Perfectoid rings, almost mathematics, and the cotangent complex

Daniel Miller

January 17, 2015

Scholze introduced perfectoid fields (and more generally, perfectoid spaces) in his paper [Sch12], in which he proved a wide range of special cases of Deligne’s Weight Mondromy Conjecture for $p$-adic fields.

1 Perfectoid fields

Recall that a valued field is a field $k$ together with a homomorphism $|\cdot| : k^\times \to \Gamma$ for some totally ordered abelian group $\Gamma$ (whose operation we will write multiplicatively). One requires $|\cdot|$ to satisfy the ultrametric inequality:

$$|x + y| \leq \max\{|x|, |y|\}.$$

As is standard, write 1 for the unit in $\Gamma$, and introduce an extra element 0, stipulating that $0 < \gamma$ for all $\gamma \in \Gamma$. We define $|0| = 0$, thus extending $|\cdot|$ to a function $k \to \Gamma \cup \{0\}$. One calls the rank of $|\cdot|$ the dimension $\dim_Q (k^\times \otimes Q)$. We will say that the valuation $|\cdot|$ is non-discrete if $k^\times \not\cong \mathbb{Z}$.

Put $k^\circ = \{x \in k : |x| \leq 1\}$.

This is called the ring of integers of $k$. It is a valuation ring with (unique) maximal ideal

$$k^+ = \{x \in k : |x| < 1\}.$$

We call $k^\circ / k^+$ the residue field of $k$.

**Definition 1.1.** A perfectoid field is a complete valued field $k$ with respect to a non-discrete rank-one valuation, with residue characteristic $p > 0$, such that the Frobenius $Fr : k^\circ / p \to k^\circ / p$ is surjective.

A typical example of a perfectoid field is $\mathbb{Q}_p(\zeta_p^\infty)^\wedge$, the completion of $\mathbb{Q}_p(\zeta_p^\infty : n \geq 1)$ with respect to the $p$-adic topology. Similarly, $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $\mathbb{C}_p = (\overline{\mathbb{Q}_p})^\wedge$ are perfectoid. An example in characteristic $p$ is the $t$-adic completion of $\mathbb{F}_p(t^{1/p^\infty})$. For a perfectoid field $k$ of residue characteristic $p$, choose $\pi \in k^\times$ with $|p| \leq |\pi| < 1$. Note that the Frobenius map $a \mapsto a^p$ is defined on $A/\pi$ for any $k^\circ$-algebra $A$.

**Definition 1.2.** Let $k$ be a perfectoid field. A perfectoid $k$-algebra is a Banach $k$-algebra $A$ such that $A^\circ = \{x \in A : |x| \leq 1\}$ is open and for which $Fr : A^\circ / \pi \to A^\circ / \pi$ is surjective.

If $k$ is a perfectoid field, let $\text{Perf}(k)$ denote the category of perfectoid $k$-algebras, with continuous $k$-maps as morphisms. We will construct, for any perfectoid field $k$ with residue characteristic $p$, a perfectoid field $k^\circ$ of characteristic $p$. Start by defining

$$k^{\circ \circ} = \lim_{\text{Fr}} (k^\circ / \pi) = \left\{(x_i) \in \prod_{i \geq 0} k^\circ / \pi : x^p_{i+1} = x^p_i \right\}.$$
It is not too difficult to check directly that \( k^{\circ \circ} \) is a valuation ring, and we put \( k^b = \text{Frac}(k^{\circ \circ}) \). There is a canonical map \((-)^2 : k^{\circ \circ} \to k^b \) defined by

\[
(x_0, x_1, \ldots)^2 = \lim_{n \to \infty} x_n p^n,
\]

where \( x_n \) is an arbitrary lift of \( x_n \in k^b / \pi \) to \( k^b \). The map \((-)^2 \) is not additive unless \( k \) already has characteristic \( p \), in which case \( k = k^b \). In general, \((-)^2 \) extends to an isomorphism of multiplicative groups \( k^{\circ \circ} \to k^b \), and we can use this to define a valuation on \( k^b \) by \( |x|_{\cdot} = |x^2|_k \). See Lemma 3.4 of [Sch12] for a proof that \((-)^2 \) has the claimed properties, and that \( k^{\circ \circ} \) is a perfectoid field with the same value group as \( k \).

**Example 1.3.** Let \( N \) be a lattice (i.e. a finite free \( \mathbb{Z} \)-module) and let \( \sigma \subset N^\vee \) be a strongly convex polyhedral cone. Let \( \sigma^\wedge \subset N^\vee \) be its dual. If we put \( M = N^\vee \), then the spectra of algebras of the form \( k[\sigma] = k[\sigma^\vee \cap M] \) form affine charts for toric varieties over \( k \). There is a “perfectoid version” of this. Write \( k[\sigma] = k[\langle \sigma \cap M[1/p] \rangle] \) for the ring

\[
\left( k^b[\sigma^\vee \cap M \otimes \mathbb{Z}[\frac{1}{p}]]^\wedge \right) \otimes k.
\]

Then \( k[\sigma] \) is a perfectoid algebra over \( k \). (Note: \( k[\sigma] \) is not, in this context, a non-commutative polynomial algebra over \( k \).

If \( A \) is a perfectoid \( k \)-algebra, define \( A^b \) in much the same way, via

\[
A^b = \left( \lim_{\text{Fr}} A^\circ / \pi \right) \otimes_{k^b} k^b.
\]

It turns out that if \( A \) is perfectoid, then \( A^b \) is also perfectoid, and we have the following deep theorem:

**Theorem 1.4 (Scholze).** The functor \((-)^b : \text{Perf}(k) \to \text{Perf}(k^b) \) is an equivalence of categories.

In fact, much more can be shown, e.g. \((-)^b \) induces equivalences of categories between \( \text{F\acute{e}t}(A) \) and \( \text{F\acute{e}t}(A^b) \) for all \( A \), where \( \text{F\acute{e}t}(A) \) is the category of finite étale algebras over \( A \) (it turns out that such algebras are perfectoid).

**Example 1.5.** If \( A = k[\sigma] = k[\sigma^\vee \cap M[1/p]] \) as in Example 1.3, then \( A^b = k^b[\sigma] = k^b[\langle \sigma \cap M[1/p] \rangle] \).

Theorem 1.4 is proved without introducing much heavy machinery in Section 3.6 of [KL]. The basic idea is that the map \((-)^2 : k^{\circ \circ} \to k^b \) induces a ring homomorphism \( \theta : \text{W}(k^{\circ \circ}) \to k^b \), where \( \text{W}(\cdot) \) is the ring of \( p \)-typical Witt vectors. The inverse to the functor \((-)^b \) is \( A^b = \text{W}(A^\circ) \otimes_{\text{W}(k^{\circ \circ})} k \). Scholze’s proof is more conceptual, and passes through a diagram

\[
\begin{array}{ccc}
\text{Perf}(k) & \xrightarrow{\sim} & \text{Perf}(k^{\circ \circ}) \\
\downarrow \cdot^b & & \downarrow \cdot^b \\
\text{Perf}(k^b) & \xrightarrow{\sim} & \text{Perf}(k^{\circ \circ})
\end{array}
\]

in which most of the categories have yet to be defined. The superscript \((-)^a \) should be read “almost,” following the “almost mathematics” initially created by Faltings, and developed systematically in [GR03].
2 Almost mathematics

Almost mathematics was first introduced by Faltings in [Fal88], where he proved a deep conjecture of Fontaine on the étale cohomology of varieties over p-adic fields. We follow the treatment in Section 2.2 of [GR03].

Let \( V \) be a valuation ring with maximal ideal \( m \). Throughout this section, we assume that the value group of \( V \) is non-discrete. This implies \( m^2 = m \). In fact, much of the theory works for any unital ring \( V \) with idempotent two-sided ideal \( m \), but we have no need to work at this level of generality. The reader should keep in mind the example \( V = k^p \) for a perfectoid field \( k \).

Let \( \text{Mod}(V) \) be the category of all \( V \)-modules, and let \( \text{Ann}(m) \) be the full subcategory consisting of those modules killed by \( m \).

**Lemma 2.1.** \( \text{Ann}(m) \) is a Serre subcategory of \( \text{Mod}(V) \).

**Proof.** We need to show that if \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) is an exact sequence of \( V \)-modules for which \( M' \) and \( M'' \) are killed by \( m \), then \( M \) is also killed by \( m \). We trivially have \( m^2 M = 0 \), but \( m^2 = m \), whence the result. \( \square \)

**Definition 2.2.** The category of \( V^a \)-modules is the Serre quotient \( \text{Mod}(V^a) = \text{Mod}(V)/\text{Ann}(m) \).

Write \((-)^a: \text{Mod}(V) \rightarrow \text{Mod}(V^a)\) for the quotient functor. Even though \(“V^a”\) does not exist as a ring we will write \( \text{hom}_{V^a}(-,-) \) for hom-sets in \( \text{Mod}(V^a) \). While in general, hom-sets in quotient categories can be difficult to describe, the category \( \text{Mod}(V^a) \) is relatively easy to understand via the following theorem.

**Theorem 2.3.** There is a natural isomorphism \( \text{hom}_{V^a}(M^a,N^a) = \text{hom}_{V}(m \otimes M,N) \).

It follows that \( \text{Mod}(V^a) \) is naturally a \( \text{Mod}(V) \)-enriched category. We define two functors \((-)_t,(-)_s : \text{Mod}(V^a) \rightarrow \text{Mod}(V)\):

\[
\begin{align*}
M_t &= \text{hom}_{V^a}(V^a,M) \\
M_s &= m \otimes M_s.
\end{align*}
\]

**Theorem 2.4.** The triple \((-)_t,(-)^a,(-)_s\) is adjoint. Moreover, these adjunctions induce natural isomorphisms \( (M_t)^a = M = (M_s)^a \).

This is suggestive of the situation in which \( j : U \hookrightarrow X \) is an open embedding of topological spaces. The restriction functor \( j^* : \text{Sh}(X) \rightarrow \text{Sh}(U) \) fits into an adjoint triple \( (j_! j^*, j_* j^*) \) in which \( j^* j_* = 1 = j_* j^* \). So we should think of \( \text{Mod}(V^a) \) as the category of quasi-coherent sheaves on some subscheme of \( \text{Spec} V \), even though there is no such subscheme. In fact, since \( \text{Mod}(V^a) \) does not contain enough projectives, it should be thought of as some kind of “non-affine” object.

The category \( \text{Mod}(V^a) \) inherits the structure of a tensor category from \( \text{Mod}(V) \). In fact, we have internal tensor and hom defined by

\[
\begin{align*}
M^a \otimes N^a &= (M \otimes N)^a \\
\text{hom}^a(M^a,N^a) &= \text{hom}(M,N)^a
\end{align*}
\]

There is a tensor-hom adjunction \( \text{hom}(L \otimes N,M) = \text{hom}(L_!, \text{hom}^a(M,N)) \). This allows us to speak of algebra objects in \( \text{Mod}(V^a) \) as commutative unital monoid objects in the tensor category \( \text{Mod}(V^a)_\otimes \). We call such objects \( V^a \)-algebras. It is easy to check that they are all of the form \( A^a \), for \( A \) some \( V \)-algebra.

If \( A \) is a \( V^a \)-algebra, we can form the category \( \text{Mod}(A)_0 \) of \( A \)-modules in the obvious way. This is also an abelian tensor category, so it makes sense to speak of “flat objects” in the usual way, i.e. an \( A \)-module \( M \) is flat if the functor \( M \otimes_A - \) is exact.

**Definition 2.5.** Let \( k \) be a perfectoid field. A perfectoid \( k^a \)-algebra is a flat, \( \pi \)-adically complete \( k^a \)-algebra \( A \) for which \( \text{Fr}: A/\pi^{1/p} \rightarrow A/\pi \) is an isomorphism.
Note that even though $\pi^{1/p}$ may not exist as an actual element of $k^0$, the ideal $(\pi^{1/p})$ is well-defined, so it makes sense to write $M/\pi^{1/p}$ if $M$ is any $k^0$- (or $k^{\infty}$)-module. Write $\text{Perf}(k^{\infty})$ for the category of perfectoid $k^{\infty}$-algebras.

**Theorem 2.6.** The functor $A \mapsto A^{\infty}$ induces an equivalence of categories $\text{Perf}(k) \to \tilde{\to} \text{Perf}(k^{\infty})$.

**Idea of proof.** See the first part of Section 5 in [Sch12]. Besides some technicalities, one has the existence of an inverse functor $A \mapsto A^\bullet \otimes_{k^0} k$.

### 3 The cotangent complex

We have seen in the last section that localization induces an equivalence between the category of perfectoid $k$-algebras and the category of perfectoid $k^{\infty}$-algebras. Since $k^\times$ and $k^{\infty} \times$ are canonically isomorphic via $(-)^\natural$, also write $\pi$ for the element of $k^\times$ corresponding to $\pi \in k$. It is easy to check that $k^0/\pi = k^{\infty}/\pi$. It easily follows that $\text{Mod}(k^{\infty}/\pi) = \text{Mod}(k^{\infty}/\pi)$. The idea of this section is to pass from $\text{Perf}(k^{\infty})$ to a suitable category of “perfectoid $k^{\infty}/\pi$-algebras.”

**Definition 3.1.** Let $k$ be a perfectoid field. A $k^{\infty}/\pi$-algebra $A$ is perfectoid if it is flat and $\text{Fr} : A/\pi^{1/p} \to A$ is an isomorphism.

Let $\text{Perf}(k^{\infty}/\pi)$ denote the category of perfectoid $k^{\infty}/\pi$-algebras. We will show that the functor $A \mapsto A/\pi$ from $\text{Perf}(k^{\infty})$ to $\text{Perf}(k^{\infty}/\pi)$ is an equivalence of categories by introducing an “almost version” of the cotangent complex. Let’s start by recalling the classical theory:

**Theorem 3.2.** There is a functorial way of assigning to a flat map $A \to B$ of (commutative, unital) rings an object (the cotangent complex) $L_{B/A}$ of $D^{\le 0}(B)$. This complex satisfies the following properties:

1. There is a functorial way of assigning to a square-zero extension $0 \to I \to \tilde{A} \to A$ is a square-zero extension an obstruction class

   $o(I) \in \text{Ext}^2(L_{B/A}, I_B)$

   such that $o(I) = 0$ if and only if there is a flat $\tilde{A}$-algebra $\tilde{B}$ such that $\tilde{B} \otimes_{\tilde{A}} A = B$. (One calls such a $\tilde{B}$ a deformation of $B$ to $\tilde{A}$.)

2. If a deformation of $B$ to $\tilde{A}$ exists, then the set $\text{Def}_{\tilde{A}}(B)$ of deformations of $B$ to $\tilde{A}$ is a $\text{Ext}^1(L_{B/A}, I_B)$-torsor.

3. If $\tilde{B}$ and $\tilde{B}'$ are deformations of $A$-algebras $B, B'$ to $\tilde{A}$, and if $f : B \to B'$ is an $A$-algebra map, then there is a functorial way of assigning to $f$ an obstruction class

   $o(f) \in \text{Ext}^1(L_{B/A}, I_{B'})$

   such that the set $\text{Def}^{\tilde{A}}(f)$ of isomorphism classes of lifts of $f$ to $\tilde{f} : \tilde{B} \to \tilde{B}'$ is nonempty if and only if $o(f) \neq 0$.

4. If a lift of $f$ to $\tilde{A}$ exists, then $\text{Def}^{\tilde{A}}(f)$ is a hom $(L_{B/A}, I_{B'})$-torsor.

**Proof.** See Proposition 2.1.2.3 in Chapter III of [Ill71] for parts 1 and 2.

Gabber and Ramero were able to generalize the “classical” theory of the cotangent complex to an almost setting. To be precise, Theorem 3.2 remains true if we work $V^{\infty}$-algebras, for any non-discrete valuation ring $V$. In other words, there is a canonical object $L_{B/A}^{\tilde{A}} \in D^{\le 0}(B)$ such that the theorem still works.

**Theorem 3.3 (Scholze).** Let $k$ be a perfectoid field. If $A$ is a perfectoid $k^{\infty}/\pi$-algebra, then $L_{A/(k^{\infty}/\pi)}^{\tilde{A}} = 0$ as an object of $D(A)$. 

Idea of proof. This is a deep result, but at the heart of its proof is the fact that if \( R \) is a smooth perfect \( \mathbb{F}_p \)-algebra, then \( L_{R/F_p} = 0 \). Smoothness yields \( L_{R/F_p} = \Omega^1_{R/F_p}[0] \), and from \( d(r^p) = pdr^{p-1} = 0 \) we see that \( \Omega^1_{R/F_p} = 0 \).

Corollary 3.4 (Scholze). The functor \( A \mapsto A/\pi \) induces an equivalence of categories \( \text{Perf}(k^{\text{oa}}) \overset{\sim}{\rightarrow} \text{Perf}(k^{\text{oa}}/\pi) \).

Proof sketch. Let \( A_n = k^{\text{oa}}/\pi^{n+1} \). We content ourselves with showing that objects and morphisms in \( \text{Alg}(A_0) \) lift uniquely to each \( \text{Alg}(A_n) \). Let \( B_0 \) be a perfectoid \( A_0 \)-algebra. Theorem 3.3 tells us that \( L^a_{B_0/A_0} = 0 \), so \( B_0 \) and any morphisms from \( B_0 \) to other perfectoid \( A_0 \)-algebras lift uniquely to \( A_1 \) by the almost version of Theorem 3.2. All that remains for the induction to work is to show that \( L^a_{B_n/A_n} = 0 \) implies \( L^a_{B_{n+1}/A_{n+1}} = 0 \). There is an exact sequence

\[ 0 \longrightarrow B_0 \overset{\pi^n}{\longrightarrow} B_{n+1} \longrightarrow B_n \longrightarrow 0, \]

and general theory gives us an exact triangle in \( D(A_{n+1}) \):

\[ L^a_{B_0/A_0} \longrightarrow L^a_{B_{n+1}/A_{n+1}} \longrightarrow L^a_{B_n/A_n} \longrightarrow 0. \]

Since \( L^a_{B_0/A_0} = 0 \) by Theorem 3.3 and \( L^a_{B_n/A_n} = 0 \) by assumption, we get that \( L^a_{B_{n+1}/A_{n+1}} = 0 \).

References