



Math 4310

Prelim 2

Name: \_\_\_\_\_

Due: 12:20pm April 15, 2016

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**INSTRUCTIONS — PLEASE READ THIS NOW**

- This take-home exam consists of **six** questions.
- You should answer the questions as you would answer normal homework questions. Please write up the answers **in the correct order** of the problems. Also, please **attach a copy of this page** to the front.
- This take-home exam is **open book**. You may look through your textbook or any other book you might want. You may also search for things on the internet (but not post the problems anywhere to actively ask for help on them). You may also use calculators, computer algebra programs, etc. If you use any source besides the textbook or your course notes, you should cite it as you would on your homework.
- You may **not** discuss the problems and their solutions with your classmates, friends, or others, either in person or electronically (besides me and the TAs).
- This exam will be due at or before the **beginning of class** on Friday, April 15. (If you get there a few minutes late that's fine, but please don't work through the class and show up at the end to hand it in). If you want to hand it in some other way than in person in class, talk to me.
- **Academic integrity** is expected of every Cornell University student at all times, whether in the presence or absence of a member of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination. I will not discuss this exam with other students until after it is turned in.

**Please sign below to indicate that you have read and agree to these instructions.**

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Signature of Student

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**OFFICIAL USE ONLY**

1. \_\_\_\_\_ / 15 \_\_\_\_\_

2. \_\_\_\_\_ / 15 \_\_\_\_\_

3. \_\_\_\_\_ / 15 \_\_\_\_\_

4. \_\_\_\_\_ / 15 \_\_\_\_\_

5. \_\_\_\_\_ / 20 \_\_\_\_\_

6. \_\_\_\_\_ / 20 \_\_\_\_\_

Total: \_\_\_\_\_ / 100 \_\_\_\_\_

**Question 1.** Let  $P_2(\mathbb{R})$  be the space of quadratic polynomials over the real numbers, which we make into an inner product space in the usual way:

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx.$$

Since  $T_0 : P_2(\mathbb{R}) \rightarrow \mathbb{R}$  given by  $T_0(p) = p(0)$  is a linear transformation, the Riesz representation theorem guarantees there exists a unique polynomial  $q_0(x) \in P_2(\mathbb{R})$  such that  $\langle p(x), q_0(x) \rangle = p(0)$  for every  $p \in P_2(\mathbb{R})$ . Example 4 of Section 15.4 of the textbook computes this  $q_0(x)$ .

More generally, for any  $a \in \mathbb{R}$  we have a linear transformation  $T_a : P_2(\mathbb{R}) \rightarrow \mathbb{R}$  given by  $T_a(p) = p(a)$ , and the Riesz representation theorem tells us there exists a unique  $q_a(x) \in P_2(\mathbb{R})$  such that  $\langle p(x), q_a(x) \rangle = p(a)$  for every  $p(x) \in P_2(\mathbb{R})$ . In other words, there's a function  $q(a, x) = q_a(x)$  of two variables such that

$$p(a) = \int_{-1}^1 p(x)q(a, x)dx$$

for every  $a \in \mathbb{R}$  and every quadratic polynomial  $p(x) \in P_2(\mathbb{R})$ . Compute this function  $q(a, x)$ . (You probably want to start by writing  $q_a(x) = c_0(a) + c_1(a)x + c_2(a)x^2$ , and thinking of a good way to determine the coefficients  $c_i(a)$ ). [15 points]

**Question 2.** Let  $F$  be a field. In this problem we consider linear transformations  $T : M_2(F) \rightarrow M_2(F)$ . Note that  $M_2(F)$  is a 4-dimensional vector space, so the space of such linear transformations is 16-dimensional! [15 points]

- (a) Consider a  $2 \times 2$  matrix (i.e. an *element* of  $M_2(F)$ )

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove that the map  $L_A : M_2(F) \rightarrow M_2(F)$  defined by  $L_A(X) = AX$  ("left multiplication by  $A$ ") is a linear transformation. Write down its coordinate matrix  $[L_A]$  in terms of the basis of elementary matrices  $E_{11}, E_{12}, E_{21}, E_{22}$ . What's the determinant of  $[L_A]$ , and when is  $L_A$  invertible?

- (b) Similarly  $R_B$  defined by  $R_B(X) = XB$  is a linear transformation. Prove that we have  $L_A = R_B$  if and only if  $A = B$  is a scalar matrix.

**Question 3.** Consider the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by the matrix

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & a \end{bmatrix} \in M_3(\mathbb{R}),$$

where  $a \in \mathbb{R}$  is a parameter. Determine (with proof) for which values of  $a$  the transformation  $T$  is cyclic and for which it isn't. [15 points]

**Question 4.** Consider a  $2 \times 2$  matrix  $A$  over the field  $\mathbb{C}$ : [15 points]

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- (a) If the characteristic polynomial of  $A$  has two distinct roots  $\lambda_1 \neq \lambda_2$ , then we know  $A$  is similar to the diagonal matrix

$$A \sim \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

If the characteristic polynomial has a repeated root, i.e.  $c_A(x) = (x - \lambda)^2$ , then one possibility is that  $A$  is similar to (and thus equal to) the scalar matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

Prove that if  $c_A$  has a repeated root  $\lambda$ , and  $A$  is not equal to this scalar matrix, then  $A$  is not diagonalizable but is similar to the matrix

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

- (b) Write down a polynomial  $P(a, b, c, d)$  in the entries  $a, b, c, d$  such that  $P(a, b, c, d) \neq 0$  iff  $A$  is diagonalizable with distinct eigenvalues.

**Question 5.** Suppose we have a matrix  $A \in M_2(\mathbb{Q})$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Q})$$

where the entries  $a, b, c, d$  are *integers*. [20 points]

- (a) Prove that any eigenvalue  $\lambda \in \mathbb{Q}$  of  $A$  is actually an integer itself.  
 (b) If  $A$  has an eigenvalue  $\lambda \in \mathbb{Z}$ , prove that it has a corresponding eigenvector with entries in  $\mathbb{Z}$ .  
 (c) Show that if  $A$  has an eigenvalue  $\lambda \in \mathbb{Z}$ , then for any prime  $p$  its “reduction mod  $p$ ”

$$\bar{A} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \in M_2(\mathbb{Z}/p\mathbb{Z}) = M_2(\mathbb{F}_p)$$

has  $\bar{\lambda}$  as an eigenvalue.

- (d) If  $A$  is diagonalizable over  $\mathbb{Q}$ , is it true that  $\bar{A}$  must be diagonalizable over  $\mathbb{F}_p$ ?

**Question 6.** For the following, let  $(\mathbb{Q}^n)^N$  denote the product of  $N$  copies of  $\mathbb{Q}^n$ , so we can consider functions  $\varphi : (\mathbb{Q}^n)^N \rightarrow \mathbb{Q}$  which take  $N$  vectors in  $\mathbb{Q}^n$  as inputs. Remember such a function was called *multilinear* if whenever we fix  $N - 1$  of the inputs it's linear in the remaining one, and such a function is *alternating* if, whenever we swap two vectors on the list of inputs, the output changes sign. Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{Q}^n$ . [20 points]

- (a) Recall that  $\det : (\mathbb{Q}^n)^n \rightarrow \mathbb{Q}$  was the unique multilinear, alternating function satisfying  $\det(e_1, \dots, e_n) = 1$ . Prove that if  $N > n$ , any multilinear alternating function  $\varphi : (\mathbb{Q}^n)^N \rightarrow \mathbb{Q}$  is identically zero.

- (b) Consider an alternating multilinear function  $\varphi : (\mathbb{Q}^n)^{n-1} \rightarrow \mathbb{Q}$  with  $n - 1$  arguments. If  $a_1, \dots, a_{n-1}$  are vectors in  $\mathbb{Q}^n$ , and we decompose each as  $a_i = \sum_{j=1}^n a_{ji} e_j$ , write down a formula for the value  $\varphi(a_1, \dots, a_{n-1})$  in terms of the coordinates  $a_{ji}$  and the  $n$  specific values

$$\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n)$$

for  $1 \leq k \leq n$ . (This is a list of  $n - 1$  of the  $n$  basis vectors in order - you have  $n$  choices of which vector to leave out, and thus  $n$  such lists).

- (c) Prove that if  $x = (x_1, \dots, x_n)$  is any list of  $n$  values in  $\mathbb{Q}$ , there's a unique multilinear alternating function  $\varphi : (\mathbb{Q}^n)^{n-1} \rightarrow \mathbb{Q}$  with

$$\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n) = x_k$$

for  $1 \leq k \leq n$ .