Math 4310 Prelim 2 Solutions

Problem 1. As suggested, write

$$q_a(x) = c_0(a) + c_1(a)x + c_2(a)x^2$$

and use the facts that we know what T_a should do to the polynomials 1, $(x-a)$, and $(x-a)^2$ (and therefore how q_a should pair with them).

Starting with the first one, we have

$$1 = T_a(1) = \langle q_a(x), 1\rangle = \int_{-1}^{1} q_a(x)dx = \int_{-1}^{1} \big(c_0(a) + c_1(a)x + c_2(a)x^2\big)dx.$$

This is just an integral of a polynomial which we can evaluate as

$$1 = 2c_0(a) + \frac{2}{3}c_2(a).$$

We can't solve for $c_0(a)$ or $c_2(a)$ yet until we have a bit more information (from pairing with $(x-a)^2$).

Similarly we can work with

$$0 = T_a(x-a) = \langle q_a(x), x-a\rangle = \int_{-1}^{1} q_a(x)(x-a)dx = \int_{-1}^{1} q_a(x)xdx - \int_{-1}^{1} q_a(x)dx.$$

We already know that the latter integral is equal to 1 (by our calculation above). For the former one we have

$$\int_{-1}^{1} q_a(x)xdx = \int_{-1}^{1} \big(c_0(a)x + c_1(a)x^2 + c_2(a)x^3\big)dx = \frac{2}{3}c_1(a).$$

Plugging this back in we get $\frac{2}{3}c_1(a) - a = 0$ and thus $c_1(a) = 3a/2$.

Finally we look at

$$0 = T_a((x-a)^2) = \langle q_a(x), (x-a)^2\rangle = \int_{-1}^{1} q_a(x)x^2dx - 2a\int_{-1}^{1} q_a(x)xdx + a^2\int_{-1}^{1} q_a(x)dx.$$

We've already evaluated the latter two integrals to get $\int q_a(x)dx = 1$ and $\int q_a(x)xdx = a$, so we can plug this in and find

$$a^2 = \int_{-1}^{1} q_a(x)x^2dx = \int_{-1}^{1} \big(c_0(a)x^2 + c_1(a)x^3 + c_2(a)x^4\big)dx = \frac{2}{3}c_0(a) + \frac{2}{5}c_2(a).$$

At this point we have two equations in the variables $c_0(a)$ and $c_2(a)$; clearing denominators we can write them as

$$6c_0(a) + 2c_2(a) = 3 \qquad 10c_0(a) + 6c_2(a) = 15a^2.$$

Solving this we get

$$c_0(a) = \frac{9-15a^2}{8} \qquad c_2(a) = \frac{45a^2-15}{8}.$$

Plugging this and our earlier value of $c_1(a)$ we find

$$q(x,a) = \frac{9-15a^2}{8} + \frac{3a}{2}x + \frac{45a^2-15}{8}x^2.$$

Problem 2. (a) To prove linearity we just need to check:

$$L_A(aX+bY) = A(aX+bY) = a(AX) + b(AY) = aL_A(X) + bL_A(Y)$$

which follows from basic algebraic properties of matrix multiplication. To see what the matrix is, recall that the first column will be the coordinates of

$$L_A(E_{11}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = aE_{11} + cE_{21}$$

and similarly for the others, so the coordinate matrix is

$$[L_A] = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

If $c = 0$ this is diagonal and $\det[L_A] = a^2d^2$. If $c \neq 0$ we can add $-a^{-1}c$ times the first row to the third and similarly for the second and fourth and get

$$\det[L_A] = \det \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & d - a^{-1}cb & 0 \\ 0 & 0 & 0 & d - a^{-1}cb \end{bmatrix}$$

which is $(ad-bc)^2$. So in either case we get $\det[L_A] = (ad-bc)^2 = \det(A)^2$, so L_A is invertible iff A is invertible.

(b) Suppose $L_A = R_B$; this means $AX = L_A(X) = R_B(X) = XB$ for every $X \in M_2(F)$. Taking $X = I$ we conclude we must have $A = B$. Then we want to prove that $L_A = R_A$ means A is a scalar matrix. Expanding out $AE_{11} = E_{11}A$ and comparing coefficients lets us conclude $b = c = 0$ so A is diagonal. Then expanding $AE_{12} = E_{12}A$ gives $a = d$ so A is a scalar matrix. (And conversely, if $A = aI$ is a scalar matrix then $L_A(X) = R_A(X) = aX$).

Problem 3. We claim that A is noncyclic iff $a = 1$ or $a = 2$. (This matrix is actually an example of a *rational canonical form*, and from the theory of rational canonical forms one can identify A must be noncyclic in those cases and that it must be cyclic in the others).

To prove this directly, we want to either prove that there either does or doesn't exist a vector $b \in \mathbb{R}^3$ such that b, Ab, A^2b spans $\mathbb{R}^3$. So consider an arbitrary b:

$$b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad Ab = \begin{bmatrix} -2y \\ x+3y \\ az \end{bmatrix} \qquad A^2b = \begin{bmatrix} -2x-6y \\ 3x+7y \\ a^2z \end{bmatrix}.$$

Now, we want to see if it's possible for this to span. First of all, just from looking at it we can see that if there's any hope for it to span we need the vector

$$v_0 = \begin{bmatrix} x \\ y \end{bmatrix}$$

to be nonzero, and also the entry z to be nonzero, so we make these assumptions.

We next claim we moreover need the vectors

$$v_0 = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad v_1 = \begin{bmatrix} -2y \\ x+3y \end{bmatrix}$$

to be linearly independent. If not, then $v_1 = \lambda v_0$ is a scalar multiple of v_0, and by looking at how the vectors are constructed we can find b, Ab, A^2b all lie in the span of the two vectors

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and thus don't span.

So, we've shown that if $\{b, Ab, A^2b\}$ has a possibility of spanning then $\{v_0, v_1\}$ must be linearly independent and $z \neq 0$. Given the linear independence of v_0 and v_1, there's a unique possible linear combination of b and Ab that can agree with the first two coordinates of A^2b, namely

$$-2 \cdot b + 3 \cdot Ab = \begin{bmatrix} -2x - 6y \\ 3x + 7y \\ (-2 + 3a)z \end{bmatrix}.$$

So the only possible way that A^2b can be a linear combination of Ab and b is if $A^2b = 2b + 3 \cdot Ab$, and by comparing the last coordinates this is true iff $(-2 + 3a)z = a^2z$. Since $z \neq 0$ by our assumption, then this forces $2 + 3a = a^2$, i.e. $a^2 - 3a + 2 = (a-2)(a-1) = 0$. So if $a \neq 1, 2$ then our vectors give something that spans, and if $a = 1, 2$ they don't.

So, to summarize, if $a = 1, 2$ then we've proven it's *impossible* to find b with $\{b, Ab, A^2b\}$ spanning, and thus A is not cyclic. On the other hand if $a \neq 1, 2$ we've given a recipe for finding b such that this set spans: we just need to pick the first two coordinates so v_0, v_1 are linearly independent, and pick $z \neq 0$. So an example basis is

$$b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad Ab = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} \qquad A^2b = \begin{bmatrix} -2 \\ 3 \\ a^2 \end{bmatrix}.$$

Problem 4. (a) Suppose A has characteristic polynomial $c_A(x) = (x - \lambda)^2$ but is not equal to the scalar matrix λI. Then there is an eigenvector v_1 of A with eigenvalue λ. Then take a vector v_2 linearly independent of v_1, so $\{v_1, v_2\}$ is a basis. Note v_2 *cannot* be an eigenvector (since otherwise it would have to have eigenvalue λ and that would force $A = \lambda I$), so $Av_2 = \alpha v_1 + \beta v_2$ for some $\alpha \neq 0$ and some β. Moreover by replacing v_2 by v_2/α we in can assume that $\alpha = 1$, i.e. $Av_1 = v_1 + \beta v_2$. Then the coordinate matrix of A with respect to this new basis is

$$\begin{bmatrix} \lambda & 1 \\ 0 & \alpha \end{bmatrix}.$$

But the fact that characteristic polynomials are independent of coordinates means that this has to have characteristic polynomial $(x - \lambda)^2$, so $\lambda = \alpha$, and thus we've shown A is similar to the matrix we want.

(b) We know A is diagonalizable with distinct eigenvalues iff $c_A(x)$ is not of the form $(x - \lambda)^2$. On the other hand computing c_A directly from the matrix A we have

$$c_A(x) = x^2 - (a - d)x + (ad - bc).$$

If you remember back to elementary algebra, a way to tell when a quadratic polynomial $\alpha x^2 + \beta x + \gamma$ has two distinct roots iff the *discriminant* $\beta^2 - 4\alpha\gamma$ is nonzero. So in our case, $c_A(x)$ has a repeated root iff

$$(a + d)^2 - 4(ad - bc) = a^2 - 2ad + d^2 + 4bc \neq 0.$$

Problem 5. (a) Since a, b, c, d are integers, the characteristic polynomial $x^2 + \beta x + \gamma$ has integer coefficients. Moreover, if it has a rational eigenvalue then this polynomial factors over $\mathbb{Q}$, so we have two rational numbers $\lambda_1, \lambda_2 \in \mathbb{Q}$ with

$$(x^2 + \beta x + \gamma) = (x - \lambda_1)(x - \lambda_2).$$

Multiplying this out and comparing coefficients, we get $\lambda_1 + \lambda_2 = -\beta \in \mathbb{Z}$ and $\lambda_1\lambda_2 = \gamma \in \mathbb{Z}$.

Now, suppose λ_1 is *not* an integer. Then there's some prime p showing up in the denominator but not in the numerator. For $\lambda_1\lambda_2$ to be an integer this means p shows up in the numerator of λ_2 and not the denominator. But then $\lambda_1 + \lambda_2$ will have p in the denominator, a contradiction. So λ_1 must be an integer (and by symmetry λ_2 must be too).

(b) Remember any scalar multiple of an eigenvector for λ is again an eigenvector for λ! So if

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Q}^2,$$

is an eigenvector with eigenvalue λ, we can just multiply it by a big enough integer to clear denominators and get an eigenvector in $\mathbb{Z}^2$.

(c) If λ is an eigenvalue in $\mathbb{Z}$, by (b) there's an eigenvector

$$v = \begin{bmatrix} x & y \end{bmatrix} \in \mathbb{Q}^2$$

with eigenvalue λ and entries in $\mathbb{Z}$. Then we can consider

$$\overline{v} = \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \in \mathbb{F}_p^2,$$

and since reduction-mod-p is compatible with the algebraic operations on $\mathbb{Z}$ we get that $Av = \lambda v$ implies $\overline{A}\overline{v} = \overline{\lambda}\overline{v}$.

(d) Perhaps surprisingly, no! A problem can come up when two *distinct* eigenvalues over $\mathbb{Z}$ become the same eigenvalue when reducing mod p (and their distinct eigenvectors can become the same). For instance if we take

$$\begin{bmatrix} p+1 & 1 \\ 0 & 1 \end{bmatrix}$$

then this has distinct eigenvalues 1 and $p+1$ over $\mathbb{Q}$, which means it is diagonalizable. But the reduction mod p is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which is never diagonalizable.

Problem 6. (a) If $N \geq n+1$, then any N vectors $v_1, \ldots, v_N$ are linearly dependent. Swapping the order so v_N is a linear combination of the others if necessary (note that at worst this changes the sign of $\varphi(v_1, \ldots, v_N)$) we can write $v_N = \sum_{i=1}^{n-1} a_i v_i$ and then by multilinearity

$$\varphi(v_1, \ldots, v_N) = \sum_{i=1}^{n-1} \varphi(v_1, \ldots, v_{n-1}, v_i) = 0$$

because alternating implies that whenever the same vector shows up twice in a list then φ of that list must be zero.

(b) We can imitate the proof of uniqueness of the determinant. We can expand out each coordinate of $\varphi(a_1, \ldots, a_{n-1})$ by multilinearity, and get

$$\varphi(a_1, \ldots, a_{n-1}) = \sum_{\sigma:[n-1]\to[n]} \left(\prod_{i=1}^{n-1} a_{\sigma(i),i} \right) \varphi(e_{\sigma(1)}, \ldots, e_{\sigma(n-1)})$$

where we take the notation that $[N]$ denotes the set $\{1, \ldots, N\}$. Now, the alternating property implies that if φ has two of the same vectors as inputs then the value is zero, so we can restrict to σ that are *injective*.

$$\varphi(a_1, \ldots, a_{n-1}) = \sum_{\sigma:[n-1]\hookrightarrow[n]} \left(\prod_{i=1}^{n-1} a_{\sigma(i),i} \right) \varphi(e_{\sigma(1)}, \ldots, e_{\sigma(n-1)}).$$

Now we think about what an injective map from $[n-1] = \{1, \dots, n-1\}$ into $[n] = \{1, \dots, n\}$ looks like. It has to miss exactly one element k and be onto $[n] \setminus k = [n] \setminus \{k\}$, so

$$\varphi(a_1, \dots, a_{n-1}) = \sum_{k=1}^{n} \sum_{\sigma : [n-1] \cong [n] \setminus k} \left(\prod_{i=1}^{n-1} a_{\sigma(i),i} \right) \varphi(e_{\sigma(1)}, \dots, e_{\sigma(n-1)}).$$

Now each σ left is a permutation $\tau \in S_{n-1}$ composed with the order-preserving bijection $\iota_k : \{1, \dots, n-1\} \leftrightarrow \{1, \dots, k-1, k+1, \dots, n\}$. By rearranging things we can write this as

$$\varphi(a_1, \dots, a_{n-1}) = \sum_{k=1}^{n} \left(\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \left(\prod_{i=1}^{n-1} a_{\iota_k(\tau(i)),i} \right) \right) \varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n)$$

which is the formula we we want.

(c) From the formula in part (b) it's obvious what we want to do: define φ by the formula

$$\varphi(a_1, \dots, a_{n-1}) = \sum_{k=1}^{n} \left(\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \left(\prod_{i=1}^{n-1} a_{\iota_k(\tau(i)),i} \right) \right) x_k.$$

Certainly any φ satisfying the properties we want has to be given by this formula by part (b), proving uniqueness. However, for existence we need to justify that this formula is actually an alternating multilinear function which has the values $\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n) = x_k$. (In part (b) we were *assuming* those properties, and we need to justify that the formula we wrote down *actually satisfies* them). This would proceed much like how one shows the formula for the determinant as a sum over permutations is alternating and multilinear.

However, we can alternatively use the fact that we already *know* the determinant is alternating and multilinear to define a new alternating and multilinear function $\varphi : (\mathbb{Q}^n)^{n-1} \to \mathbb{Q}$ satisfying the properties we want; this will prove existence, and the formula from (b) lets us conclude uniqueness. A particularly slick way to do this is as follows: take a vector $y \in \mathbb{Q}^n$ with coordinates given by $y_i = (-1)^{n-i} x_i$. Then define a function $\varphi : (\mathbb{Q}^n)^{n-1} \to \mathbb{Q}$

$$\varphi(a_1, \dots, a_{n-1}) = \det(a_1, \dots, a_{n-1}, y).$$

Then we can conclude that φ is multilinear and alternating from the fact that det is (for instance, switching the order of two arguments in φ switches the order of two arguments in det and thus reverses the sign of the determinant). So we just need to check that φ defined this way has the right value of $\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n)$. But we just compute

$$\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n) = \det(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n, y) = (-1)^{n-k} \det(e_1, \dots, e_{k-1}, y, e_{k+1}, \dots, e_n)$$

using that det is alternating, and then substituting $y = \sum y_i e_i$ we get

$$\varphi(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n) = (-1)^{n-k} \det(e_1, \dots, e_{k-1}, \sum y_i e_i, e_{k+1}, \dots, e_n) = (-1)^{n-k} y_k,$$

which is x_k by definition.