1 Motivation

Throughout, $k$ is an arbitrary field of characteristic zero. We will work over $k$, so all maps are tacitly assumed to be $k$-linear and all tensor product will be over $k$. Consider the following categories.

1.1 Representations of an algebraic group

For $G/k$ an algebraic group, the category $\text{Rep}(G)$ has as objects pairs $(V, \rho)$, where $V$ is a finite-dimensional $k$-vector space and $\rho : G \to \text{GL}(V)$ is a homomorphism of $k$-groups. A morphism $(V_1, \rho_1) \to (V_2, \rho_2)$ in $\text{Rep}(G)$ is a $k$-linear map $f : V_1 \to V_2$ such that for all $k$-algebras $A$ and $g \in G(A)$, one has $f\rho_1(g) = \rho_2(g)f$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
V_1 \otimes A & \xrightarrow{f} & V_2 \otimes A \\
\downarrow{\rho_1(g)} & & \downarrow{\rho_2(g)} \\
V_1 \otimes A & \xrightarrow{f} & V_2 \otimes A.
\end{array}
\]

1.2 Representations of a Hopf algebra

Let $H$ be a co-commutative Hopf algebra. The category $\text{Rep}(H)$ has as objects $H$-modules that are finite-dimensional over $k$, and morphisms are $k$-linear maps. The algebra $H$ acts on a tensor product $U \otimes V$ via its comultiplication $\Delta : H \to H \otimes H$.

1.3 Representations of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over $k$. The category $\text{Rep}(\mathfrak{g})$ has as objects $\mathfrak{g}$-representations that are finite-dimensional as a $k$-vector space. There is a canonical isomorphism $\text{Rep}(\mathfrak{g}) = \text{Rep}(\mathcal{U}\mathfrak{g})$, where $\mathcal{U}\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

1.4 Continuous representations of a compact Lie group

Let $K$ be a compact Lie group. The category $\text{Rep}_C(K)$ has as objects pairs $(V, \rho)$, where $V$ is a finite-dimensional complex vector space and $\rho : K \to \text{GL}(V)$ is a continuous (hence smooth, by Cartan’s theorem) homomorphism. Morphisms $(V_1, \rho_1) \to (V_2, \rho_2)$ are $K$-equivariant $\mathbb{C}$-linear maps $V_1 \to V_2$.

1.5 Graded vector spaces

Consider the category whose objects are finite-dimensional $k$-vector spaces $V$ together with a direct sum decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$. Morphisms $U \to V$ are $k$-linear maps $f : U \to V$ such that $f(U_n) \subset V_n$. 

Tannakian categories

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1.6 Hodge structures

Let \( V \) be a finite-dimensional \( \mathbb{R} \)-vector space. A Hodge structure on \( V \) is a direct sum decomposition \( V = \bigoplus V_{p,q} \) such that \( V_{p,q} = V_{q,p} \). If \( U, V \) are vector spaces with Hodge structures, a morphism \( U \to V \) is a \( \mathbb{R} \)-linear map \( f : U \to V \) such that \( f(U_{p,q}) \subset V_{p,q} \). Write \( \text{Hdg} \) for the category of vector spaces with Hodge structure.

Let \( \text{Vec}(k) \) be the category of finite-dimensional \( k \)-vector spaces. For any of the categories above, there is a faithful functor \( \omega : C \to \text{Vec}(k) \). In our examples, it is just the forgetful functor. The main theorem will be that for \( \pi = \text{Aut}(\omega) \), the functor \( \omega \) induces an equivalence of categories \( C \cong \text{Rep}(\pi) \). We proceed to make sense of the undefined terms in this theorem.

2 Main definitions

Our definitions follow [DM82]. As before, \( k \) is an arbitrary field of characteristic zero.

2.1 Tannakian category

A \( k \)-linear category is an abelian category \( C \) such that each \( V_1, V_2 \), the group \( \text{hom}(V_1, V_2) \) has the structure of a \( k \)-vector space in such a way that the composition map \( \text{hom}(V_2, V_3) \otimes \text{hom}(V_1, V_2) \to \text{hom}(V_1, V_3) \) is \( k \)-linear. For us, a rigid \( k \)-linear tensor category is a \( k \)-linear category \( C \) together with the following data:

1. An exact faithful functor \( \omega : C \to \text{Vec}(k) \).
2. A bi-additive functor \( \otimes : C \times C \to C \).
3. Natural isomorphisms \( \omega(V_1 \otimes V_2) \cong \omega(V_1) \otimes \omega(V_2) \).
4. Isomorphisms \( V_1 \otimes V_2 \cong V_2 \otimes V_1 \) for all \( V_i \in C \).
5. Isomorphisms \( (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \)

These data are required to satisfy the following conditions:

1. There exists an object \( 1 \in C \) such that \( \omega(1) \) is one-dimensional and such that the natural map \( k \to \text{hom}(1, 1) \) is an isomorphism.
2. If \( \omega(V) \) is one-dimensional, there exists \( V^{-1} \in C \) such that \( V \otimes V^{-1} \cong 1 \).
3. Under \( \omega \), the isomorphisms 3 and 4 are the obvious ones.

By [DM82, Pr. 1.20], this is equivalent to the standard (more abstract) definition. Note that all our examples in section 1 are rigid \( k \)-linear tensor categories. One calls the functor \( \omega \) a fiber functor.

2.2 Automorphisms of a functor

Let \( (C, \otimes) \) be a rigid \( k \)-linear tensor category. In this setting, define a functor \( \text{Aut}(\omega) \) from \( k \)-algebras to groups by setting:

\[
\text{Aut}^\otimes(\omega)(A) = \text{Aut}^\otimes(\omega : C \otimes A \to \text{Rep}(A)) = \left\{ (g_V) \in \prod_{V \in C} \text{GL}(\omega(V) \otimes A) : g_1 = 1, g_{V_1} \otimes g_{V_2} = g_{V_1 \otimes V_2}, \text{ and } fg_{V_1} = g_{V_1}f \text{ for all } f, V_1, V_2 \right\}.
\]

In other words, an element of \( \text{Aut}(\omega)(A) \) consists of a collection \( (g_V) \) of \( A \)-linear automorphisms \( g_V : \omega(V) \otimes A \to \omega(V) \otimes A \), where \( V \) ranges over objects in \( C \). This collection must satisfy:
1. $g_1 = 1_{\omega(1)}$

2. $g_{V_1 \otimes V_2} = g_{V_1} \otimes g_{V_2}$ for all $V_1, V_2 \in \mathcal{C}$, and

3. whenever $f : V_1 \to V_2$ is a morphism in $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
\omega(V_1)_A & \xrightarrow{f} & \omega(V_2)_A \\
\downarrow^{g_{V_1}} & & \downarrow^{g_{V_2}} \\
\omega(V_1)_A & \xrightarrow{f} & \omega(V_2)_A.
\end{array}
$$

### 2.3 Pro-algebraic group

Typically one only considers affine group schemes $G_{/k}$ that are algebraic, i.e. whose coordinate ring $\mathcal{O}(G)$ is a finitely generated $k$-algebra, or equivalently that admit a finite-dimensional faithful representation. Let $G_{/k}$ be an arbitrary affine group scheme, $V$ an arbitrary representation of $G$ over $k$. By [DM82, Cor. 2.4], one has $V = \varprojlim V_i$, where $V_i$ ranges over the finite-dimensional subrepresentations of $V$. Applying this to the regular representation $G \to \text{GL}(\mathcal{O}(G))$, we see that $\mathcal{O}(G) = \varprojlim \mathcal{O}(G_i)$, where $G_i$ ranges over the algebraic quotients of $G$. That is, an arbitrary affine group scheme $G_{/k}$ can be written as a filtered projective limit $G = \varprojlim G_i$, where each $G_i$ is an affine algebraic group over $k$. So we will speak of pro-algebraic groups instead of arbitrary affine group schemes.

If $V$ is a finite-dimensional $k$-vector space and $G = \varprojlim G_i$ is a pro-algebraic $k$-group, representations $G \to \text{GL}(V)$ factor through some algebraic quotient $G_i$. That is, $\text{hom}(G, \text{GL}(V)) = \varinjlim \text{hom}(G_i, \text{GL}(V))$. As a basic example of this, let $\Gamma$ be a profinite group, i.e. a projective limit of finite groups. If we think of $\Gamma$ as a pro-algebraic group, then algebraic representations $\Gamma \to \text{GL}(V)$ are exactly those representations that are continuous when $V$ is given the discrete topology.

### 3 Reconstruction theorem

First, suppose $\mathcal{C} = \text{Rep}(G)$ for a pro-algebraic group $G$, and that $\omega : \text{Rep}(G) \to \text{Vec}(k)$ is the forgetful functor. Then the Tannakian fundamental group $\text{Aut}^\otimes (\omega)$ carries no new information [DM82, Pr. 2.8]:

**Theorem 3.1.** Let $G_{/k}$ be a pro-algebraic group, $\omega : \text{Rep}(G) \to \text{Vec}(k)$ the forgetful functor. Then $G \overset{\sim}{\to} \text{Aut}^\otimes (G)$.

The main theorem is the following, taken essentially verbatim from [DM82, Th. 2.11].

**Theorem 3.2.** Let $(\mathcal{C}, \otimes, \omega)$ be a rigid $k$-linear tensor category. Then $\pi = \text{Aut}^\otimes (\omega)$ is represented by a pro-algebraic group, and $\omega : \mathcal{C} \to \text{Rep}(\pi)$ is an equivalence of categories.

Often, the group $\pi_1(\mathcal{C})$ is “too large” to handle directly. For example, if $\mathcal{C}$ contains infinitely many simple objects, probably $\pi_1(\mathcal{C})$ will be infinite-dimensional. For $V \in \mathcal{C}$, let $\mathcal{C}(V)$ be the Tannakian subcategory of $\mathcal{C}$ generated by $V$. One puts $\pi_1(\mathcal{C}/V) = \pi_1(\mathcal{C}(V))$. It turns out that $\pi_1(\mathcal{C}/V) \subset \text{GL}(\omega V)$, so $\pi_1(\mathcal{C}/V)$ is finite-dimensional. One has $\pi_1(\mathcal{C}) = \varprojlim \pi_1(\mathcal{C}/V)$.

### 4 Examples

#### 4.1 Pro-algebraic groups

If $G_{/k}$ is a pro-algebraic group, then Theorem 3.1 tells us that if $\omega : \text{Rep}(G) \to \text{Vec}(k)$ is the forgetful functor, then $G = \text{Aut}^\otimes (G)$. That is, $G = \pi_1(\text{Rep}(G)$.
4.2 Hopf algebras

Suppose \( H \) is a co-commutative Hopf algebra over \( k \). Then \( \pi_1(\text{Rep} H) = \text{Spec}(H^\circ) \), where \( H^\circ \) is the reduced dual defined in [Car07]. Namely, for any \( k \)-algebra \( A \), \( A^\circ \) is the set of \( k \)-linear maps \( \lambda : A \to k \) such that \( \lambda(a) = 0 \) for some two-sided ideal \( a \subseteq A \) of finite codimension. The key fact here is that \((A \otimes B)^\circ = A^\circ \otimes B^\circ \), so that we can use multiplication \( m : H \otimes H \to H \) to define comultiplication \( m^\ast : H^\circ \to (H \otimes H)^\circ = H^\circ \otimes H^\circ \). From [DG80, II \S 6 1.1], if \( G \) is a linear algebraic group over an algebraically closed field \( k \) of characteristic zero, we get an isomorphism \( \mathcal{O}(G)^\circ = k[G(k)] \otimes U(\mathfrak{g}) \). Here \( k[G(k)] \) is the usual group algebra of the abstract group \( G(k) \), and \( U(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} = \text{Lie}(G) \), both with their standard Hopf structures.

[Note: one often calls \( \mathcal{O}(G)^\circ \) the “space of distributions on \( G \).” If instead \( G \) is a real Lie group, then one often writes \( \mathcal{H}(G) \) for the space of distributions on \( G \). Let \( K \subseteq G \) be a maximal compact subgroup, \( M(K) \) the space of finite measures on \( K \). Then convolution \( D \otimes \mu \mapsto D \ast \mu \) induces an isomorphism \( U(\mathfrak{g}) \otimes M(K) \xrightarrow{\sim} \mathcal{H}(G) \). In the algebraic setting, \( k[G(k)] \) is the appropriate replacement for \( M(K) \).]

4.3 Lie algebras

Let \( \mathfrak{g} \) be a semisimple Lie algebra over \( k \). Then by [Mil07], \( G = \pi_1(\text{Rep} \mathfrak{g}) \) is the unique connected, simply connected algebraic group with \( \text{Lie}(G) = \mathfrak{g} \). If \( \mathfrak{g} \) is not semisimple, e.g. \( \mathfrak{g} = k \), then things get a lot nastier. See the above example.

4.4 Compact Lie groups

By definition, the complexification of a real Lie group \( K \) is a complex Lie group \( K_C \) such that all morphisms \( K \to \text{GL}(V) \) factor uniquely through \( K_C \to \text{GL}(V) \). It turns out that \( K_C \) is a complex algebraic group, and so \( \pi_1(\text{Rep} K) = K_C \).

4.5 Graded vector spaces

To give a grading \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) on a vector space is equivalent to giving an action of the split rank-one torus \( \mathbb{G}_m \). On each \( V_n \), \( \mathbb{G}_m \) acts via the character \( g \mapsto g^n \). Thus \( \pi_1(\text{graded vector spaces}) = \mathbb{G}_m \).

4.6 Hodge structures

Let \( S = R_{C/R} \mathbb{G}_m ; \) this is defined by \( S(A) = (A \otimes C)^\times \) for \( R \)-algebras \( A \). One can check that the category \( \text{Hdg} \) of Hodge structures is equivalent to \( \text{Rep}_R(S) \). Thus \( \pi_1(\text{Hdg}) = S \).

References


