

# Generators of the Arc Algebra

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# The Arc Algebra

## Background:

- ▶ generalization of the Kauffman bracket skein algebra
- ▶ defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- ▶ applies to thickened surfaces with punctures

# The Arc Algebra

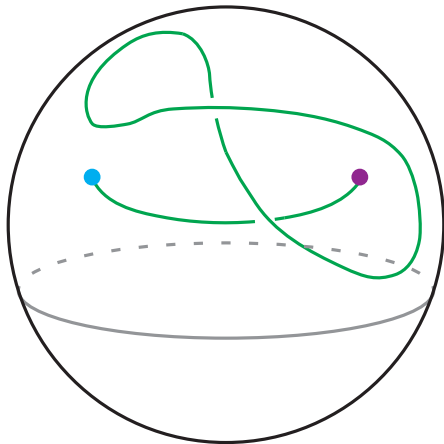
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- ▶ generalization of the Kauffman bracket skein algebra
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## Results:

- ▶ full presentations for small examples
- ▶ a finite set of generators for any surface with punctures

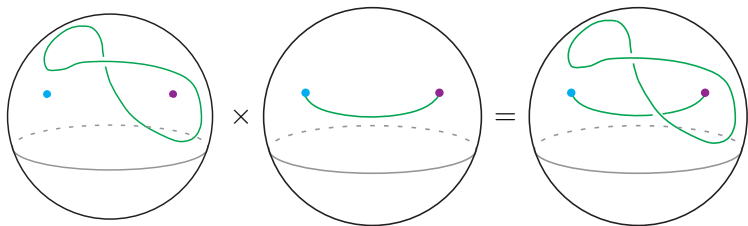
# The Arc Algebra - An Element



# The Arc Algebra - Definition

- ▶ Let  $F_{g,n}$  denote the surface of genus  $g$  with  $n$  punctures.
- ▶ Let  $R_n$  be the ring  $\mathbb{Z}[A^{\pm\frac{1}{2}}][v_1, \dots, v_n]$ .
- ▶ The arc algebra  $\mathcal{A}(F_{g,n})$  consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface  $F_{g,n} \times [0, 1]$  subject to four relations.
- ▶ Multiplication is by stacking, induced by 
$$F_{g,n} \times [0, 1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1].$$

# The Arc Algebra - Multiplication



# The Arc Algebra - Kauffman Bracket Relations

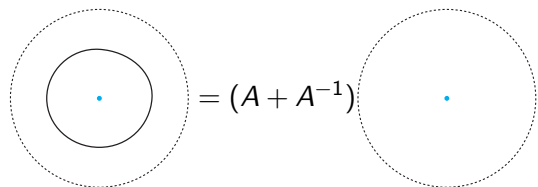
Two of the four relations are the same as the skein algebra:

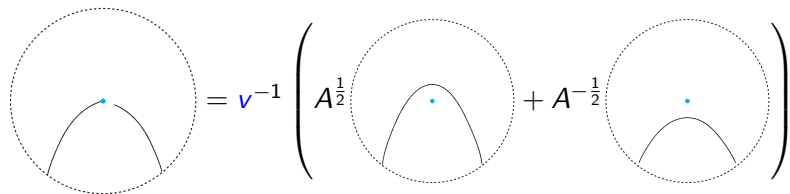
The image shows two equations involving diagrams of circles with dashed outer boundaries. The first equation shows a circle with a smaller circle inside it, equal to  $(-A^2 - A^{-2})$  times a single circle. The second equation shows a circle with two crossing lines, equal to  $A$  times a circle with two lines that meet at a cusp (forming a 'V' shape), plus  $A^{-1}$  times a circle with two lines that meet at a cusp (forming an inverted 'V' shape).

$$\text{Circle with inner circle} = (-A^2 - A^{-2}) \text{Circle}$$
$$\text{Circle with crossing lines} = A \text{Circle with cusp} + A^{-1} \text{Circle with cusp}$$

# The Arc Algebra - Puncture Relations

There are two more relations for punctured surfaces:


$$\text{Diagram 1} = (A + A^{-1}) \text{Diagram 2}$$


$$\text{Diagram 3} = v^{-1} \left( A^{\frac{1}{2}} \text{Diagram 4} + A^{-\frac{1}{2}} \text{Diagram 5} \right)$$



# The Arc Algebra - All Relations

$$\text{Diagram 1} = (-A^2 - A^{-2}) \text{Diagram 2}$$

$$\text{Diagram 3} = A \text{Diagram 4} + A^{-1} \text{Diagram 5}$$

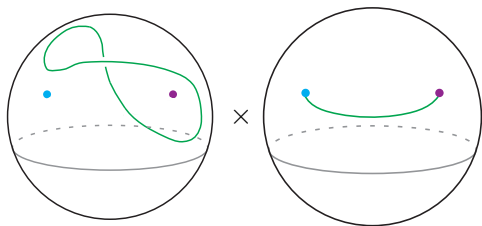
$$\text{Diagram 6} = (A + A^{-1}) \text{Diagram 7}$$

$$\text{Diagram 8} = v^{-1} \left[ A^{\frac{1}{2}} \text{Diagram 9} + A^{-\frac{1}{2}} \text{Diagram 10} \right]$$

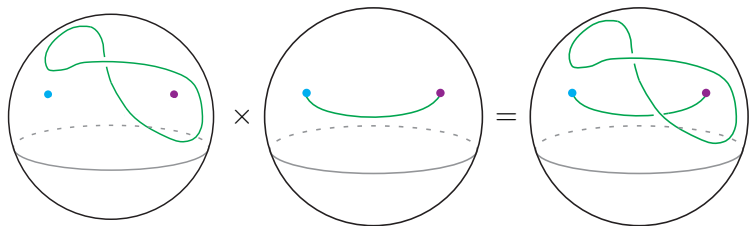
# Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

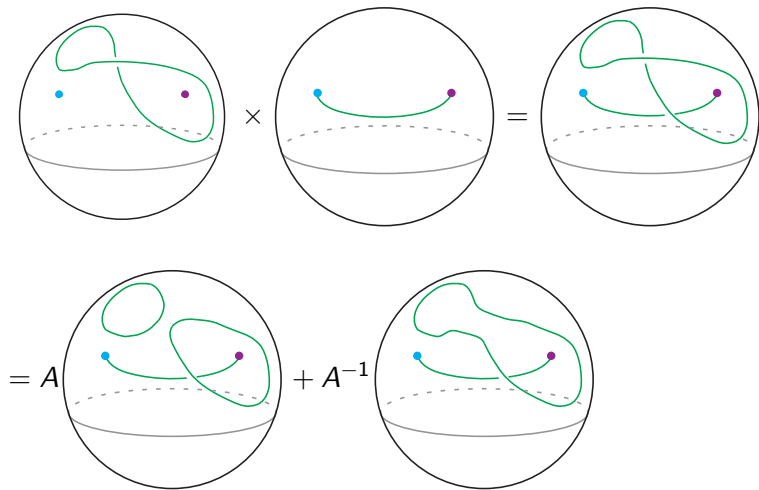
## Example on the Twice-Punctured Sphere



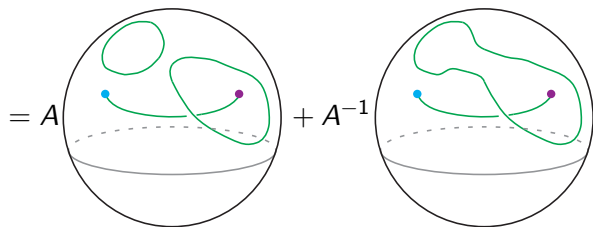
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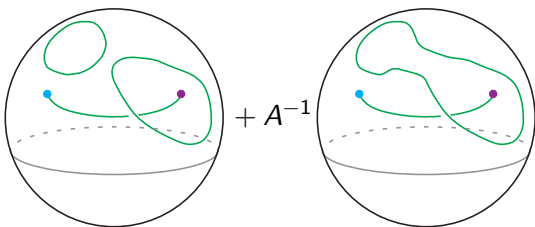
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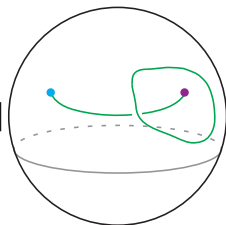


# Example on the Twice-Punctured Sphere



# Example on the Twice-Punctured Sphere

$$= A \left[ \text{Diagram 1} \right] + A^{-1} \left[ \text{Diagram 2} \right]$$


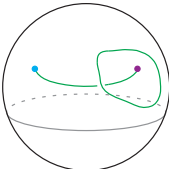
$$= [A(-A^2 - A^{-2}) + A^{-1}] \left[ \text{Diagram 3} \right]$$


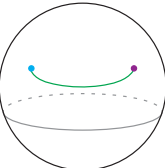
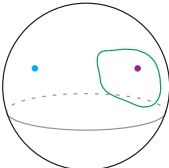
## Example on the Twice-Punctured Sphere

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

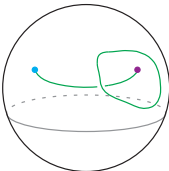


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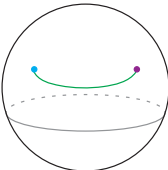
$$= [A(-A^2 - A^{-2}) + A^{-1}] \left( \text{Diagram 1} \right)$$


$$= [A(-A^2 - A^{-2}) + A^{-1}] \left( \text{Diagram 2} \right) \times \left( \text{Diagram 3} \right)$$


## Example on the Twice-Punctured Sphere

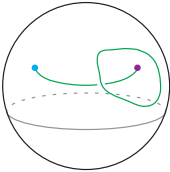
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$


A diagram of a twice-punctured sphere, represented as a circle with a horizontal line through the center and a dashed line below it. Two small dots, one blue and one purple, are placed on the upper surface. A green loop starts at the blue dot, goes to the right, loops back to the left, and ends at the purple dot.

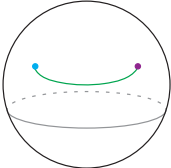
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


A diagram of a twice-punctured sphere, represented as a circle with a horizontal line through the center and a dashed line below it. Two small dots, one blue and one purple, are placed on the upper surface. A green arc connects the blue dot to the purple dot.

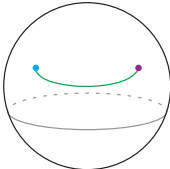
## Example on the Twice-Punctured Sphere

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$


A diagram of a sphere with a dashed line representing the equator. Two points, one blue and one purple, are marked on the sphere's surface. A green curve connects these two points, forming a loop that crosses itself once.

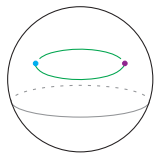
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


A diagram of a sphere with a dashed line representing the equator. Two points, one blue and one purple, are marked on the sphere's surface. A green curve connects these two points, forming a simple arc.

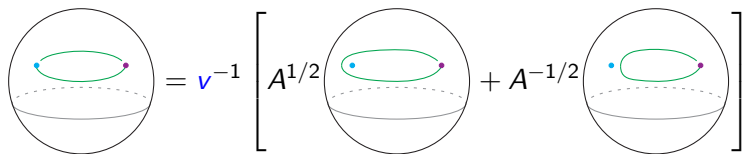
$$= (-A^4 - A^2)$$


A diagram of a sphere with a dashed line representing the equator. Two points, one blue and one purple, are marked on the sphere's surface. A green curve connects these two points, forming a simple arc.

## Another Example on the Twice-Punctured Sphere



## Another Example on the Twice-Punctured Sphere



The diagram shows an equation involving four spheres, each representing a twice-punctured sphere with a blue dot on the left and a purple dot on the right. A green loop encircles the punctures. The equation is:

$$\text{Sphere} = v^{-1} \left[ A^{1/2} \text{Sphere} + A^{-1/2} \text{Sphere} \right]$$

The first sphere on the left has a green loop with an arrow pointing clockwise. The two spheres inside the brackets have green loops with arrows pointing counter-clockwise.

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \left[ \text{Sphere with blue dot, purple dot, and large green loop} \right] = v^{-1} \left[ A^{1/2} \left[ \text{Sphere with blue dot, purple dot, and large green loop} \right] + A^{-1/2} \left[ \text{Sphere with blue dot, purple dot, and large green loop} \right] \right] = \\
 & (vv)^{-1} \left[ A \left[ \text{Sphere with blue dot, purple dot, and small green loop} \right] + \left[ \text{Sphere with blue dot, purple dot, and large green loop} \right] + \left[ \text{Sphere with blue dot, purple dot, and medium green loop} \right] + A^{-1} \left[ \text{Sphere with blue dot, purple dot, and small green loop} \right] \right]
 \end{aligned}$$

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \left[ \text{Sphere with large loop} \right] = v^{-1} \left[ A^{1/2} \left[ \text{Sphere with large loop} \right] + A^{-1/2} \left[ \text{Sphere with large loop} \right] \right] = \\
 & (vv)^{-1} \left[ A \left[ \text{Sphere with small loop} \right] + \left[ \text{Sphere with large loop} \right] + \left[ \text{Sphere with small loop} \right] + A^{-1} \left[ \text{Sphere with small loop} \right] \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})]
 \end{aligned}$$

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \left[ \text{Diagram} \right] = v^{-1} \left[ A^{1/2} \left[ \text{Diagram} \right] + A^{-1/2} \left[ \text{Diagram} \right] \right] = \\
 & (vv)^{-1} \left[ A \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + A^{-1} \left[ \text{Diagram} \right] \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \\
 & = (vv)^{-1} (-A^2 + 2 - A^{-2})
 \end{aligned}$$



# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.

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3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.

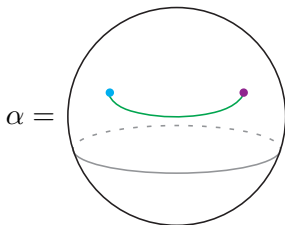
# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

# Presentation of the Twice-Punctured Sphere

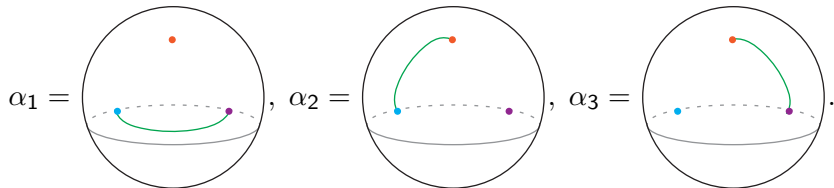
## Theorem

The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures,  $\alpha$ , with the relation  $\alpha^2 = -\frac{1}{v_1 v_2} (A - A^{-1})^2$ .



# Algebra of the Thrice-Punctured Sphere

By the same process as in the twice-punctured sphere, a generating set is:



# Algebra of the Thrice-Punctured Sphere

As before, squaring a generator results in unknots which can be removed.

$$\begin{aligned}
 \alpha_i^2 &= \text{diagram: a dashed circle containing a dot labeled } i \text{ at the top and a smaller solid circle below it with dots labeled } i+1 \text{ and } i+2 \text{ on its left and right sides respectively.} \\
 &= v_{i+1}^{-1} v_{i+2}^{-1} \left( A \text{diagram: dashed circle with } i \text{ at top, } i+1 \text{ at bottom-left, } i+2 \text{ at bottom-right, and a smaller dashed circle below it with } i+1 \text{ and } i+2 \text{ on its left and right sides.} + \text{diagram: dashed circle with } i \text{ at top, } i+1 \text{ at bottom-left, } i+2 \text{ at bottom-right, and a smaller solid circle below it with } i+1 \text{ and } i+2 \text{ on its left and right sides.} + \text{diagram: dashed circle with } i \text{ at top, } i+1 \text{ at bottom-left, } i+2 \text{ at bottom-right, and a smaller dashed circle above it with } i+1 \text{ and } i+2 \text{ on its left and right sides.} + A^{-1} \text{diagram: dashed circle with } i \text{ at top, } i+1 \text{ at bottom-left, } i+2 \text{ at bottom-right, and a smaller solid circle below it with } i+1 \text{ and } i+2 \text{ on its left and right sides.} \right) \\
 &= v_{i+1}^{-1} v_{i+2}^{-1} (A(A + A^{-1}) + (-A^2 - A^{-2}) + (A + A^{-1}) \\
 &\quad + A^{-1}(A + A^{-1})) \\
 &= v_{i+1}^{-1} v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2 \\
 &= v_{i+1}^{-1} v_{i+2}^{-1} \delta^2.
 \end{aligned}$$



# Algebra of the Thrice-Punctured Sphere

Also,

$$\begin{aligned}\alpha_i \alpha_{i+1} &= \text{Diagram} = v_{i+2}^{-1} \left( A^{\frac{1}{2}} \text{Diagram} + A^{-\frac{1}{2}} \text{Diagram} \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2}\end{aligned}$$

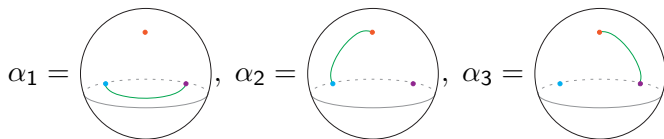
and similarly

$$\begin{aligned}\alpha_{i+1} \alpha_i &= \text{Diagram} = v_{i+2}^{-1} \left( A^{\frac{1}{2}} \text{Diagram} + A^{-\frac{1}{2}} \text{Diagram} \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2}.\end{aligned}$$

# Presentation of the Thrice-Punctured Sphere

## Theorem

*The arc algebra for the thrice-punctured sphere is generated by three simple arcs*



*and has relations*

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

*where subscripts are interpreted modulo 3.*

# Algebra of the Thrice-Punctured Sphere in Matrices

$$\rho(\alpha_1) = \begin{bmatrix} 0 & v_2^{-1}v_3^{-1}\delta^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2^{-1}\delta \\ 0 & 0 & v_3^{-1}\delta & 0 \end{bmatrix},$$

$$\rho(\alpha_2) = \begin{bmatrix} 0 & 0 & v_1^{-1}v_3^{-1}\delta^2 & 0 \\ 0 & 0 & 0 & v_1^{-1}\delta \\ 1 & 0 & 0 & 0 \\ 0 & v_3^{-1}\delta & 0 & 0 \end{bmatrix},$$

$$\rho(\alpha_3) = \begin{bmatrix} 0 & 0 & 0 & v_1^{-1}v_2^{-1}\delta^2 \\ 0 & 0 & v_1^{-1}\delta & 0 \\ 0 & v_2^{-1}\delta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

# The Arc Algebra is Finitely Generated for all $F_{g,n}$

## Theorem

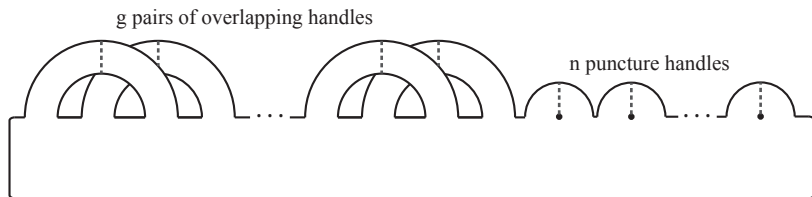
*When  $n = 0$  or  $n = 1$ , the arc algebra  $\mathcal{A}(F_{g,n})$  can be generated by  $2^{2g} - 1$  knots. For  $n > 1$ , it can be generated by a set of  $(2^{2g} - 1)(n)$  knots and  $2^{2g} \binom{n}{2}$  arcs.*

## Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams with minimal complexity. These generators can be counted combinatorially. □

## Proof Sketch - Setting Up

Remove a small disk from  $F_{g,n}$  to make  $F_{g,n}^*$ .

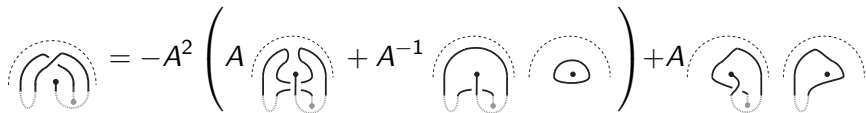


Generators of  $\mathcal{A}(F_{g,n}^*)$  will generate  $\mathcal{A}(F_{g,n})$ .

## Proof Sketch - Complexity 1 of 3

Reduce to curves passing through each handle (genus or puncture) at most once. There are several cases. Example:

$$\text{Diagram 1} = -A^2 \text{Diagram 2} + A \text{Diagram 3}$$


$$\text{Diagram 4} = -A^2 \left( A \text{Diagram 5} + A^{-1} \text{Diagram 6} \text{Diagram 7} \right) + A \text{Diagram 8} \text{Diagram 9}$$


Each of the diagrams with crossings can be further simplified by a straightforward application of the skein relation.

## Proof Sketch - Complexity 2 of 3

Reduce to curves passing through pairs of genus handles in only the "good" way. There are two cases. Example:



## Proof Sketch - Complexity 3 of 3

Reduce to curves that pass through puncture handles a total of 0 (if arc) or 1 (if knot) times. There are many cases. Example:

The diagram illustrates the expansion of a curve in a genus-2 surface. The top part shows a curve consisting of two loops, one around puncture  $i$  and one around puncture  $j$ , connected by a horizontal arc. This curve is equal to a sum of four terms, each representing a different configuration of the curve passing through the punctures:

$$\begin{aligned} &= v_i v_j A^{-1} \left( \text{Diagram 1} \right) - A^{-1} \left( \text{Diagram 2} \right) \\ &\quad - A^{-1} \left( \text{Diagram 3} \right) - A^{-2} \left( \text{Diagram 4} \right) \end{aligned}$$

Diagram 1: The curve passes through puncture  $i$  once and puncture  $j$  once, with the arc connecting them.

Diagram 2: The curve passes through puncture  $i$  once and puncture  $j$  once, with the arc connecting them, but the orientation is reversed.

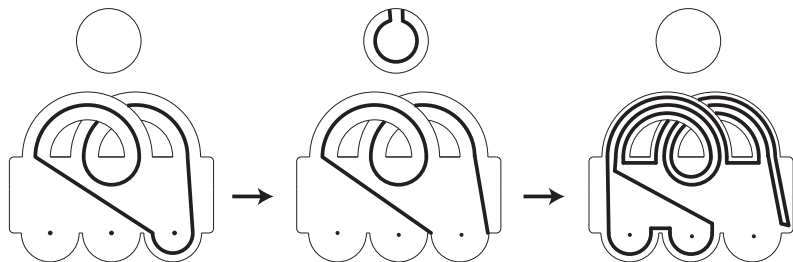
Diagram 3: The curve passes through puncture  $i$  once and puncture  $j$  once, with the arc connecting them, but the orientation is reversed and the weight is  $A^{-1}$ .

Diagram 4: The curve passes through puncture  $i$  once and puncture  $j$  once, with the arc connecting them, but the orientation is reversed and the weight is  $A^{-2}$ .

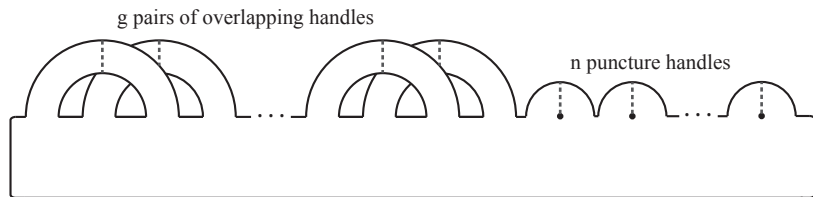


## Proof Sketch - Finishing

In addition, we can force the curves to avoid one of the puncture handles using the fact that we are generating  $\mathcal{A}(F_{g,n})$  and not  $\mathcal{A}(F_{g,n}^*)$ .



# Proof Sketch - Counting Generators



A generator can

- ▶ pass through each genus handle at most once
- ▶ pass through a pair of genus handles in only the good way
- ▶ pass through puncture handles a total of 1 time if knot
- ▶ pass through puncture handles a total of 0 times if arc
- ▶ start and end at any distinct pair of punctures if arc

these choices uniquely determine the generator.

# The Arc Algebra is Finitely Generated for all $F_{g,n}$

## Theorem

*When  $n = 0$  or  $n = 1$ , the arc algebra  $\mathcal{A}(F_{g,n})$  can be generated by  $2^{2g} - 1$  knots. For  $n > 1$ , it can be generated by a set of  $(2^{2g} - 1)(n)$  knots and  $2^{2g} \binom{n}{2}$  arcs.*

# Acknowledgements

Thanks to Helen Wong, Stephen Kennedy, Martin Bobb, and the Carleton College Mathematics Department. This project was partially supported by NSF Grant DMS-1105692.

## References

- [1] Doug Bullock. A finite set of generators for the Kauffman bracket skein algebra. *Math. Z.*, 231(1):91-101, 1999.
- [2] Doug Bullock and Józef H. Przytycki. Multiplicative structure of the Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.*, 128(3):923-931, 2000.
- [3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space. *J. Differential Geom.*, 96(1):95-140, 2014.

