Math 4310 Prelim 1 Solutions

Problem 1. (a) False; since 0 is a rational number, $0 \notin \mathbb{I}$ and thus $\mathbb{I}$ doesn't have an additive identity! (Among many other problems).

(b) False; if F has characteristic 2 (e.g. $F = \mathbb{F}_2$) then $v + v = 2_F v = 0v = 0$.

(c) True. It's easy to see it contains 0 and is closed under addition and scalar multiplication.

(d) False. The rank-nullity theorem in this case tells us $\dim\ker(T) + \dim\operatorname{img}(T) = 9$, so if $\ker(T) = \operatorname{img}(T)$ then we get $2\dim\ker(T) = 9$, which is impossible since dimensions have to be integers.

(e) True. The linear transformation $T : \mathbb{R}[x] \to \mathbb{R}^{k+1}$ given by $f \mapsto (f(0), \frac{df}{dx}(0), \ldots, \frac{d^k f}{dx^k}(0))$ has kernel U_k, and then the first isomorphism theorem tells us $\mathbb{R}[x]/U_k \cong \operatorname{img} T$ where $\operatorname{img} T$ is some subspace of $\mathbb{R}^{k+1}$ and thus is finite-dimensional. (In fact $\operatorname{img} T = \mathbb{R}^{k+1}$).

Problem 1. (a) W being finite-dimensional means that it has a finite spanning set $w_1, \ldots, w_m$, and V/W being finite-dimensional means that it has a finite spanning set $v_1 + W, \ldots, v_n + W$. We claim that $w_1, \ldots, w_m, v_1, \ldots, v_n$ is a finite spanning set for V (where we've taken the spanning set for W together with *coset representatives* for the spanning set for V).

To see this, suppose $v \in V$. Then $v + W$ is in V/W and can be written as a linear combination of the $v_i + W$'s:

$$v + W = \sum_{i=1}^{n} a_i(v_i + W) = \left(\sum_{i=1}^{n} a_i v_i\right) + W.$$

By definition of V/W as a quotient space, this equality means that $v - \sum a_i v_i \in W$. But since the w_i's span W, we have $v - \sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{m} b_j w_j$ which rearranges to let us write our arbitrary $v \in V$ in the span of the v_i's and w_j's.

(b) For $V = F[x]$, we can take:

- $W = 0$ (the trivial subspace) gives W finite-dimensional and $V/W \cong F[x]$ infinite-dimensional.
- $W = F[x]$ gives W infinite-dimensional and $V = F[x]$ finite-dimensional.
- If we let $W = L(x, x^3, x^5, \cdots)$ be the span of even-power monomials, then W and V/W are both infinite-dimensional: W is infinite-dimensional because it's the span of an infinite subset of a basis of $F[x]$, and moreover if we define $T : F[x] \to F[x]$ by $\sum a_i x^i \mapsto \sum a_{2i} x^i$ we find that W is the kernel of T and T is surjective so the first isomorphism theorem gives $V/W \cong F[x]$ is infinite-dimensional.

Problem 2. (a) There are "none" or "exactly one". To see that there's at most one (i.e. that "more than one" is impossible) note that E contains a basis B; then if $T : V \to W$ is a linear extension of $T_0 : E \to W$, it has to be a linear extension of the restriction $T_0 : B \to W$, and there is exactly one such restriction.

For examples, let $V = W = \mathbb{R}$ and $E = \{1, 2\}$. Then if $T_0 : E \to \mathbb{R}$ is defined by $T_0(1) = 1$ and $T_0(2) = 2$ it has exactly one extension, $T(x) = x$. But if $T_0 : E \to \mathbb{R}$ is defined by $T_0(1) = T_0(2) = 1$ then it has no extensions: any linear extension T would have to satisfy

$$1 = T_0(2) = T_0(2 \cdot 1) = 2 \cdot T_0(1) = 2 \cdot 1 = 2,$$

which is a contradiction.

(b) There will always be "more than one". Since D is linearly independent but does not span, it can be extended to a basis $D' = D \cup \{v_1, \ldots, v_k\}$ where $k \geq 1$. Then we can define many extensions of the function $T_0 : D \to W$ to functions $T_1 : D' \to W$ (because we can choose $T_1(v_1)$ to be any vector in W we want), and each of these many extensions T_1 has a unique extension further to a linear transformation $T : V \to W$ that extends T_0.