Math 4310 | Name: ______________________

Prelim 1 | March 4, 2016 | Time: 50 minutes

INSTRUCTIONS — PLEASE READ THIS NOW

- This exam consists of **five** True/False questions and **two** long-answer questions with **two parts** each, on separate pages.
- Please write each final answer on the page where the question is posed. You should include a complete logical justification, written in grammatically correct mathematical language. There are basic definitions and two blank pages for scratch work at the end of the exam; we **will not read** your work on those pages.
- Write your name **right now**.
- Look over your test packet **as soon as the exam begins**. If you find any missing pages or problems please ask for another test booklet.
- You have 50 minutes to complete this exam. You may leave early, but if you finish within the last 10 minutes, please remain in your seat.
- This is a closed book exam. You are **NOT** allowed to use a calculator, cell phone, or any other electronic device (not even as a time keeping device).
- **Academic integrity** is expected of every Cornell University student at all times, whether in the presence or absence of a member of the faculty. Understanding this, I declare I shall not give, use, or receive unaccepted aid in this examination. I will not discuss this exam with other students until both sections have taken the exam.

Please sign below to indicate that you have read and agree to these instructions.

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Signature of Student

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T/F	____	/ 25	____
1.	____	/ 35	____
2.	____	/ 40	____
Total:	____	/ 100	____

True/False. Please circle **TRUE** if the statement is always true, or **FALSE** if it fails in at least one example. You do **not** need to justify your answer, and I will not read what you write in the spaces below. [5 points each]

(a) The subset $\mathbb{I}$ of $\mathbb{R}$ consisting of irrational numbers is a field (under the usual algebraic operations for real numbers).

TRUE	FALSE

(b) If V is a vector space over a field and $v \in V$ is a nonzero vector, $v + v$ must be nonzero.

TRUE	FALSE

(c) The set of polynomials

$$\{a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x] : a_i = 0 \text{ for all odd numbers } i\}$$

is a subspace of the $\mathbb{R}$-vector space $\mathbb{R}[x]$ of all real polynomials.

TRUE	FALSE

(d) It's possible for there to be an $\mathbb{R}$-linear transformation $T : \mathbb{R}^9 \to \mathbb{R}^9$ with $\ker(T) = \text{img}(T)$.

TRUE	FALSE

(e) For any integer k, the quotient space $\mathbb{R}[x]/U_k$ is finite-dimensional over $\mathbb{R}$, where U_k is the subspace of polynomials that vanish to order $k+1$ at $x = 0$:

$$U_k = \left\{ f(x) \in \mathbb{R}[x] : f(0) = 0, \frac{df}{dx}(0) = 0, \ldots, \frac{d^k f}{dx^k}(0) = 0 \right\}.$$

TRUE	FALSE

Question 1. Let V be a vector space over a field F, and let $W \subseteq V$ be a subspace.

(a) Suppose both W and V/W are finite-dimensional. Prove that V is also finite-dimensional. (Warning: you can't use formulas about dimension since we only stated those under the assumption V was already finite-dimensional!) [20 points]

(b) If V is infinite-dimensional, then we must be in one of the following three cases:

(1) W is finite-dimensional and V/W is infinite-dimensional.

(2) W is infinite-dimensional and V/W is finite-dimensional.

(3) W is infinite-dimensional and V/W is infinite-dimensional.

For each of these three possibilities, write down an explicit example satisfying it with $V = F[x]$ (and W a subspace you choose). Justify that your spaces W and V/W are finite- or infinite-dimensional as appropriate in each case. [15 points]

Question 2. Let V, W be *finite-dimensional* and *nontrivial* (i.e. $V, W \neq 0$) vector spaces over a field F. Recall that the Linear Extension Theorem told us that if B was a basis of V, then any function $T_0 : B \to W$ extended to **exactly one** linear transformations $T : V \to W$.

(a) Suppose that E is a spanning set of V that is **not** linearly independent, and $T_0 : E \to W$ is a function. Which of "none", "exactly one", or "more than one" are possibilities for the number of extensions of T_0 to a linear transformation $T : V \to W$? Justify your answer (show examples of the ones you say are possibilities, and prove that the others cannot happen). [20 points]

(b) Suppose that D is a linearly independent set of V that does **not** span, and $T_0 : D \to W$ is a function. Which of "none", "exactly one", or "more than one" are possibilities for the number of extensions of T_0 to a linear transformation $T : V \to W$? Justify your answer (show examples of the ones you say are possibilities, and prove that the others cannot happen). [20 points]

Definition 1. A **field** is a set F together with two binary operations[1] $+$ and $\cdot$ which satisfy

1. (Associativity of addition): For any $a, b, c \in F$ we have $(a + b) + c = a + (b + c)$.
2. (Commutativity of addition): For any $a, b \in F$ we have $a + b = b + a$.
3. (Additive identity): There exists an element $0 \in F$ such that for every $a \in F$, we have $a + 0 = a$.
4. (Additive inverses): For each $a \in F$ there exists an element $-a \in F$ such that $a + (-a) = 0$.
5. (Distributivity of multiplication over addition): For any $a, b, c \in F$ we have $(a+b)c = ac+bc$.
6. (Associativity of multiplication): For any $a, b, c \in F$ we have $(ab)c = a(bc)$.
7. (Commutativity of multiplication): For any $a, b \in F$ we have $ab = ba$.
8. (Multiplicative identity): There exists an element $1 \in F$ (which is not equal to 0) such that for every $a \in F$, we have $a \cdot 1 = a$.
9. (Multiplicative inverses): For each nonzero $a \in F$ there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

Definition 2. A **vector space** over a field F is a set V together with two operations[2] $+ : V \times V \to V$ and $\cdot : F \times V \to V$ which satisfy

1. (Associativity of addition): For any $u, v, w \in V$ we have $(u + v) + w = u + (v + w)$.
2. (Commutativity of addition): For any $v, w \in V$ we have $v + w = w + v$.
3. (Additive identity): There exists an element $0 \in V$ such that for every $v \in V$, we have $v + 0 = v$.
4. (Additive inverses): For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = 0$.
5. (Distributivity of scalar multiplication over vector addition): For for any $a \in F$ and $v, w \in V$ we have $a(v + w) = av + aw$.
6. (Distributivity of scalar multiplication over scalar addition): For for any $a, b \in F$ and $v \in V$ we have $(a + b)v = av + bv$.
7. (Compatibility of multiplications): For any $a, b \in F$ and $v \in V$ we have $(ab)v = a(bv)$.
8. (Compatibility of the multiplicative identity): For any $v \in V$ we have $1v = v$.

[1]Saying that $+$ and $\cdot$ are binary operations implicitly assumes that they are well-defined, and that the sum and product of two elements of F is again in F.

[2]Saying that $+$ and $\cdot$ are binary operations implicitly assumes that they are well-defined, and that the sum and scalar product is again a vector in V.

This page is for scratch work.

Don't forget to transfer your final work to the page where the question is posed!

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