

A Finite Set of Generators for the Arc Algebra

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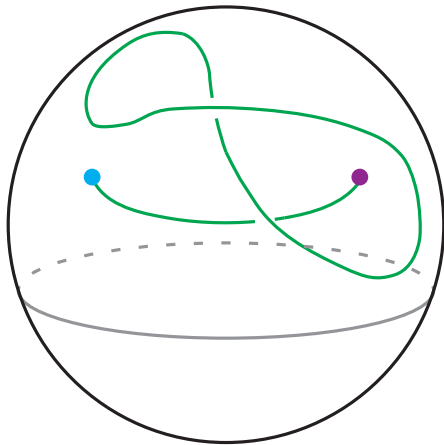
Cornell University

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The Arc Algebra of a Surface

- ▶ defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- ▶ generalization of the Jones polynomial (specifically the Kauffman bracket skein algebra)
- ▶ applies to thickened surfaces with punctures

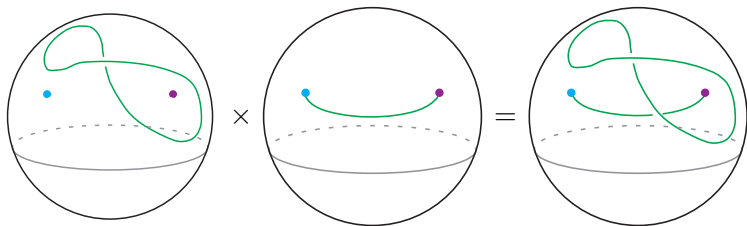
The Arc Algebra - An Element



The Arc Algebra - Definition

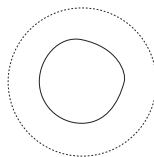
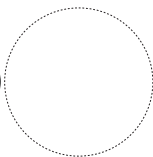
- ▶ Let $F_{g,n}$ denote the surface of genus g with n punctures.
- ▶ Let R_n be the ring $\mathbb{Z}[A^{\pm\frac{1}{2}}][v_1, \dots, v_n]$.
- ▶ The arc algebra $\mathcal{A}(F_{g,n})$ consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface $F_{g,n} \times [0, 1]$ subject to four relations.
- ▶ Multiplication is by stacking, induced by $F_{g,n} \times [0, 1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1]$.

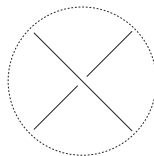
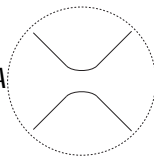
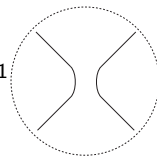
The Arc Algebra - Multiplication



The Arc Algebra - Kauffman Bracket Relations

Two of the four relations are the same as the skein algebra:


$$= (-A^2 - A^{-2})$$


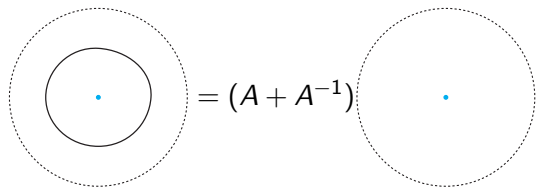

$$= A$$

$$+ A^{-1}$$


The Arc Algebra - Puncture Relations

There are two more relations for punctured surfaces:

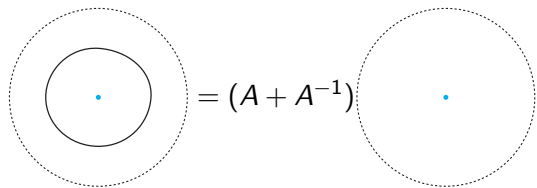
The Arc Algebra - Puncture Relations

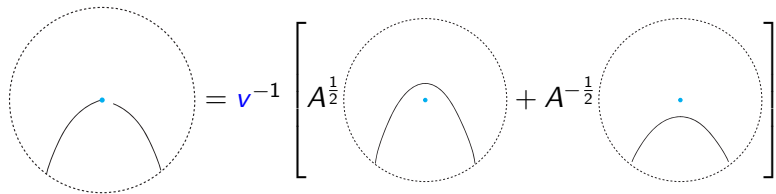
There are two more relations for punctured surfaces:



The Arc Algebra - Puncture Relations

There are two more relations for punctured surfaces:


$$\text{Punctured disk with inner circle} = (A + A^{-1}) \text{Punctured disk}$$


$$\text{Punctured disk with arcs} = v^{-1} \left[A^{\frac{1}{2}} \text{Punctured disk with arc} + A^{-\frac{1}{2}} \text{Punctured disk with arc} \right]$$

The Arc Algebra - All Relations

$$\text{Diagram 1} = (-A^2 - A^{-2}) \text{Diagram 2}$$

$$\text{Diagram 3} = A \text{Diagram 4} + A^{-1} \text{Diagram 5}$$

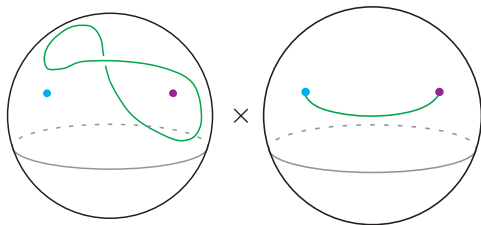
$$\text{Diagram 6} = (A + A^{-1}) \text{Diagram 7}$$

$$\text{Diagram 8} = v^{-1} \left[A^{\frac{1}{2}} \text{Diagram 9} + A^{-\frac{1}{2}} \text{Diagram 10} \right]$$

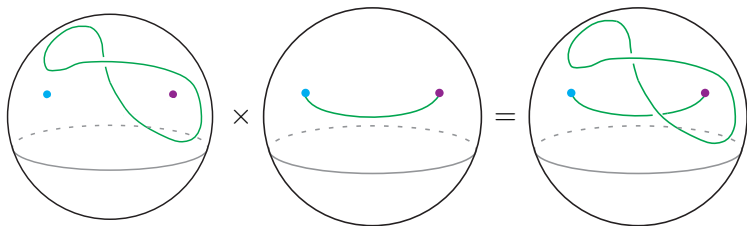
Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

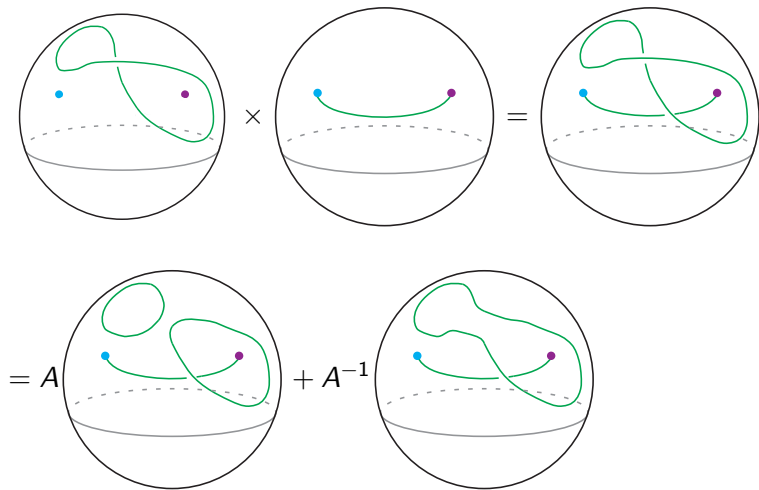
Example on the Twice-Punctured Sphere



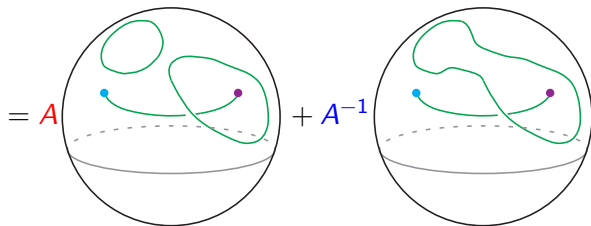
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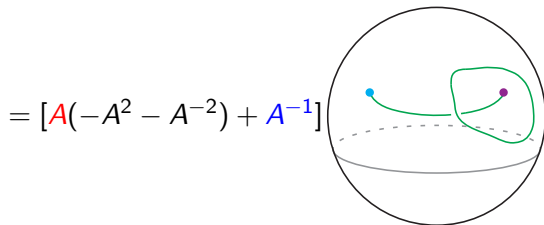
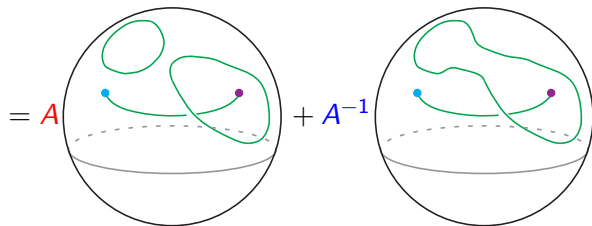
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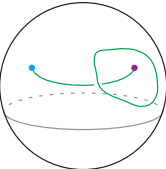
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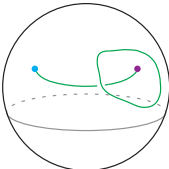
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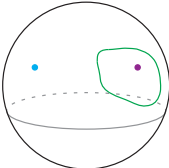
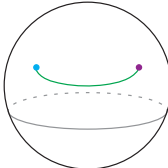


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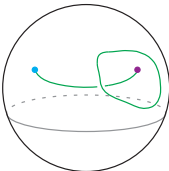
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$


Example on the Twice-Punctured Sphere

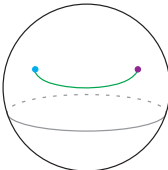
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$$\times$$


Example on the Twice-Punctured Sphere

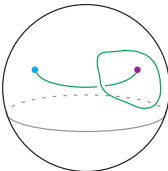
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$


A diagram of a sphere with a horizontal line representing the equator. A dashed line indicates the back half of the sphere. A green loop is drawn on the sphere, starting from a blue dot on the left, going up and around to a purple dot on the right, and then returning to the blue dot. The loop is oriented such that it encircles the sphere.

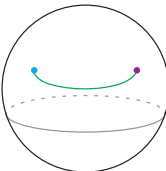
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


A diagram of a sphere with a horizontal line representing the equator. A dashed line indicates the back half of the sphere. A green arc is drawn on the sphere, starting from a blue dot on the left and ending at a purple dot on the right, representing a path between the two punctures.

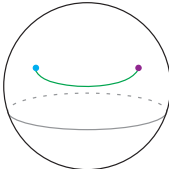
Example on the Twice-Punctured Sphere

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$


A diagram of a sphere with a horizontal equator line. A dashed line represents the back half of the equator. Two points, one blue and one purple, are located on the sphere's surface. A green loop starts at the blue point, goes up and around the sphere, and ends at the purple point.

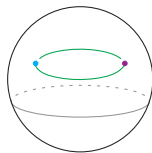
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


A diagram of a sphere with a horizontal equator line. A dashed line represents the back half of the equator. Two points, one blue and one purple, are located on the sphere's surface. A green arc connects the blue point to the purple point, passing above the sphere.

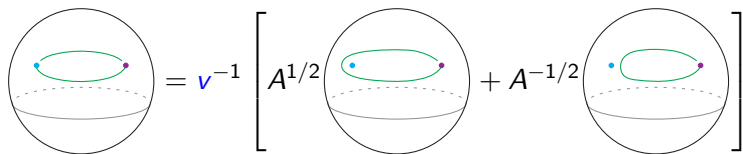
$$= (-A^4 - A^2)$$


A diagram of a sphere with a horizontal equator line. A dashed line represents the back half of the equator. Two points, one blue and one purple, are located on the sphere's surface. A green arc connects the blue point to the purple point, passing above the sphere.

Another Example on the Twice-Punctured Sphere



Another Example on the Twice-Punctured Sphere



The diagram illustrates an equation on a twice-punctured sphere. On the left, a sphere with a dashed equator and a solid lower hemisphere contains a green horizontal ellipse with a blue dot on the left and a purple dot on the right. An arrow on the ellipse points from the purple dot to the blue dot. This is equal to v^{-1} times a bracketed sum of two identical spheres. Each sphere in the sum has the same green ellipse and dots, but the arrow on the ellipse points from the blue dot to the purple dot. The first sphere in the sum is multiplied by $A^{1/2}$ and the second by $A^{-1/2}$.

$$\text{Sphere with arrow from purple to blue} = v^{-1} \left[A^{1/2} \text{Sphere with arrow from blue to purple} + A^{-1/2} \text{Sphere with arrow from blue to purple} \right]$$

Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \left(\text{Diagram 1} \right) = v^{-1} \left[A^{1/2} \left(\text{Diagram 2} \right) + A^{-1/2} \left(\text{Diagram 3} \right) \right] = \\
 & (vv)^{-1} \left[A \left(\text{Diagram 4} \right) + \left(\text{Diagram 5} \right) + \left(\text{Diagram 6} \right) + A^{-1} \left(\text{Diagram 7} \right) \right]
 \end{aligned}$$

The diagrams represent spheres with a dashed line for the equator. Each sphere contains two points: a blue dot on the left and a purple dot on the right.

- Diagram 1:** A large green ellipse encircling the equator, with arrows indicating a counter-clockwise orientation.
- Diagram 2:** A large green ellipse encircling the equator, with arrows indicating a counter-clockwise orientation.
- Diagram 3:** A large green ellipse encircling the equator, with arrows indicating a counter-clockwise orientation.
- Diagram 4:** A small green circle encircling the blue dot.
- Diagram 5:** A large green ellipse encircling the equator, with arrows indicating a counter-clockwise orientation.
- Diagram 6:** A small green circle encircling the purple dot.
- Diagram 7:** A small green circle encircling the purple dot.

Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \left(\text{Diagram 1} \right) = v^{-1} \left[A^{1/2} \left(\text{Diagram 2} \right) + A^{-1/2} \left(\text{Diagram 3} \right) \right] = \\
 & (vv)^{-1} \left[A \left(\text{Diagram 4} \right) + \left(\text{Diagram 5} \right) + \left(\text{Diagram 6} \right) + A^{-1} \left(\text{Diagram 7} \right) \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})]
 \end{aligned}$$

The diagrams represent elements of the skein algebra of a twice-punctured sphere. Each diagram is a sphere with a horizontal equator (solid in front, dashed in back). Two points, a blue dot and a purple dot, are located on the sphere. Green loops are drawn around these points. In the first row, the loops are large and enclose both points. In the second row, the loops are smaller and enclose only one point at a time. The diagrams are arranged in a sequence that represents a linear combination of terms in the skein algebra.

Another Example on the Twice-Punctured Sphere

$$\begin{aligned}
 & \text{Diagram 1} = v^{-1} \left[A^{1/2} \text{Diagram 2} + A^{-1/2} \text{Diagram 3} \right] = \\
 & (vv)^{-1} \left[A \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + A^{-1} \text{Diagram 7} \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \\
 & = (vv)^{-1} (-A^2 + 2 - A^{-2})
 \end{aligned}$$

The diagrams represent elements of the skein algebra of a twice-punctured sphere. Each diagram is a sphere with a horizontal equator (solid in front, dashed in back). Two points, a blue dot and a purple dot, are located on the sphere. Green loops are drawn around these points. In the first diagram, a single large loop encircles both points. In the subsequent diagrams, the loops are more complex, involving multiple components and crossings, representing different algebraic terms in the expansion.

Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.

Algebra of the Twice-Punctured Sphere

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2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.

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3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.

Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.

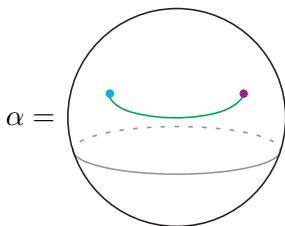
Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

Presentation of the Twice-Punctured Sphere

Theorem

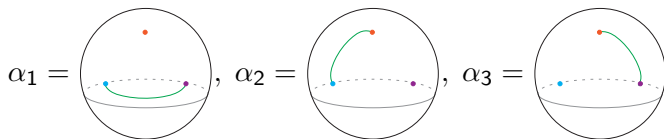
The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures, α , with the relation $\alpha^2 = -\frac{1}{v_1 v_2} (A - A^{-1})^2$.



Presentation of the Thrice-Punctured Sphere

Theorem

The arc algebra for the thrice-punctured sphere is generated by three simple arcs



and has relations

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

where subscripts are interpreted modulo 3.

The Arc Algebra is Finitely Generated for all $F_{g,n}$

Theorem

When $n = 0$ or $n = 1$, the arc algebra $\mathcal{A}(F_{g,n})$ can be generated by 2^{2g} knots. For $n > 1$, it can be generated by a set of $(2^{2g} - 1)(n)$ knots and $2^{2g} \binom{n}{2}$ arcs.

Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams. These generators can be counted combinatorially. □

Acknowledgements

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References

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- [3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space, arXiv:1110.2748v2, 2012.

