Math 4310 Final Exam Solutions

True/False. (a) True. Remember that having an eigenvector is the same as having an eigenvalue, i.e. that the characteristic polynomial $c_A(x)$ has a root; since $c_A(x)$ is a degree 3 polynomial, basic calculus tells us it must cross the x-axis.

(b) True. This is the uniqueness part of the Riesz representation theorem.

(c) False. The eigenvalues of A (and thus the Jordan canonical form) may lie outside of $\mathbb{Q}$.

(d) True. The function $\omega = 3\det$ is satisfies these properties, and conversely if ω satisfies them then $\frac{1}{3}\omega$ is multilinear, alternating, and normalized so must equal det.

(e) True. Since this system corresponds to a symmetric matrix, it is diagonalizable over $\mathbb{R}$, and if the eigenvalues are $\lambda_1, \lambda_2 \in \mathbb{R}$ then we know y_1 and y_2 must be linear combinations of $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$. (In fact the eigenvalues are 1 and 3).

Short answer. (a) $\dim(V_1) = 4$. Remember $\mathbb{C}$ is 2-dimensional as a real vector space and thus a product of two copies of $\mathbb{C}$ is 4-dimensional.

(b) $\dim(V_2) = 15$. Here $M_4(\mathbb{R})$ is a 16-dimensional vector space, and $\mathrm{tr} : M_4(\mathbb{R}) \to \mathbb{R}$ is a linear transformation with kernel equal to V_2. By the rank-nullity theorem (and noting that $\mathrm{img}\,\mathrm{tr} = \mathbb{R}$ so tr has rank 1) we get $\dim(V_2) = 15$.

(c) $\dim(V_3) = \infty$. One way to see this is remembering that the polynomial $p_0(x) = x^{17} - x$ takes the value zero everywhere, and thus so does each polynomial $p_k(x) = x^k(x^{17} - x)$, and all of these are linearly independent.

(d) $\dim(V_4) = 17$. To see this (which implies the answer to part (c)) we can use the first isomorphism theorem for the linear map $\mathbb{F}_{17}[x] \to (\mathbb{F}_{17})^{17}$ which takes a polynomial to its values at each of the 17 elements of $\mathbb{F}_{17}$. This map has kernel V_3, so $\mathbb{F}_{17}[x]/V_3$ is isomorphic to the image, and it's not too hard to see it's surjective (this can be done by looking at Lagrange interpolation polynomials, for instance).

(e) $\dim(V_5) = 5$. Remember we have a formula

$$\dim(V_5 + W) = \dim(V_5) + \dim(W) - \dim(V_5 \cap W);$$

plugging in what we have gives $7 = 2\dim(V_5) - 3$ which we then solve.

Problem 1. (a) Denote the matrix in question by V_n; assume that we know the desired identity for V_{n-1}. To compute the determinant of V_n, we start by subtracting the first row from each other row:

$$\det(V_n) = \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 0 & \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^{n-1} - \lambda_1^{n-1} \\ 0 & \lambda_3 - \lambda_1 & \lambda_3^2 - \lambda_1^2 & \cdots & \lambda_3^{n-1} - \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_n - \lambda_1 & \lambda_n^2 - \lambda_1^2 & \cdots & \lambda_n^{n-1} - \lambda_1^{n-1} \end{bmatrix}.$$

Since the first column has just one nonzero entry we get

$$\det(V_n) = \det \begin{bmatrix} \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^{n-1} - \lambda_1^{n-1} \\ \lambda_3 - \lambda_1 & \lambda_3^2 - \lambda_1^2 & \cdots & \lambda_3^{n-1} - \lambda_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n - \lambda_1 & \lambda_n^2 - \lambda_1^2 & \cdots & \lambda_n^{n-1} - \lambda_1^{n-1} \end{bmatrix}.$$

This matrix isn't quite V_{n-1}, so we're not done yet. However, we can fix that with column operations: if we add λ_1 times the $(n-2)$-th column to the $(n-1)$-th column it turns the $\lambda_i^{n-1} - \lambda_1^{n-1}$ entries into

$(\lambda_i - \lambda_1)\lambda_2^{n-2}$, and then we can add λ_1 times the $(n-3)$-th column to the $(n-2)$-th column, and so on. We get

$$\det(V_n) = \det \begin{bmatrix} \lambda_2 - \lambda_1 & (\lambda_2 - \lambda_1)\lambda_2 & \cdots & (\lambda_2 - \lambda_1)\lambda_2^{n-2} \\ \lambda_3 - \lambda_1 & (\lambda_3 - \lambda_1)\lambda_3 & \cdots & (\lambda_3 - \lambda_1)\lambda_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n - \lambda_1 & (\lambda_n - \lambda_1)\lambda_n & \cdots & (\lambda_n - \lambda_1)\lambda_n^{n-2} \end{bmatrix}.$$

Factoring out $\lambda_i - \lambda_1$ from each row we get

$$\det(V_n) = \prod_{i=1}^{n}(\lambda_i - \lambda_1) \cdot \det \begin{bmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{n-2} \\ 1 & \lambda_3 & \cdots & \lambda_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-2} \end{bmatrix}.$$

But this is an $(n-1) \times (n-1)$ Vandermonde determinant (with entries $\lambda_2, \ldots, \lambda_n$) so by induction we have

$$\det(V_n) = \prod_{j=1}^{n}(\lambda_j - \lambda_1) \cdot \prod_{2 \le i < j \le n}(\lambda_j - \lambda_i),$$

which combines to the formula we want.

(b) By definition, we have $T^j(b) = \sum_i T^j(v_i) = \sum_i \lambda_i^j b_i$. So if we put $b, T(b), \ldots, T^{n-1}(b)$ into coordinates for the basis $\{v_1, \ldots, v_n\}$, the coordinate vectors we get form the columns of the Vandermonde matrix

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

Part (a) gives a formula for the determinant of this matrix, which is nonzero since we're assuming all of the λ_i's are distinct. So the matrix is nonsingular, which means the columns are a basis of $\mathbb{R}^n$, and thus the vectors $b, T(b), \ldots, T(b)^{n-1}$ they represent are a basis of V.

(c) Since V is diagonalizable, let $v_1, \ldots, v_n$ be a basis of eigenvectors, with $T(v_i) = \lambda_i v_i$. Moreover, since we're assuming there's a repeated eigenvalue, assume $\lambda_1 = \lambda_2$.

We now want to show that an arbitrary vector

$$v = a_1 v_1 + \cdots + a_n v_n$$

is not a cyclic vector. To do this, we note that v lies in the subspace

$$W = L(a_1 v_1 + a_2 v_2, v_3, \ldots, v_n) \subsetneq V,$$

and we claim $T[W] \subseteq W$; if this is true then certainly $\{v, T(v), \ldots, T^{n-1}(v)\}$ must stay inside W as well, so v can't be cyclic! But this follows by seeing that T keeps each of the vectors in the spanning set inside W; we have $T(v_i) = v_i \in W$ for $3 \le i \le n$, and also

$$T(a_1 v_1 + a_2 v_2) = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = \lambda_1(a_1 v_1 + a_2 v_2) \in W$$

because $\lambda_1 = \lambda_2$.

Problem 2. Computing out and factoring $\det(A - xI)$ results in

$$c_A(x) = -x(x-3)(x-6),$$

so we know the diagonal matrix is

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To find the matrix U, we just need to take its columns to be eigenvectors v_1, v_2, v_3 with eigenvalues $6, 3, 0$, respectively, which are normalized with respect to the dot product on $\mathbb{R}^3$.

The vector v_3 is something in the span of $\ker(A)$ itself; row-reduction lets us get that the kernel is spanned by $[-1\ 1\ 2]$ and normalizing we find

$$v_3 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Similarly, for v_2 we need to find something that spans the kernel of $A - 3I$:

$$A - 3I = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

For v_1 we look at $A - 6I$:

$$A - 6I = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & -1 \\ 1 & -1 & -4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Putting these together in the right order we find that we can take:

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}.$$

Problem 3. (a) Since we're told the characteristic polynomial, we know that the eigenvalues will be 1 and 2. The eigenvalue 1 has multiplicity 1, so there will be a unique eigenvector for it; we'll call that v_4. It's easy enough to find by looking at $A - 1 \cdot I$ and finding its kernel:

$$A - I = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The eigenvalue 2 has multiplicity 3, so we need to look at the kernel of $A - 2I$ and its powers to see what Jordan blocks we get. We have:

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 \end{bmatrix} \qquad \ker(A - 2I) = L(e_1 + e_2, e_3).$$

From this we can see $\dim \ker(A - 2I) = 2$, so there are two Jordan blocks; so at this point we know we have Jordan canonical form

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, our change-of-basis matrix P will have columns $[v_1\ v_2\ v_3\ v_4]$ where v_4 is our eigenvector for $\lambda = 1$ from before, and v_1, v_2, v_3 are generalized eigenvectors for $\lambda = 2$. In particular, for it to be of this form we need to have that v_1 is something in $\ker(A - 2I)^2$ but not $\ker(A - 2I)$, that $v_2 = (A - 2I)v_1$, and that v_3 is an element of $\ker(A - 2I)$ (i.e. an eigenvector with eigenvalue 2) linearly independent from v_2.

So we need to find $\ker(A - 2I)^2$; this is a computation again:

$$(A-2I)^2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \qquad \ker(A-2I)^2 = L(e_1 + e_2, e_3, e_1 - e_4).$$

We have plenty of choices of vectors in $\ker(A - 2I)^2$ but not in $\ker(A - 2I)$; let's pick $v_1 = e_1 - e_4$, so we then have

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad v_2 = (A-2I)v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, we need to pick v_3 in $\ker(A-2I) = L(e_1+e_2, e_3)$ that's linearly independent from $v_2 = e_1+e_2$; the obvious choice is $v_3 = e_3$. Thus our change-of-basis matrix is

$$P = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The minimal polynomial is $(x - 2)^2(x - 1)$; this is easiest to see from the Jordan canonical form. We need at least one factor of $(x-2)$ and of $(x-1)$ in our polynomial to kill off the diagonal elements, and since we have a Jordan block of size 2 it's not hard to see that $(A-2I)^2(A-I) = 0$ but $(A-2I)(A-I) \neq 0$.

Expanding this out, the minimal polynomial is $m_A(x) = x^3 - 5x^2 + 8x - 4$, so we know

$$A^3 - 5A^2 + 8A - 4I = 0.$$

Multiplying through by A^{-1} and rearranging we get the formula

$$A^{-1} = \frac{1}{4}(A^2 - 5A + 8I).$$

(c) Since we have an expression $A = PJP^{-1}$, we can find a square root of A by finding a square root $\sqrt{J}$ of J and taking $\sqrt{A} = P\sqrt{J}P^{-1}$. It's easy to find a square root of J; we take a square root of each of the diagonal elements, and then it's not hard to find what to fill in below the diagonal to get back the one nontrivial Jordan block we have:

$$\sqrt{J} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We then just need to multiply $P\sqrt{J}P^{-1}$. Computing P^{-1} out isn't too hard, and we get

$$\sqrt{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 \end{bmatrix}.$$

Multiplying this would be a huge pain, of course.

Problem 4. (a) Nilpotent matrices are exactly the ones with all eigenvalues 0, so this just amounts to writing down the possible combinations of Jordan block sizes that add up to 4; there are 5 of them:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) The answer is trivially yes for J_1 since it's the zero matrix; we can take the zero matrix to be a square root as well. It's also straightforward to show that the answer is no for J_4 and J_5; both of those matrices satisfy $J^2 \neq 0$, so if they had a square root A it would satisfy $A^4 \neq 0$ but still be nilpotent, and we know no 4×4 nilpotent matrix can satisfy $A^4 \neq 0$.

For J_2 and J_3 the answer turns out to be yes; we could cook up matrices that work, but it's probably more instructive to think in terms of the linear transformations. For instance J_2 gives a linear transformation $T : \mathbb{C}^4 \to \mathbb{C}^4$ which satisfies $T(e_2) = T(e_3) = T(e_4) = 0$ and $T(e_1) = e_2$. If S is a square root of that linear transformation, then we need $S(S(e_1)) = e_2$; to accomplish this we might as well define $S(e_1) = e_3$ and $S(e_3) = e_2$. Then we need $S(S(e_3)) = S(e_2) = 0$ as well, which gives us $S(S(e_2)) = 0$ for free. If we set $S(e_4) = 0$, we've defined a linear transformation S by specifying it on a basis, and shown that it satisfies $S^2 = T$ on this basis. So S is a square root of the linear transformation T, and thus the matrix of S is a square root of J_2.

The logic for J_3 is similar: this one determines a linear transformation T with $T(e_2) = T(e_4) = 0$, $T(e_1) = e_2$, and $T(e_3) = T(e_4)$. To get S satisfying $S^2 = T$, we can set $S(e_1) = e_3$, $S(e_2) = 0$, and $S(e_3) = e_2$ again, which makes S^2 agree with T on e_1, e_2, e_3. Finally, we need $S(S(e_4)) = e_3$ as well. But we've already set $S(e_2) = e_3$, so if we define $S(e_4) = e_2$ then we have $S(S(e_4)) = e_3$, which is what we want! So this specifies an S which is a square root of T, and thus determines a square root of J_3.