A Finite Set of Generators for the Arc Algebra

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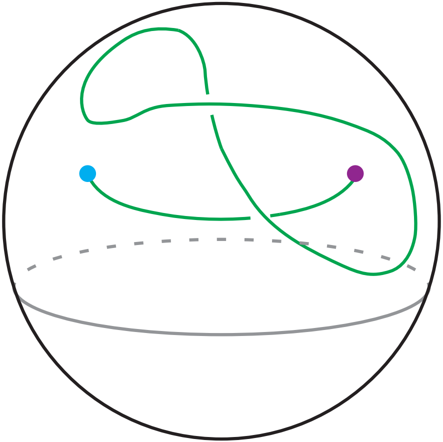
The Arc Algebra of a Surface

- defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- generalization of the Jones polynomial (specifically the Kauffman bracket skein algebra)
- applies to thickened surfaces with punctures

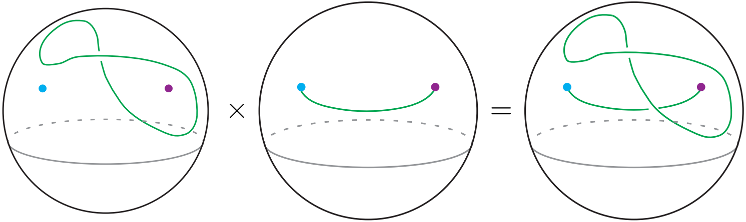
The Arc Algebra - An Element



The Arc Algebra - Definition

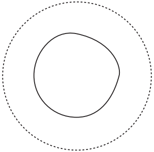
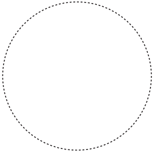
- Let $F_{g,n}$ denote the surface of genus g with n punctures.
- Let R_n be the ring $\mathbb{Z}[A^{\pm\frac{1}{2}}][v_1, \ldots, v_n]$.
- The arc algebra $\mathcal{A}(F_{g,n})$ consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface $F_{g,n} \times [0,1]$ subject to four relations.
- Multiplication is by stacking, induced by $F_{g,n} \times [0,1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1]$.

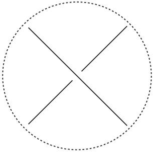
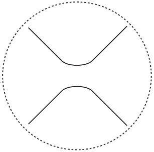
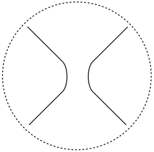
The Arc Algebra - Multiplication



The Arc Algebra - Kauffman Bracket Relations

Two of the four relations are the same as the skein algebra:

 $= (-A^2 - A^{-2})$ 

 $= A$  $+ A^{-1}$ 

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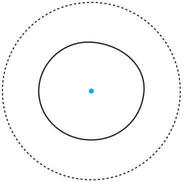
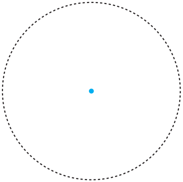


$= (A + A^{-1})$



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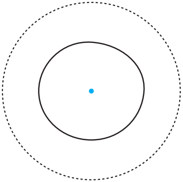

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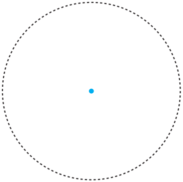
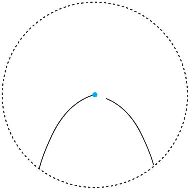
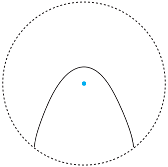




$= v^{-1} \Big[A^{\frac{1}{2}}$



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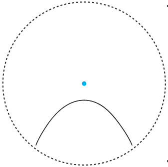

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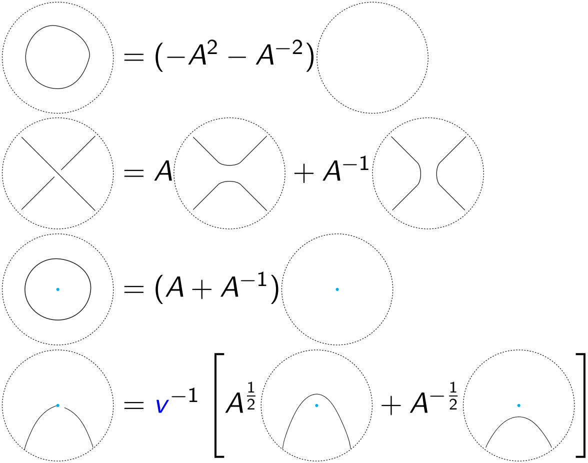
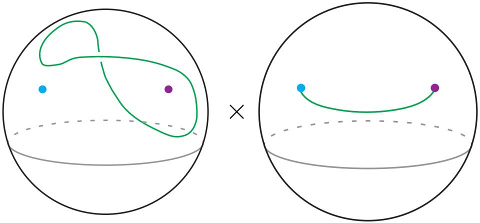
The Arc Algebra - All Relations



Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

Example on the Twice-Punctured Sphere



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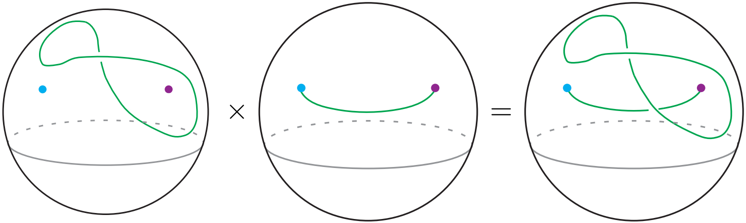

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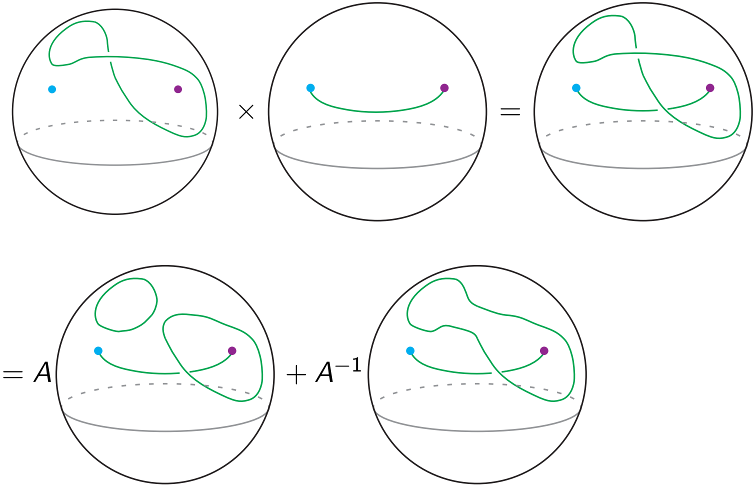

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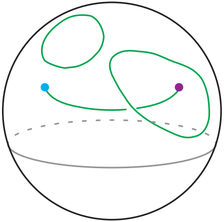
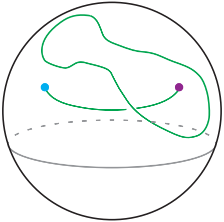
$= A$  $+ A^{-1}$ 

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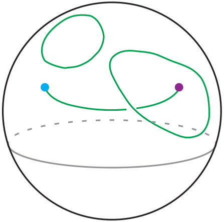
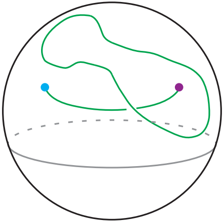
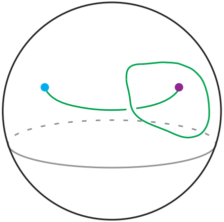
$= A$  $+ A^{-1}$ 

$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

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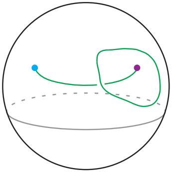

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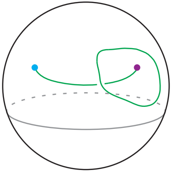
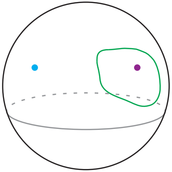
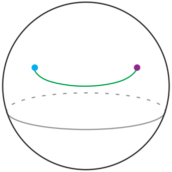
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$ 

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$  $\times$ 

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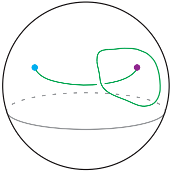
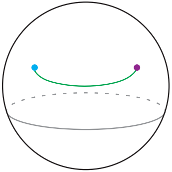
$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$ 

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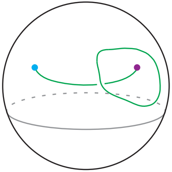
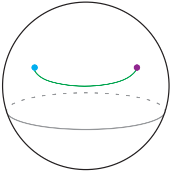
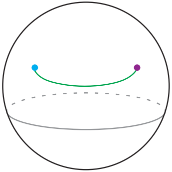
$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$ 

$= (-A^4 - A^2)$ 

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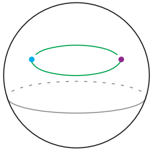

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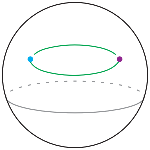
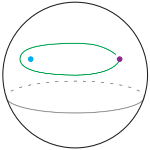
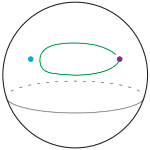
 $= v^{-1} \Big[A^{1/2}$  $+ A^{-1/2}$  $\Big]$

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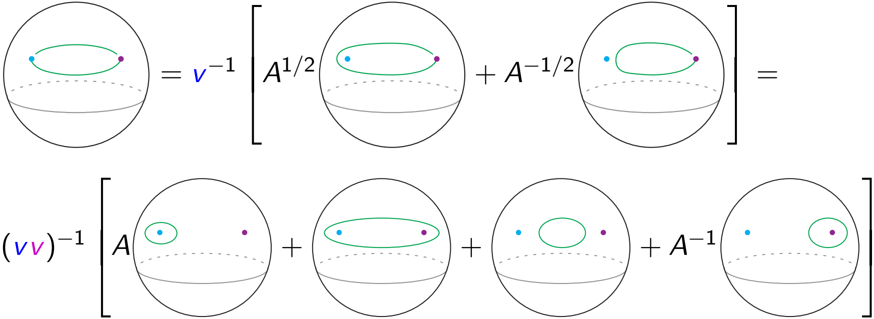

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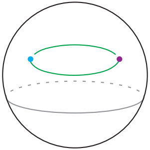
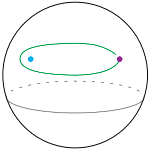
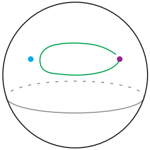
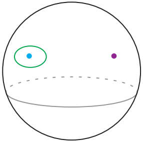
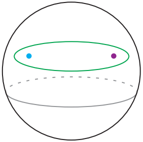
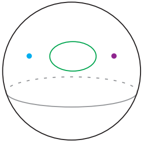
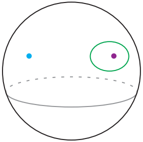
 $= v^{-1} \Big[A^{1/2}$  $+ A^{-1/2}$  $\Big] =$

$(vv)^{-1} \Big[A$  $+$  $+$  $+ A^{-1}$  $\Big]$

$$= (vv)^{-1}[A(A+A^{-1})+(-A^2-A^{-2})+(-A^2-A^{-2})+A^{-1}(A+A^{-1})]$$

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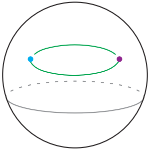

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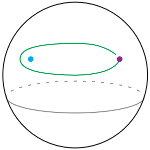

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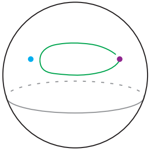

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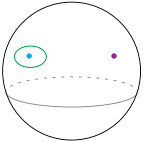

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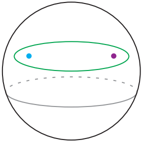

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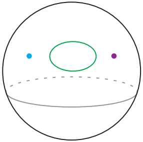

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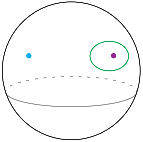

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$$= (vv)^{-1}[A(A+A^{-1})+(-A^2-A^{-2})+(-A^2-A^{-2})+A^{-1}(A+A^{-1})]$$

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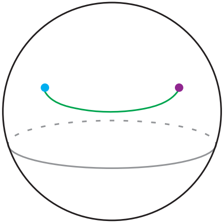
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

Presentation of the Twice-Punctured Sphere

Theorem

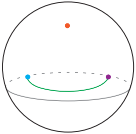
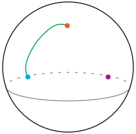
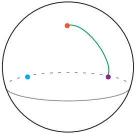
The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures, α, with the relation $\alpha^2 = -\frac{1}{v_1 v_2}(A - A^{-1})^2$.

$\alpha =$ 

Presentation of the Thrice-Punctured Sphere

Theorem

The arc algebra for the thrice-punctured sphere is generated by three simple arcs

$\alpha_1 =$ , $\alpha_2 =$ , $\alpha_3 =$ 

and has relations

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} \left(A^{\frac{1}{2}} + A^{-\frac{1}{2}}\right) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

where subscripts are interpreted modulo 3.

The Arc Algebra is Finitely Generated for all $F_{g,n}$

Theorem

When $n = 0$ or $n = 1$, the arc algebra $\mathcal{A}(F_{g,n})$ can be generated by 2^{2g} knots. For $n > 1$, it can be generated by a set of $(2^{2g} - 1)(n)$ knots and $2^{2g}\binom{n}{2}$ arcs.

Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams. These generators can be counted combinatorially. □

Acknowledgements

We would like to thank our advisors Helen Wong and Stephen Kennedy for their guidance; Francis Bonahon, Eric Egge, and Tommy Occhipinti for helpful discussions; and the Carleton College mathematics department for their support and encouragement throughout this research. This project was partially supported by NSF Grant DMS-1105692 and this talk partially supported by Cornell University and the University of Texas at Austin.

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[2] Doug Bullock and Józef H. Przytycki. Multiplicative structure of the Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.*, 128(3):923-931, 2000.

[3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space, arXiv:1110.2748v2, 2012.