Math 4310

Final Exam

Name: ______________________

May 18, 2016

Time: 150 minutes

INSTRUCTIONS — PLEASE READ THIS NOW

• This exam consists of **five** True/False questions, **five** Short Answer questions, and **four** long-answer questions with **two or three parts** each, on separate pages.

• Please write each final answer on the page where the question is posed. You should include a complete logical justification, written in grammatically correct mathematical language. There are two blank pages for scratch work at the end of the exam; we **will not read** your work on those pages.

• Write your name **right now**.

• Look over your test packet **as soon as the exam begins**. If you find any missing pages or problems please ask for another test booklet.

• You have 2 hours and 30 minutes to complete this exam. You may leave early, but if you finish within the last 10 minutes, please remain in your seat.

• This is a closed book exam. You are **NOT** allowed to use a calculator, cell phone, or any other electronic device (not even as a time keeping device).

• **Academic integrity** is expected of every Cornell University student at all times, whether in the presence or absence of a member of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination. I will not discuss this exam with other students until both sections have taken the exam.

Please sign below to indicate that you have read and agree to these instructions.

____

Signature of Student

OFFICIAL USE ONLY

T/F	______	/ 15	______
SA	______	/ 15	______
1.	______	/ 20	______
2.	______	/ 15	______
3.	______	/ 20	______
4.	______	/ 15	______
Total:	______	/ 100	______

True/False. Please circle **TRUE** if the statement is always true, or **FALSE** if it fails in at least one example. You do **not** need to justify your answer, and I will not read what you write in the spaces below.

(a) Any matrix $A \in M_3(\mathbb{R})$ has an eigenvector in $\mathbb{R}^n$. TRUE | FALSE

(b) If V is a finite-dimensional inner product space, a vector $v \in V$ is uniquely determined by the values of the function $w \mapsto \langle v, w \rangle$ (for all $w \in W$). TRUE | FALSE

(c) Every matrix $A \in M_n(\mathbb{Q})$ has a Jordan canonical form $J \in M_n(\mathbb{Q})$. TRUE | FALSE

(d) There is a unique alternating multilinear function $\omega : (\mathbb{R}^n)^n \to \mathbb{R}$ satisfying $\omega(e_1, \ldots, e_n) = 3$. TRUE | FALSE

(e) Any solutions to the system of linear differential equations $dy_1/dx = 2y_1 + y_2$ and $dy_2/dx = y_1 + 2y_2$ must consist of linear combinations of exponential functions $e^{\lambda x}$ for $\lambda \in \mathbb{R}$. TRUE | FALSE

Short answer. What are the **dimensions** of the following vector spaces? (Your answer should either be "infinity" or a nonnegative integer). You do not need to justify your responses. We will only grade your final answer, circled or written in the space provided.

(a) $V_1 = \mathbb{C}^2$ as an $\mathbb{R}$-vector space. $\boxed{\dim(V_1) = \quad}$

(b) $V_2 = \{A \in M_4(\mathbb{R}) : \mathrm{tr}(A) = 0\}$ as an $\mathbb{R}$-vector space. $\boxed{\dim(V_2) = \quad}$
Here tr is the trace function, taking A to the sum of its diagonal entries.

(c) $V_3 = \{f \in \mathbb{F}_p[x] : f(a) = 0 \text{ for all } a \in \mathbb{F}_{17}\}$ (i.e. the set of all polynomials over the finite field $\mathbb{F}_{17}$ that have value zero everywhere), as an $\mathbb{F}_{17}$-vector space. $\boxed{\dim(V_3) = \quad}$

(d) $V_4 = \mathbb{F}_{17}[x]/V_3$ (a quotient space involving the subspace defined in part (c)), as an $\mathbb{F}_{17}$-vector space. $\boxed{\dim(V_4) = \quad}$

(e) V_5 is a subspace of $\mathbb{R}^9$, such that there's another subspace W satisfying $\dim(V_5) = \dim(W)$, $\dim(V_5 + W) = 7$, and $\dim(V_5 \cap W) = 3$. $\boxed{\dim(V_5) = \quad}$

Question 1.

(a) Prove that given n scalars $\lambda_1, \dots, \lambda_n$ in a field F, the formula

$$\det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i).$$

holds, where the product is over all pairs of integers (i, j) satisfying $1 \le i < j \le n$. You probably want to proceed by induction on n. (This is called the *Vandermonde determinant*).

(b) Suppose that V is an n-dimensional vector space over F, and $T : V \to V$ is diagonalizable with n **distinct** eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated basis of eigenvectors $v_1, \ldots, v_n$. Prove that $b = v_1 + \cdots + v_n$ is a cyclic vector, i.e. that $\{b, T(b), \ldots, T^{n-1}(b)\}$ spans V.

(c) Conversely, show that if $T : V \to V$ is diagonalizable and has a **repeated** eigenvalue then T is not cyclic. (Remember that this means you have to rule out *every possible* vector from being a cyclic vector).

Question 2. Consider the symmetric matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 1 \end{bmatrix} \in M_3(\mathbb{R}).$$

Compute and factor the characteristic polynomial of A, find a diagonal matrix D similar to A, and an orthogonal matrix U such that $A = UDU^{\top}$. (Remember an orthogonal matrix must have *orthonormal* columns!)

Question 3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 1 & 2 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \in M_4(\mathbb{C}),$$

which has characteristic polynomial

$$c_A(x) = (x-2)^3(x-1).$$

(a) Compute a Jordan canonical form J for A, and the change-of-basis matrix P such that $A = PJP^{-1}$.

(b) Write down the minimal polynomial $m_A(x)$ for the matrix A, and justify why it is the minimal polynomial. Use the minimal polynomial to write a formula expressing A^{-1} as a linear combination of I, A, and A^2. (You don't need to actually compute out the matrix A^{-1} from this formula).

(c) Compute a square root of J, compute P^{-1}, and write out the three matrices P, $\sqrt{J}$, and P^{-1}. If you multiplied them together you would get a square root $P\sqrt{J}P^{-1}$ of A, but the multiplication is pretty ugly so you **do not** need to do it.

Question 4.

(a) Write down the Jordan canonical forms representing similarity classes of **nilpotent** 4×4 **matrices** in $M_4(\mathbb{C})$. (So every 4×4 nilpotent matrix should be similar to *exactly one* of the matrices you listed!).

(b) Which of the JCFs from part (a) have a square root, i.e. for which of the J that you wrote down does there exist $A \in M_4(\mathbb{C})$ with $A^2 = J$? (Remember, the theorem from class about square root existing doesn't apply to nilpotent matrices! But for each matrix you wrote down, you should either be able to find such an A or prove that none exists).

This page is for scratch work.

Don't forget to transfer your final work to the page where the question is posed!

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