Generators of the Arc Algebra

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The Arc Algebra

Background:

- generalization of the Kauffman bracket skein algebra
- defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- applies to thickened surfaces with punctures

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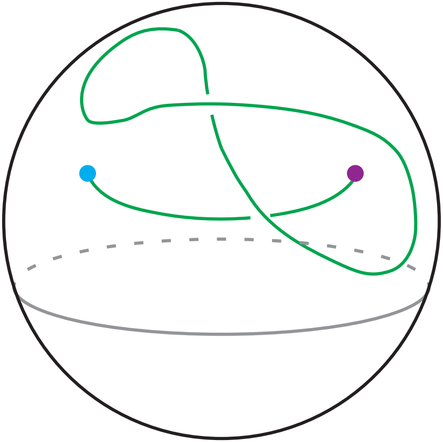
Results:

- full presentations for small examples
- a finite set of generators for any surface with punctures

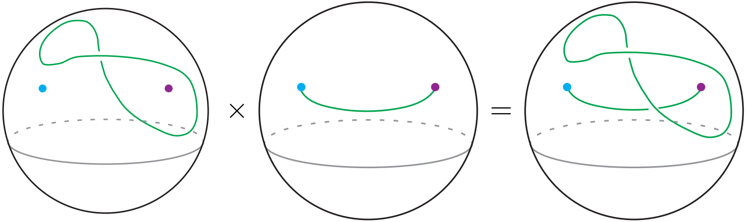
The Arc Algebra - An Element



The Arc Algebra - Definition

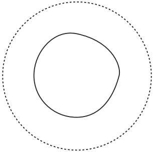
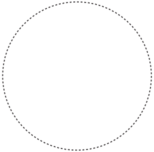
- Let $F_{g,n}$ denote the surface of genus g with n punctures.
- Let R_n be the ring $\mathbb{Z}[A^{\pm\frac{1}{2}}][v_1, \ldots, v_n]$.
- The arc algebra $\mathcal{A}(F_{g,n})$ consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface $F_{g,n} \times [0,1]$ subject to four relations.
- Multiplication is by stacking, induced by $F_{g,n} \times [0,1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1]$.

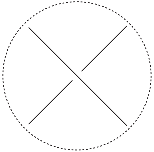
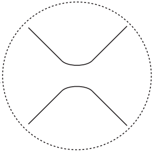
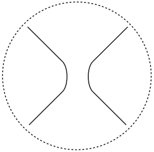
The Arc Algebra - Multiplication



The Arc Algebra - Kauffman Bracket Relations

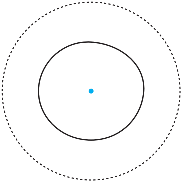
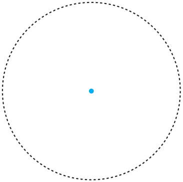
Two of the four relations are the same as the skein algebra:

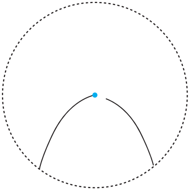
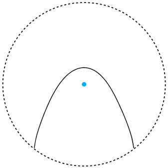
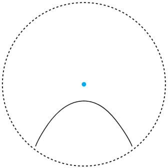
 $= (-A^2 - A^{-2})$ 

 $= A$  $+ A^{-1}$ 

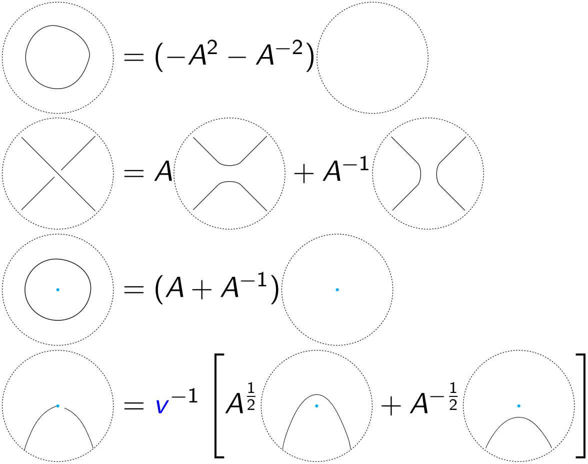
The Arc Algebra - Puncture Relations

There are two more relations for punctured surfaces:

 $= (A + A^{-1})$ 

 $= v^{-1} \Big(A^{\frac{1}{2}}$  $+ A^{-\frac{1}{2}}$  $\Big)$

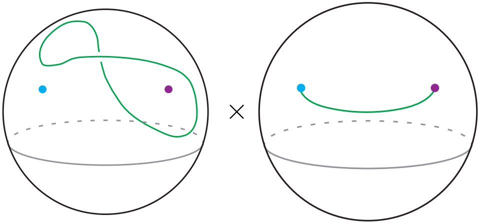
The Arc Algebra - All Relations



Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

Example on the Twice-Punctured Sphere



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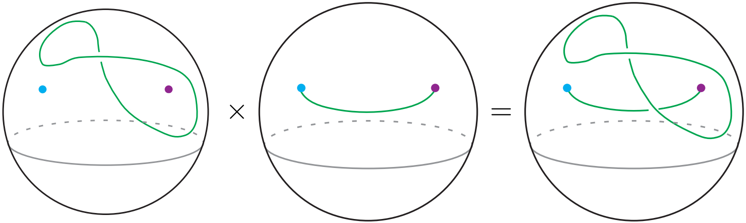

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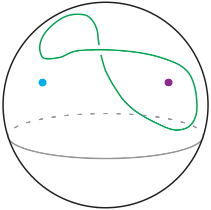
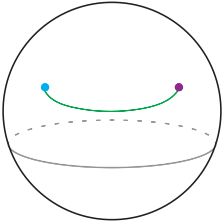
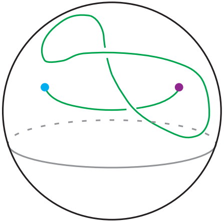
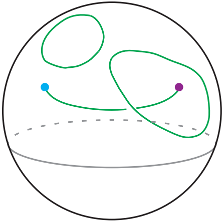
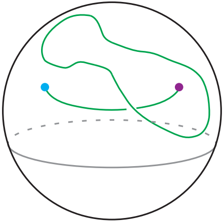
 $\times$  $=$ 

$= A$  $+ A^{-1}$ 

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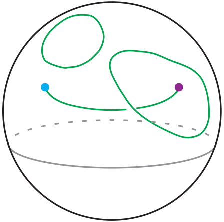
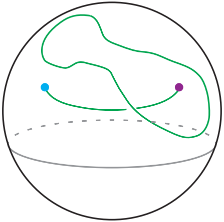
$= A$  $+ A^{-1}$ 

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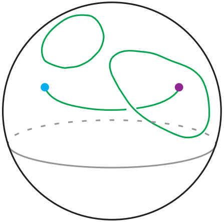
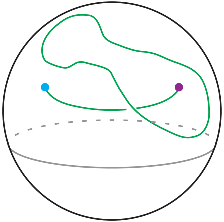
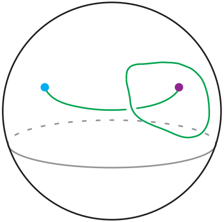
$= A$  $+ A^{-1}$ 

$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

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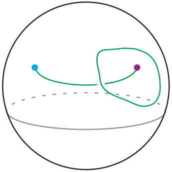
$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

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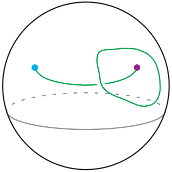
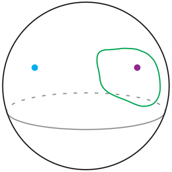
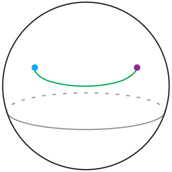
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$ 

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$  $\times$ 

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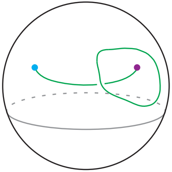

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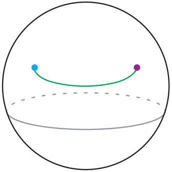

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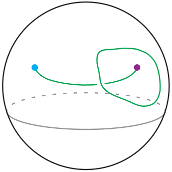
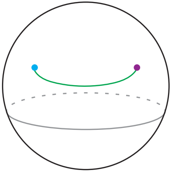
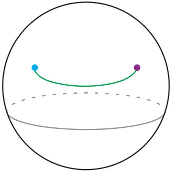
$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$ 

$= (-A^4 - A^2)$ 

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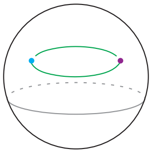

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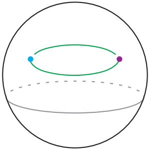
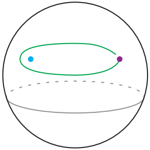
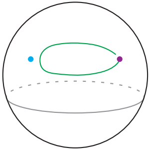
 $= v^{-1} \Big[A^{1/2}$  $+ A^{-1/2}$  $\Big]$

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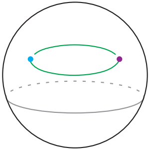
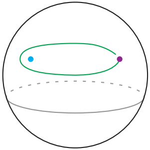
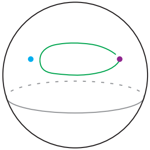


$= v^{-1} \Big[A^{1/2}$



$+ A^{-1/2}$



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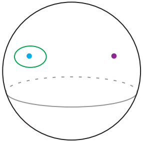

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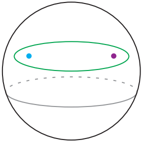

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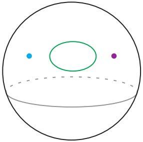

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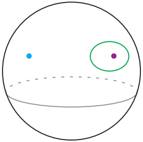

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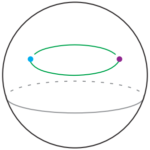
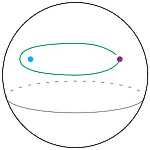
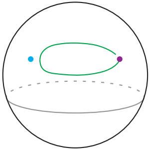
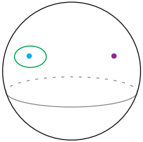
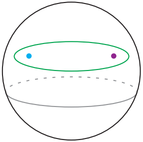
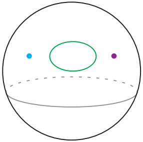
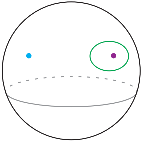
 $= v^{-1} \Big[A^{1/2}$  $+ A^{-1/2}$  $\Big] =$

$(vv)^{-1} \Big[A$  $+$  $+$  $+ A^{-1}$  $\Big]$

$$= (vv)^{-1}[A(A+A^{-1})+(-A^2-A^{-2})+(-A^2-A^{-2})+A^{-1}(A+A^{-1})]$$

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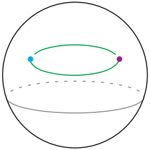
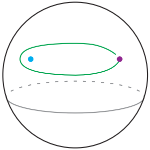
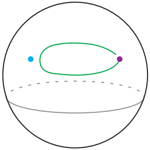
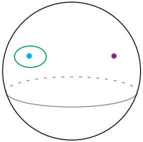
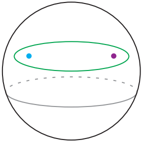
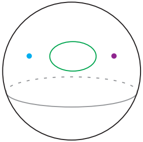
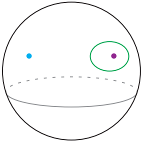
 $= v^{-1} \Bigg[A^{1/2}$  $+ A^{-1/2}$  $\Bigg] =$

$(vv)^{-1} \Bigg[A$  $+$  $+$  $+ A^{-1}$  $\Bigg]$

$$= (vv)^{-1}[A(A+A^{-1})+(-A^2-A^{-2})+(-A^2-A^{-2})+A^{-1}(A+A^{-1})]$$

$$= (vv)^{-1}(-A^2+2-A^{-2})$$

Algebra of the Twice-Punctured Sphere

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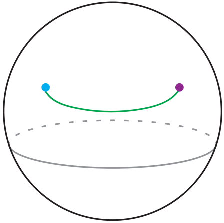
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

Presentation of the Twice-Punctured Sphere

Theorem

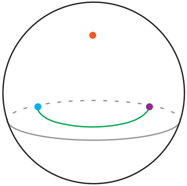
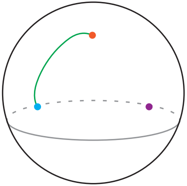
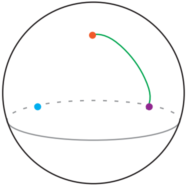
The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures, α, with the relation $\alpha^2 = -\frac{1}{v_1 v_2}(A - A^{-1})^2$.

$\alpha =$



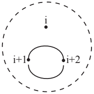
Algebra of the Thrice-Punctured Sphere

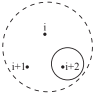
By the same process as in the twice-punctured sphere, a generating set is:

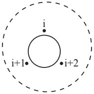
$\alpha_1 =$  , $\alpha_2 =$  , $\alpha_3 =$  .

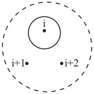
Algebra of the Thrice-Punctured Sphere

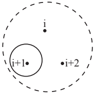
As before, squaring a generator results in unknots which can be removed.

$\alpha_i^2 =$ 

$= v_{i+1}^{-1} v_{i+2}^{-1} \Bigg(A$ 

$+$ 

$+$ 

$+ A^{-1}$ 

$\Bigg)$

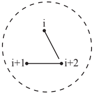
$$= v_{i+1}^{-1} v_{i+2}^{-1} (A(A + A^{-1}) + (-A^2 - A^{-2}) + (A + A^{-1}) + A^{-1}(A + A^{-1}))$$

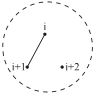
$$= v_{i+1}^{-1} v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

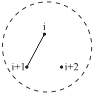
$$= v_{i+1}^{-1} v_{i+2}^{-1} \delta^2.$$

Algebra of the Thrice-Punctured Sphere

Also,

$\alpha_i \alpha_{i+1} =$ 

$= v_{i+2}^{-1} \Bigg(A^{\frac{1}{2}}$ 

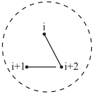
$+ A^{-\frac{1}{2}}$ 

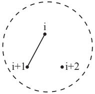
$\Bigg)$

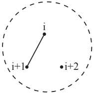
$$= v_{i+2}^{-1}(A^{\frac{1}{2}} + A^{-\frac{1}{2}})\alpha_{i+2}$$

$$= v_{i+2}^{-1}\delta\alpha_{i+2}$$

and similarly

$\alpha_{i+1} \alpha_i =$ 

$= v_{i+2}^{-1} \Bigg(A^{\frac{1}{2}}$ 

$+ A^{-\frac{1}{2}}$ 

$\Bigg)$

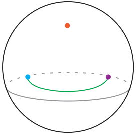
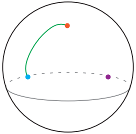
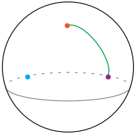
$$= v_{i+2}^{-1}(A^{\frac{1}{2}} + A^{-\frac{1}{2}})\alpha_{i+2}$$

$$= v_{i+2}^{-1}\delta\alpha_{i+2}.$$

Presentation of the Thrice-Punctured Sphere

Theorem

The arc algebra for the thrice-punctured sphere is generated by three simple arcs

$\alpha_1 =$  , $\alpha_2 =$  , $\alpha_3 =$ 

and has relations

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} \left(A^{\frac{1}{2}} + A^{-\frac{1}{2}}\right) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

where subscripts are interpreted modulo 3.

Algebra of the Thrice-Punctured Sphere in Matrices

$$\rho(\alpha_1) = \begin{bmatrix} 0 & v_2^{-1} v_3^{-1} \delta^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2^{-1}\delta \\ 0 & 0 & v_3^{-1}\delta & 0 \end{bmatrix},$$

$$\rho(\alpha_2) = \begin{bmatrix} 0 & 0 & v_1^{-1} v_3^{-1} \delta^2 & 0 \\ 0 & 0 & 0 & v_1^{-1}\delta \\ 1 & 0 & 0 & 0 \\ 0 & v_3^{-1}\delta & 0 & 0 \end{bmatrix},$$

$$\rho(\alpha_3) = \begin{bmatrix} 0 & 0 & 0 & v_1^{-1} v_2^{-1} \delta^2 \\ 0 & 0 & v_1^{-1}\delta & 0 \\ 0 & v_2^{-1}\delta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The Arc Algebra is Finitely Generated for all $F_{g,n}$

Theorem

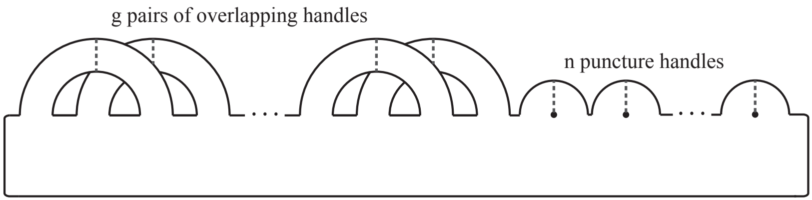
When $n = 0$ or $n = 1$, the arc algebra $\mathcal{A}(F_{g,n})$ can be generated by $2^{2g} - 1$ knots. For $n > 1$, it can be generated by a set of $(2^{2g} - 1)(n)$ knots and $2^{2g}\binom{n}{2}$ arcs.

Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams with minimal complexity. These generators can be counted combinatorially. $\square$

Proof Sketch - Setting Up

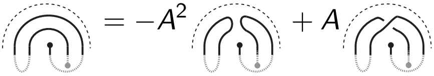
Remove a small disk from $F_{g,n}$ to make $F^*_{g,n}$.

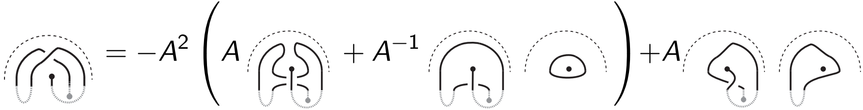


Generators of $\mathcal{A}(F^*_{g,n})$ will generate $\mathcal{A}(F_{g,n})$.

Proof Sketch - Complexity 1 of 3

Reduce to curves passing through each handle (genus or puncture) at most once. There are several cases. Example:

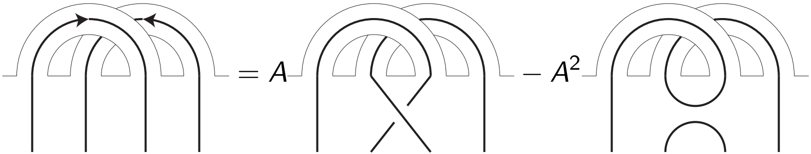




Each of the diagrams with crossings can be further simplified by a straightforward application of the skein relation.

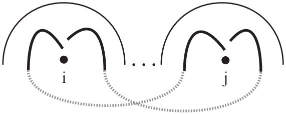
Proof Sketch - Complexity 2 of 3

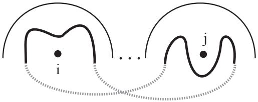
Reduce to curves passing through pairs of genus handles in only the "good" way. There are two cases. Example:

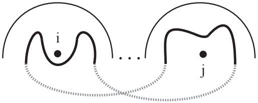


Proof Sketch - Complexity 3 of 3

Reduce to curves that pass through puncture handles a total of 0 (if arc) or 1 (if knot) times. There are many cases. Example:

$= v_i v_j A^{-1}$


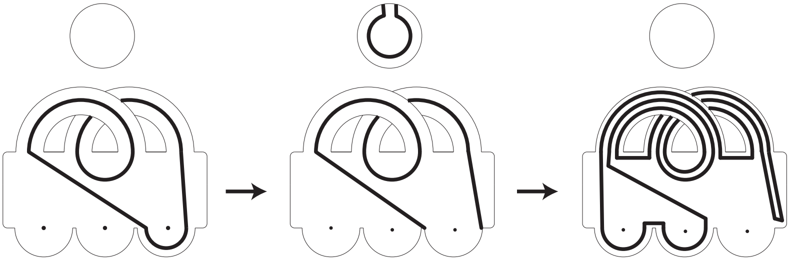
$- A^{-1}$


$- A^{-1}$


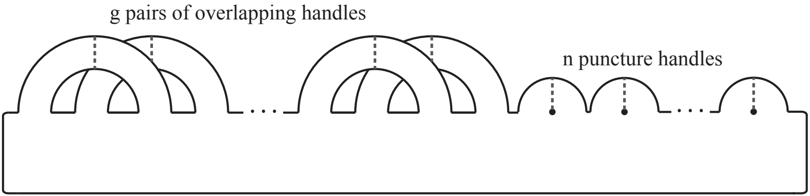
$- A^{-2}$


Proof Sketch - Finishing

In addition, we can force the curves to avoid one of the puncture handles using the fact that we are generating $\mathcal{A}(F_{g,n})$ and not $\mathcal{A}(F^*_{g,n})$.



Proof Sketch - Counting Generators



A generator can

- pass through each genus handle at most once
- pass through a pair of genus handles in only the good way
- pass through puncture handles a total of 1 time if knot
- pass through puncture handles a total of 0 times if arc
- start and end at any distinct pair of punctures if arc

these choices uniquely determine the generator.

The Arc Algebra is Finitely Generated for all $F_{g,n}$

Theorem

When $n = 0$ or $n = 1$, the arc algebra $\mathcal{A}(F_{g,n})$ can be generated by $2^{2g} - 1$ knots. For $n > 1$, it can be generated by a set of $(2^{2g} - 1)(n)$ knots and $2^{2g}\binom{n}{2}$ arcs.

Acknowledgements

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[1] Doug Bullock. A finite set of generators for the Kauffman bracket skein algebra. *Math. Z.*, 231(1):91-101, 1999.

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[3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space. *J. Differential Geom.*, 96(1):95-140, 2014.