IX. Equilibrium?

In this problem, you will try to determine whether a one-dimensional "room" with a heat source in the middle and ice cubes at the ends reaches equilibrium in a reasonable amount of time. The given problem is

$$\begin{aligned} u_t - u'' &= f, \quad t > 0, \\ u(x,0) &= 0, \\ u(0,t) = u(1,t) = 0, & \quad t > 0. \end{aligned}$$

where $f(x) = 10$ if $.5 < x < .6$ and $f(x) = 0$ otherwise.

You should use your piecewise linear finite element code along with the trapezoidal method to investigate this problem. The trapezoidal method is a one-step linear multistep method which has global second-order accuracy. The development of the trapezoidal method is not done most naturally from difference quotients, but rather by integrating a linear interpolant (and in fact it corresponds precisely to the trapezoid rule for numerical integration). The final finite element time step for this method is

$$U^{(n+1)} = (M + \frac{k_n}{2} S(t_{n+1}))^{-1}(MU^{(n)} + \frac{k_n}{2}(F(t_n) + F(t_{n+1}) - S(t_n)U^{(n)})).$$

Using your finite element code, determine whether, and at about what time, the "room" reaches a steady state (equilibrium). Of course, the room won't actually ever reach a steady state, but you should try to make a convincing argument using simulations that the temperature distribution in the room will remain constant to high precision after some point. Note that you should be able to determine the room's steady state analytically. Also, be careful to place mesh points in a "good" configuration (think about problem 2!), and balance the size of the time step and mesh size properly (they should be about the same since the trapezoid rule is second-order, but if you use a non-uniform mesh, you'll need to decide whether $k$ should be close to the smallest mesh size or the largest mesh size).