by a well-ordered set I consists of:

1. A collection of open substacks $\mathcal{X}_{\leq c} \subset \mathcal{X}$ for $c \in I$ such that $\mathcal{X}_{\leq c} \subset \mathcal{X}_{\leq c'}$ for $c < c'$; and

2. A Θ-stratum $\mathcal{S}_c \subset \mathrm{Filt}(\mathcal{X}_{\leq c})$ such that $\mathcal{X}_{\leq c} \setminus \mathrm{ev}_1(\mathcal{S}_c) = \bigcup_{c' < c} \mathcal{X}_{\leq c'}$.

Remark 17.3. A slightly more general notion is that of a *weak* Θ-stratum, in which the map $\mathrm{ev}_1 : \mathcal{S} \to \mathcal{X}$ is finite and radicial. Any weak Θ-stratum is a Θ-stratum for a stack of characteristic 0 [**????**].

   Note that condition (2) means that $\mathcal{S}_c \subset \mathrm{Filt}(\mathcal{X}_{\leq c})$ is an open substack.
   ¡¡TODO: describe stratification in terms of an open substack of $\mathrm{Filt}(\mathcal{X})$.¿¿

17.2   Numerical invariants

`defn:numerical_invariant_simple`

Definition 17.4. A *numerical invariant* μ on a stack $\mathcal{X}$ is the data of an assignment to any $p \in \mathcal{X}(k)$ and any homomorphism of k-groups $\gamma :$

$(\mathbb{G}_m^n)_k \to \operatorname{Aut}(p)$ with finite kernel a scaling-invariant continuous function $\mu_\gamma : \mathbb{R}^n \setminus 0 \to \mathbb{R}$ such that:

1. Given a homomorphism with finite kernel $\phi : (\mathbb{G}_m^q)_k \to (\mathbb{G}_m^n)_k$, $\mu_{\gamma \circ \phi}$ is the restriction of μ_γ along the inclusion $\mathbb{R}^q \hookrightarrow \mathbb{R}^n$ induced by ϕ; and

2. Given a scheme S, an S-point $\xi : S \to \mathcal{X}$, and a homomorphism of S-group-schemes $\gamma : (\mathbb{G}_m^n)_S \to \operatorname{Aut}(\xi)$ with finite kernel, if one considers the maps $\gamma_s : (\mathbb{G}_m^n)_{k(s)} \to \operatorname{Aut}(\xi(s))$ indexed by points in $s \in S$, then the function μ_{γ_s} on $\mathbb{R}^n \setminus 0$ is locally constant on S.

¡¡This defines a function from the set of filtrations in $\mathcal{X}$ ¿¿

17.2.1 Line bundles and norms on one parameter subgroups

17.2.2 The degeneration space and the HN-problem

17.3 Criteria for Θ-stratifications

Lecture 18

Beyond geometric invariant theory

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Date:
Exercises:

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