Chromatic Homotopy Theory: Journey to the Frontier

Graduate Workshop Notes

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Abstract

Collected here are notes from the nine graduate workshop lectures at the Chromatic Homotopy Theory: Journey to the Frontier conference at CU Boulder, 16-20 May 2018.

These notes were lightly edited for grammar, spelling, and some of the more obvious mathematical errors, but I'm certain that errors and omissions remain. If you spot any, I would be grateful if you could send me an email at dmehrle@math.cornell.edu.

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1 MU AND FORMAL GROUP LAWS

1.1 COMPLEX ORIENTED COHOMOLOGY THEORIES

Complex oriented cohomology theories form the heart of the chromatic approach to stable homotopy theory. We'll start by saying what it means for a cohomology theory to be complex orientable. We want to generalize orientability of manifolds to other contexts.

Definition 1.1. A cohomology theory E is **complex orientable** if there is a class $x \in E^2(\mathbb{CP}^\infty)$ that restricts to a unit under $E^2(\mathbb{CP}^\infty) \to E^2(\mathbb{CP}^1) \cong E^0(S^0)$.

The map $E^2(\mathbb{CP}^\infty) \to E^2(\mathbb{CP}^1)$ is induced by the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and $E^2(\mathbb{CP}^1) \cong E^0(S^0)$ by the suspension isomorphism $E^n(X) \cong E^{n+k}(\Sigma^k X)$.

Notice that the choice of complex orientation is not a property of the cohomology theory – it is a choice. There is a difference between knowing that there is a complex orientation and a choice of a complex orientation, roughly analogous to the difference between an abstract vector space and choosing a basis.

Definition 1.2.

(a) E is **even** if $E^{2k+1}(S^0) = 0$ for all $k \in \mathbb{Z}$.

(b) E is **even periodic** if it is even and there is a class $u \in E^2(S^0)$ that is a unit.

To say that $u \in E^2(S^0)$ is a unit is to say that multiplication by u induces a natural isomorphism $E^n(-) \to E^{n+2}(-)$ for all n.

Example 1.3. The prototype for this is complex K-theory, or KU-theory. The isomorphism $KU^n \to KU^{n+2}$ is Bott periodicity.

It's very easy to compute the E-cohomology of a space with even cells, in the case that E is even.

Proposition 1.4. *Let E be an even cohomology theory. Then:*

(a) There is an isomorphism of rings (or $E^*(S^0)$-algebras)

$$E^*(\mathbb{CP}^n) = E^*(S^0)[x]/\langle x^{n+1}\rangle,$$

where $|x| = 2$,

(b) the following diagram commutes:

$$\begin{tikzcd} E^*(\mathbb{CP}^n) \arrow[r] \arrow[d, "\cong"] & E^*(\mathbb{CP}^{n-1}) \arrow[d, "\cong"] \\ E^*(S^0)[x]/\langle x^{n+1}\rangle \arrow[r] & E^*(S^0)[x]/\langle x^n\rangle, \end{tikzcd}$$

(c) and there is a Künneth isomorphism:

$$E^*(\mathbb{CP}^{n_1}) \otimes_{E^*(S^0)} \cdots \otimes_{E^*(S^0)} E^*(\mathbb{CP}^{n_r}) \xrightarrow{\cong} E^*(\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_r}).$$

Notice that this Künneth isomorphism is not generally true for cohomology theories – we don't necessarily even have it for E.

Proof. Using the Atiyah–Hirzebruch spectral sequence, we have

$$E_2^{p,q} = H^p(\mathbb{CP}^n; E^q(S^0)) \implies E^{p+q}(\mathbb{CP}^n).$$

For any q,

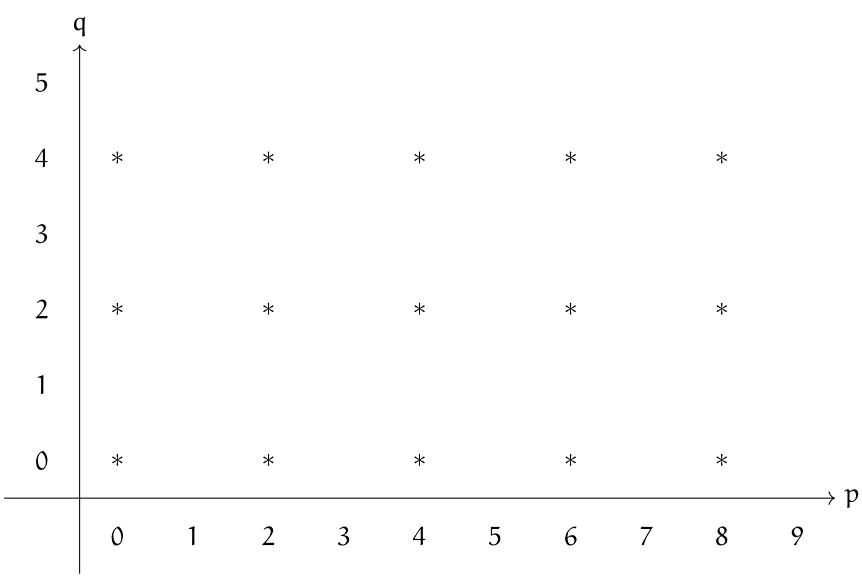
$$H^*(\mathbb{CP}^n; E^q(S^0)) \cong {E^q(S^0)[x]}/{\langle x^{n+1} \rangle}$$

where $|x| = (2, q)$.

Claim that in this spectral sequence, there are no nontrivial differentials. We can see this because, whenever either p or q is odd, we have

$$H^p(\mathbb{CP}^n; E^q(S^0)) = 0.$$

The E_2-page is pictured below. An asterisk $*$ indicates a nontrivial entry and a blank means that entry is zero.



$$E_2^{p,q} = H^p(\mathbb{CP}^n; E^q(S^0))$$

Since there are no nonzero differentials, the E_∞-page of this spectral sequence is exactly what we drew above. Since this is a free module, and there are no extensions between free modules, it is easy to reconstruct $E^{p+q}(\mathbb{CP}^n)$ from this E_∞-page.

The second two parts of the proposition follow. □

In this proof, we only used the fact that $\mathbb{CP}^n$ has only even cells. So the argument, and therefore the proposition, holds true for any space X with only even cells, such as $BU(n)$.

Corollary 1.5. *If E is even, then*

$$E^*(\mathbb{CP}^\infty) \cong \lim_n \left({E^*(S^0)[x]}/{\langle x^n \rangle} \right) \cong E^*(S^0)[\![x]\!].$$

If E is even periodic, then we may move (via $u\colon E^n(-) \to E^{n+2}(-)$) the generator x to degree zero.

Corollary 1.6. *If E is an even cohomology theory*

$$E^*(\underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k}) \cong E^*(S^0)[\![x_1, \ldots, x_k]\!]$$

To write down these isomorphisms, we chose an orientation, which in turn gives an element $x \in E^0(\mathbb{CP}^\infty)$. We think of this element x as the orientation.

Corollary 1.7. *Any even cohomology theory is complex orientable; a choice of complex orientation gives an isomorphism*

$$E^*(\mathbb{CP}^\infty) \cong E^*(S^0)[\![x]\!].$$

1.2 Formal Group Laws

There is a multiplication map on $\mathbb{CP}^\infty$, given by

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\otimes} \mathbb{CP}^\infty.$$

There are two interpretations of this multiplication map: geometric and algebraic. The geometric interpretation of this multiplication map is the fact that $\mathbb{CP}^\infty$ classifies complex line bundles, so given two complex line bundles, we can tensor them together. By the Yoneda lemma, this must come from a map between the classifying space for pairs of line bundles, $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, to the classifying space for line bundles, $\mathbb{CP}^\infty$. On the other hand, the algebraic interpretation comes from the fact that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

Applying the cohomology theory E^* to the multiplication, we have

$$E^*(\mathbb{CP}^\infty) \xrightarrow{\mu} E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

Then applying the previous isomorphism, we have

$$\begin{array}{ccc} E^*(\mathbb{CP}^\infty) & \xrightarrow{\mu} & E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ \downarrow \cong & & \downarrow \cong \\ E^*(S^0)[\![x]\!] & \longrightarrow & E^*(S^0)[\![y,z]\!] \\ x & \longmapsto & F(y,z) \end{array}$$

The choice of complex orientation specifies an isomorphism between an abstract power series ring $E^*(\mathbb{CP}^\infty)$ and an actual power series ring $E^*(S^0)[\![x]\!]$.

We codify the map $E^*(S^0)[\![x]\!] \to E^*(S^0)[\![y,z]\!]$ in a formal group law.

Definition 1.8. A **formal group law** over a ring R is a power series $F(x,y) \in R[\![x,y]\!]$ that is

(a) **unital**: $F(x,0) = F(0,x) = x$;

(b) **commutative**: $F(x,y) = F(y,z)$;

(c) **associative**: $F(x, F(y,z)) = F(F(x,y),z) \in R[\![x,y,z]\!]$

We are defining a new kind of addition on the maximal ideal of the power series ring $R[\![x]\!]$, and the properties we wrote down guarantee that this defines a unital, commutative, and associative operation. But we don't know that this has an inverse, *a priori*.

What we have seen is that an even (periodic) cohomology theory and a choice of orientation gives a formal group law. Theorems of Lazard and Quillen give the relation explicitly.

Theorem 1.9 (Lazard)**.** *The functor that sends a ring* R *to the set of formal group laws over* R *is representable by a ring* L*, where*

$$L \cong \mathbb{Z}[x_1, x_2, \ldots]$$

The fact that this functor is representable should not be surprising. The properties of the formal group law give conditions on the coefficients of power series, and applying these relations to a free $\mathbb{Z}$-algebra yields L. Maps out of this ring L capture the formal group laws. On the other hand, the fact that L is polynomial is miraculous.

Remark 1.10. We have been a bit careless about the gradings on the rings here. There is a question of how the grading on the ring interacts with the formal group law. This endows the ring L with a grading, once you trace through the formal group law properties.

Definition 1.11. The ring $L \cong \mathbb{Z}[x_1, x_2, \ldots]$ with $|x_i| = -2i$ is the **Lazard ring**.

Now consider the Thom spectrum MU. The first space in this spectrum is $MU_1 \simeq \mathbb{CP}^\infty$, so we have a map

$$\Sigma^{-2}\mathbb{CP}^\infty \to MU$$

which gives a canonical complex orientation on the cohomology theory represented by MU. Hence, this gives a formal group law over $MU^*(S^0)$.

Theorem 1.12 (Quillen)**.** $MU^*(S^0) \cong L$*, and the formal group law here is the universal one.*

Definition 1.13. A **homomorphism of formal group laws** $f\colon F \to G$ is a power series $f(x) \in \mathbb{R}[\![x]\!]$ such that

$$f(F(x,y)) = G(f(x), f(y)).$$

An **isomorphism of formal group laws** is a homomorphism $f\colon F \to G$ that has an inverse under composition.

An isomorphism of formal group laws is **strict** if $f(x) \equiv x \pmod{x^2}$.

Strict isomorphism are the ones that take coordinates to coordinates, in the analogy between vector spaces/bases and complex orientable cohomology theories/choices of these orientations.

Theorem 1.14 (Quillen)**.** *The ring* $(MU \wedge MU)^*(S^0)$ *represents strict isomorphisms and the isomorphism taking the left orientation to the right is the universal strict isomorphism.*

2 BP, Landweber exactness, and other examples

In the previous lecture, we saw that for any complex oriented cohomology theory E, there is an associated formal group law.

Remark 2.1. This formal group law tells us how to write the Chern class of the tensor product of line bundles:

$$c_1(L_1 \otimes L_2) = F_E(c_1(L_1), c_1(L_2))$$

Example 2.2.

(a) If $E = H\mathbb{Z}$, then the associated formal group law is $F_{H\mathbb{Z}}(x, y) = x + y$. This is the **additive formal group law**.

(b) If $E = KU$, then $c_1(L) = \beta([L] - 1) \in KU^0(X)$ where β is the Bott class. The associated formal group law is

$$F_{KU}(x, y) = x + y + \beta xy.$$

This is the **multiplicative formal group law**.

(c) If $E = MU$, then we have the **universal formal group law**, which starts

$$F_{MU}(x, y) = x + y + \dots.$$

In this talk, we will produce more examples, motivated by a theorem of Conner–Floyd, which roughly says that you can recover KU-homology from MU-homology.

Theorem 2.3 (Conner–Floyd)**.** *For any spectrum X, there is an isomorphism of rings*

$$MU_*(X) \otimes_{MU_*} KU_* \cong KU_*(X).$$

This leads to the question:

Question 2.4. When, given a map $MU_* \to R$ (i.e. a formal group law over R), is the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

a homology theory?

To check the Eilenberg-Steenrod axioms, we really only need to check the exactness axiom. Say, given a cofiber sequence

$$A \to B \to C,$$

can we apply $MU_*(-) \otimes_{MU_*} R$ and get an exact sequence?

This is not always true, because tensoring with R is not an exact functor. So we may ask that R is flat over MU_*, but MU_* is a polynomial ring and there aren't many interesting examples of flat algebras over polynomial rings. So the idea is instead to consider $(-) \otimes_{MU_*} R$ as a functor from a different category, consisting of MU_*-modules with extra structure.

Consider a (homotopy) ring spectrum E. For any other spectrum X, E^*X becomes an E^*E-comodule in the following way. Since E is a ring spectrum, we may use the unit map to define a map of spectra

$$E \wedge X = S^0 \wedge E \wedge X \to E \wedge E \wedge X.$$

Applying the functor $\pi_*(-)$, we get a map

$$E_*(X) \to E_*E \otimes_{E_*} E_*X$$

if E_*E is flat over E_*. This is a **coaction** of E^*E on E_*X, which gives E_*X the structure of an $E \wedge E$-comodule.

Example 2.5. If $E = H\mathbb{F}_2$, then $E_*E = \mathcal{A}_* = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$. Let $X = S^0 \cup_2 e^1$ be the zero-sphere with a cell attached via the degree 2 map $S^1 \xrightarrow{2} S^1$.

$$\begin{aligned} H_*(X) &\longrightarrow \mathcal{A}_* \otimes H_*(X) \\ e_1 &\longmapsto 1 \otimes e_1 + \zeta_1 \otimes e_0 \end{aligned}$$

Example 2.6. If $E = MU$, then the ring $MU_*(MU)$ represents isomorphisms of formal group laws. The comultiplication on $MU_* MU$ is a homomorphism

$$MU_* MU \xrightarrow{\Delta} MU_* MU \otimes_{MU_*} MU_* MU.$$

To determine what this means, apply the functor $\mathrm{Hom}(-, R)$. The module $\mathrm{Hom}(MU_* MU, R)$ consists of isomorphisms of formal group laws over R, and $\mathrm{Hom}(MU_* MU \otimes_{MU_*} MU_* MU, R)$ consists of pairs of isomorphisms of formal group laws over R, such that these group law isomorphisms are composable. Then Δ^* becomes the composition

$$\mathrm{Hom}(MU_* MU \otimes_{MU_*} MU_* MU, R) \xrightarrow{\Delta_*} \mathrm{Hom}(MU_* MU, R)$$

So $(MU_*, MU_* MU)$ corepresents a functor from rings to groupoids.

This answers our question about what we should pick as the source category for the functor $(-) \otimes_{MU_*} R$: it should be the category $\mathbf{Comod}(MU_* MU)$ of comodules over $MU_* MU$.

Question 2.7. When is $(-) \otimes_{MU_*} R\colon$ $\mathbf{Comod}(MU_* MU) \to \mathbf{Mod}(R)$ exact?

Definition 2.8 (Terminology/Notation)**.** If F is a formal group law, we write $x +_F y = F(x, y)$ for the "addition" defined by F, and write $[n]_F(x)$ for the formal power series

$$\underbrace{x +_F x +_F \cdots +_F x}_{n}.$$

Let v_i denote the coefficient of x^{p^i} in $[p]_F(x)$, for $v_i \in MU_*$.

The number v_n is an invariant that we could go and compute, in fact, it is a strict isomorphism invariant modulo the ideal $\langle p, v_1, v_2, \ldots, v_{n-1}\rangle$.

Lemma 2.9. $[p]_F(x) = ax^{p^n} +$ *higher order terms*

Definition 2.10. If $[p]_F(x) = ax^{p^n} + \ldots$, we say that F **has height at least** n**.**

If $a \in R$ is a unit, then we say that F **has height** n**.**

The Landweber theorem gives conditions under which the functor $(-) \otimes_{MU_*} R$ is exact.

Theorem 2.11 (Landweber Exact Functor Theorem)**.** *Given a formal group law* $MU_* \to R$ *such that for all primes* p*, and all* n

$$R/\langle p, v_1, \ldots, v_{n-1}\rangle \xrightarrow{\cdot v_n} R/\langle p, v_1, \ldots, v_{n-1}\rangle$$

is injective, then the functor

$$(-) \otimes_{MU_*} R\colon \mathbf{Comod}(MU_* MU) \to \mathbf{Mod}(R)$$

is exact.

Definition 2.12. If a formal group law satisfies the condition of the previous theorem, it is called **Landweber exact**.

Example 2.13. Consider the additive formal group law for $H\mathbb{Z}$,

$$F_{H\mathbb{Z}}(x, y) = x + y,$$

we can compute

$$[p]_{F_{H\mathbb{Z}}}(x) = px,$$

so $v_i = 0$. We must check that the following are all injective, for all primes p:

$$\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$$
$$\mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p$$

This is not true, so the additive formal group law is not Landweber exact.

Example 2.14. Consider rational cohomology $H\mathbb{Q}$. The multiplication-by-p-map

$$\mathbb{Q} \xrightarrow{\cdot p} \mathbb{Q}$$

is injective, and moreover $\mathbb{Q}/p = 0$, so all maps $\mathbb{Q}/p \to \mathbb{Q}/p$ are injective. Hence, the formal group law defined by $H\mathbb{Q}$ is Landweber exact.

Example 2.15. Consider complex K-theory KU. We have

$$[p]_{F_{KU}}(x) = \beta^{-1}((\beta x + 1)^p - 1) = px + \ldots + \beta^{p-1}x^p.$$

Check injectivity:

$$KU_* = \mathbb{Z}[\beta^{\pm 1}] \xrightarrow{\cdot p} \mathbb{Z}[\beta^{\pm 1}]$$

is injective, and so are the homomorphisms

$$\mathbb{Z}[\beta^{\pm 1}]/p \xrightarrow{\cdot \beta^{p-1}} \mathbb{Z}[\beta^{\pm 1}]/p$$

and

$$0 \xrightarrow{\cdot v_n} 0$$

Example 2.16. Consider $MU_* \cong \mathbb{Z}[x_1, x_2, \ldots]$. We may choose v_i at p to be x_{p^i-1}. This is Landweber exact. Then

$$(MU_*)_{(p)}/\langle x_n \mid n \neq p^i - 1\rangle \cong BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Definition 2.17. The cohomology theory defined by BP is called the **Brown-Peterson Spectrum**.

Definition 2.18. Define the **Johnson–Wilson theory** $E(n)$ with

$$E(n)_* := \left(BP_*/\langle v_{n+1}, v_{n+2}, \ldots \rangle \right)[v_n^{-1}]$$

2.1 Proof sketch of Landweber exactness

The Landweber exact functor theorem is essentially a theorem in algebra, together with some convenient facts about MU_*.

(1) With appropriate finiteness conditions, if M is an R-module, then there is a filtration of M

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

such that

$$\mathrm{gr}_*(M) = \bigoplus_i R/P_i$$

where $P_i \subseteq R$ are prime ideals. This is a vast generalization of the classification of modules.

(2) This allows us to compute Tor and Ext of these things easily:

$$\mathrm{Tor}(M, N) = 0 \iff \mathrm{Tor}({}^{R}/_{P}, N) = 0$$

for all prime ideals P of R.

(3) If M has the extra structure of an $MU_*(MU)$-comodule, then you can arrange for P to be **invariant**, i.e. given $\psi\colon MU_* \to MU_*(MU) \otimes_{MU_*} MU_*$, we have

$$\psi(P) \subseteq MU_*(MU) \otimes_{MU_*} P$$

(4) We use the **invariant prime ideal theorem**, which says that the ideals $\langle p, v_1, \ldots, v_n \rangle$ are exactly the finitely generated invariant prime ideals in MU_*.

(5) We can do some technical work to guarantee that the previous items still apply to $M = MU_*$, even though it may not meet the finiteness conditions from step (1).

Example 2.19. Another example of formal group laws comes from elliptic curves. Consider

$$g(x) = \int_0^x \frac{dt}{\sqrt{1 + \delta t^2 + \varepsilon t^4}}$$

considered as a power series in x. Then the **elliptic formal group law** is

$$g^{-1}(g(x) + g(y)) = F_{\mathrm{Ell}}(x, y).$$

This is a formal group law over $\mathbb{Z}[\frac{1}{2}][\varepsilon, \delta]$.

After inverting ε, this formal group law is Landweber exact over $\mathbb{Z}[\frac{1}{2}][\varepsilon^{\pm 1}, \delta]$.

This is due to Landweber–Ravenel–Strong, it is called $Tmf_0(2)$ in modern language.

3 Bousfield localization and the Hasse square

This talk is about Bousfield localization, but we will first talk about the geometry of localization. This should mirror the way that we localize rings, but now in the category of spectra.

We have already seen the **height stratification** of formal group laws by the **invariant prime ideals**:

$$\langle p\rangle \subseteq \langle p, v_1\rangle \subseteq \langle p, v_1, v_2\rangle \subseteq \ldots$$

So once we localize at a prime, we have a stratification of formal group laws by height.

Theorem 3.1. *For each prime p, there is a complex oriented cohomology theory* BP *with*

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

and a p-typical formal group law over R is classified by a map $BP_* \to R$ sending each v_i to the appropriate element of R defined by the formal group law.

Definition 3.2. The spectrum BP is called a **Brown–Peterson** spectrum.

We may draw a picture of the relation between MU and each BP by drawing a quotient of $\mathrm{Spec}(MU_*)$. Notice that morphisms of affine schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(MU_*)$ are in bijection with ring homomorphisms $R \to MU_*$, or with formal group laws over R. Isomorphisms of formal group laws give an action of an algebraic group $\mathbb{G}$ on $\mathrm{Spec}(MU_*)$; the quotient $\mathrm{Spec}(MU_*)/\mathbb{G}$ (which is properly thought of as a stack) collapses any two prime ideals of MU_* which can be related by an isomorphism of formal group laws. Therefore, the points of $\mathrm{Spec}(MU_*)$ are exactly the invariant prime ideals from before.

Since $MU_* \cong \mathbb{Z}[x_1, x_2, \ldots]$, $\mathrm{Spec}(MU_*)$ contains a copy of $\mathrm{Spec}(\mathbb{Z})$, and this is preserved by the action of $\mathbb{G}$. Above each nonzero prime in $\mathrm{Spec}(\mathbb{Z})$, we have a tower corresponding to $\mathrm{Spec}(BP_*)/\mathbb{G}$ for the Brown–Peterson spectrum associated to p.

Theorem 3.3. *For each prime p and each nonnegative integer n, there is a spectrum $E(n)$ with*

$$E(n)_* = BP_*[v_n^{-1}] = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}].$$

If a formal group law over $\mathbb{Z}_{(p)}$ is a map $BP_* \to R$ for the p-typical Brown–Peterson spectrum, then a formal group law of height at most n must have v_n invertible. In other words, maps $BP_*[v_n^{-1}] \to R$ classify the p-typical formal

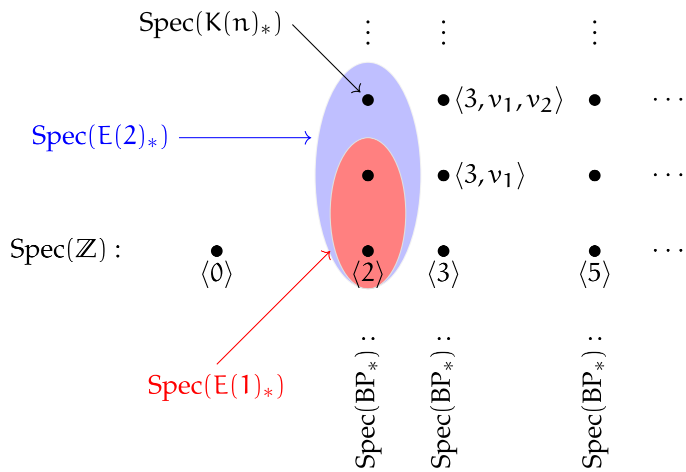


Figure 1: A picture of $\mathrm{Spec}(MU_*)/\mathbb{G}$.

group laws of height at most n. Several open sets are shaded in the picture below, representing pictorially the formal group laws of height at most n for some n. Each of these open sets corresponds to $\mathrm{Spec}(E(n)_*)/\mathbb{G}$ for some n.

Theorem 3.4. *For each prime* p *and each nonnegative integer* n*, there is a complex-orientable spectrum* $K(n)$ *with*

$$K(n)_* = {}^{\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}]}\big/_{\langle p, v_1, v_2, \ldots, v_{n-1}\rangle} \cong \mathbb{F}_p[v_n^{\pm 1}].$$

Note that if we consider only homogeneous expressions in $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, then we have a graded field. In the picture, this corresponds to a single point above the bottom row representing $\mathrm{Spec}(\mathbb{Z})$. Much as p-typical formal group laws of height at most n are classified by maps $E(n)_* \to R$, formal group laws of height exactly n are classified by maps $K(n)_* \to R$.

Theorem 3.5. *Any two formal group laws of height* n *are isomorphic over an algebraically closed field.*

3.1 Bousfield Localization

Following the analogy with algebraic geometry, we want to ask which part of an $MU_*(MU)$-comodule is seen by one of these open sets in the picture.

Definition 3.6. If E is a homology theory, then we say that a spectrum Z is E**-acyclic** if $E_*(Z) = 0$, or equivalently, $E \wedge Z \simeq \mathrm{pt}$.

If you think of the smash product as an extension of the tensor product, then this is saying that tensoring with E obtains zero, or that elements of Z are invertible in E.

Definition 3.7. A spectrum K is E**-local** if for all E-acylic spectra Z, $K^0(Z) = 0$.

Definition 3.8. A map of spectra $f\colon F \to K$ is an E**-equivalence** if $f_*\colon E_*(F) \to E_*(K)$ is an isomorphism.

Proposition 3.9. *Say that* $K \to K'$ *is an* E*-equivalence, and* J *is* E*-local. Then the induced map* $[K', J] \to [K, J]$ *is an isomorphism.*

This says that E-local spectra are precisely those that can't tell the difference between E-equivalent spectra.

Theorem 3.10 (Bousfield)**.** *There is a functor* $L_E\colon \mathbf{Sp} \to \mathbf{Sp}$*, and a natural transformation* $\eta\colon \mathrm{id} \implies L_E$ *such that*

(a) η_X *is an* E*-equivalence;*

(b) $L_E(X)$ *is always* E*-local;*

(c) η *is initial among such functors: any map* $X \to Y$ *such that* Y *is* E*-local factors through* $L_E(X)$*.*

Moreover, L_E is idempotent as a functor: $L_E(X) \to L_E(L_E(X))$ is an equivalence, and the map is $L_E(\eta_X)$.

Remark 3.11. This is built from a localization of a model category structure on **Sp**: we keep the same cofibrations, but then require that weak equivalences are E-equivalences. This forces a choice of fibrations, and fibrant objects are exactly the E-local ones. Then $L_E(-)$ is fibrant replacement in this new model structure.

Example 3.12. Let J be a set of primes. Then write

$$S^0[J^{-1}] := \operatorname{colim}\left(S^0 \xrightarrow{j_1} S^0 \xrightarrow{j_1 j_2} S^0 \to \cdots\right),$$

where the map $S^0 \xrightarrow{n} S^0$ is given by the suspension of the map of degree n from the 1-sphere to itself. $S^0[J^{-1}]$ is a ring spectrum, and

$$L_{S^0[J^{-1}]}(X) = S^0[J^{-1}] \wedge X.$$

The analogy with rings is that localization of an R-module M at a prime ideal P is given by $M_P = R_P \otimes_R M$.

Example 3.13. Let $E = S^0/p = \mathrm{cofib}\left(S^0 \xrightarrow{p} S^0\right)$. Then localization at E is a function spectrum:

$$L_{S^0/p} = F(\Sigma^{-1}\left({}^{S^0[\frac{1}{p}]}/_{S^0}\right), X) = \lim\left({}^{S^0}/_{p^n}\right) \wedge X.$$

This is a p-completion.

Theorem 3.14 (Sullivan)**.** *There is a (homotopy) pullback square*

$$\begin{tikzcd} L_{\mathbb{Z}_{(p)}}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{\mathbb{Z}/p}(X) \arrow[d] \\ L_{\mathbb{Q}}(X) \arrow[r] & L_{\mathbb{Q}}L_{\mathbb{Z}/p}(X) \end{tikzcd}$$

Exercise 3.15. Prove this theorem. Use the fact that the map $L_{\mathbb{Z}/p}(X) \to L_{\mathbb{Q}}L_{\mathbb{Z}/p}(X)$ is a rational equivalence, and the bottom map is a $\mathbb{Z}/2$-equivalence.

Theorem 3.16. *There is a natural equivalence between the localization functors*

$$L_{E(n)}(-) \simeq L_{K(0)\vee K(1)\vee\cdots\vee K(n)}(-).$$

Hence, there are natural transformations

$$\cdots \to L_{E(n)}(-) \to L_{E(n-1)}(-) \to \cdots.$$

So for any spectrum, we have a tower of spectra.

Definition 3.17. The **chromatic tower** of a spectrum X is the diagram

$$\cdots \to L_{E(n)}(X) \to L_{E(n-1)}(X) \to \cdots \to L_{E(0)}(X)$$

The fibers of the maps in the tower are the **monochromatic layers** of X.

There is a map from X to this tower from the natural transformations $\eta_X\colon X \to L_{E(n)}(X)$, so we may compare X to the limit of this system. If $X \simeq \mathrm{holim}_n(L_{E(n)}X)$, then we have **chromatic convergence**.

Following the example of the rational sphere, we have the following theorem that matches our intuition that localization looks like $-\otimes_R R_P$.

Theorem 3.18. $L_{E(n)}(X) \simeq L_{E(n)}(S^0)\wedge X$

Theorem 3.19 (Hasse Square)**.** *There is a (homotopy) pullback square*

$$\begin{tikzcd} L_{E(n)}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{K(n)}(X) \arrow[d] \\ L_{E(n-1)}(X) \arrow[r] & L_{E(n-1)}L_{K(n)}(X) \end{tikzcd}$$

?

4 Morava E-theory, Morava K-theory, and the stabilizer group

Since this is a chromatic homotopy theory conference, let's think about a map between spheres

$$S^k \xrightarrow{g} S^0.$$

More generally, for any cohomology theory E, consider applying it to this diagram:

$$E_*[k] \xrightarrow{d(g)} E_*,$$

where $E_*[k]$ is the shift of E_* in degree by k. We call the map $d(g)$ because it is given by the degree of the map g of spheres, if we take usual cohomology.

If $d(g) = 0$, then consider instead

$$S^k \xrightarrow{g} S^0 \to C(g) \to S^{k+1}$$

where $C(g)$ is the mapping cone. If we apply E_* to this sequence, we arrive at

$$E_*[k] \to E_* \to E_*(C(g)) \to E_*[k+1] \to E_*[1].$$

Since $d(g) = 0$, we the first and last terms vanish. hence, we are left with

$$0 \to E_* \to E_*(C(g)) \to E_*[k+1] \to 0$$

and this defines an element in $\mathrm{Ext}(E_*[k+1], E_*)$, which is Ext in the category of E_*E-comodules.

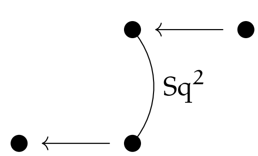
Example 4.1. If $E = H\mathbb{F}_2$, then we have

$$S^1 \xrightarrow{\eta} S^0 \to C(\eta) \to S^2$$

This defines an element $h_1 \in \mathrm{Ext}^1_{\mathcal{A}_*}((H\mathbb{F}_2)_*[2], (H\mathbb{F}_2)_*)$.

$$H^*(S^0) \leftarrow H^*(C(\eta)) \leftarrow H^*(S^2).$$

This is frequently drawn as



In general, we get a sequence of invariants of a map $S^k \to S^0$ living in $\mathrm{Ext}^S_{E_*E}(E_*[k+s], E_*)$. There is a spectral spectral sequence, called the **Adams-Novikov spectral sequence**

$$\mathrm{Ext}^S_{E_*E}(E_*[k+s], E_*) \implies \pi_*(L_E(S)).$$

This is what we want to compute. If we take $E = MU$, then (being lazy and omitting the shifts), we have

$$\mathrm{Ext}_{MU_* MU}(MU_*, MU_*) \implies \pi_*(S^0)$$

So we can use this to compute the stable homotopy groups of spheres.

4.1 Moduli stack of formal group laws

Let $\mathcal{M}_{FG}$ be the **moduli stack of formal group laws**. This is roughly a "homotopy quotient" of groupoids

$$\mathrm{Spec}(MU_*)/\,\mathrm{Spec}(MU_*(MU)).$$

A map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is a formal group law over R. If we have a map of rings $R_2 \to R_1$, then we have a diagram

$$\begin{tikzcd}
\mathrm{Spec}(R_1) \arrow[d] \arrow[dr, "F_1"] & \\
\mathrm{Spec}(R_2) \arrow[r, "F_2"'] \arrow[u, phantom, "\Uparrow"] & \mathcal{M}_{FG}
\end{tikzcd}$$

We think that $F_1 \otimes_{R_1} R_2 \cong F_2$.

$\mathcal{M}_{FG}$ classifies bundles of formal group laws. For any ring R, a point in $\mathrm{Spec}(R)$ is a residue field of R, some field k and a map $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$. So the composite

$$\mathrm{Spec}(k) \to \mathrm{Spec}(R) \xrightarrow{\alpha} \mathcal{M}_{FG}$$

is the map of a point to $\mathcal{M}_{FG}$. The map α classifies a formal group law over R, so we may say that a formal group law over R is determined by a formal group law over the residue fields R/P for each $P \in \mathrm{Spec}(R)$. The idea is that "points" in algebraic geometry are $\mathrm{Spec}(k)$, where k is an algebraically closed field.

$$(\mathcal{M}_{FG})_{(p)} = \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$
$$\longrightarrow \text{height}$$

Theorem 4.2 (Lazard)**.** *Over an algebraically closed field of characteristic p, height is a complete invariant of formal group laws.*

Remark 4.3. Over $\mathbb{Z}_{(p)}$, $[p]_F$ determines F.

Theorem 4.4. *Let R be a commutative ring. There is an equivalence of groupoids after applying* $\mathrm{Hom}(-, R)$*:*

$$((MU_*)_{(p)}, (MU_*(MU))_{(p)}) \cong (BP_*, BP_*(BP)).$$

Claim 4.5. $MU_* MU$-comodules are quasi-coherent sheaves on $\mathcal{M}_{FG}$.

Proof. What does a quasi-coherent sheaf on $\mathcal{M}_{FG}$ mean?

$$\begin{tikzcd} g^*F \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "g"] & \mathcal{M}_{FG} \end{tikzcd}$$

A quasicoherent sheaf on $\mathcal{M}_{FG}$ should consist of

- for all $\eta\colon \mathrm{Spec}(R) \to \mathcal{M}_{FG}$, an R-module M_η;
- for all diagrams

$$\begin{tikzcd} \mathrm{Spec}(R_1) \arrow[dd] \arrow[dr, "\eta_1"] & \\ & \mathcal{M}_{FG} \\ \mathrm{Spec}(R_2) \arrow[ur, "\eta_2"'] \arrow[uu, phantom, "\Rightarrow"] & \end{tikzcd}$$

  an isomorphism $M_{\eta_2} \otimes_{R_2} R_1 \cong M_{\eta_1}$.
- compatibilities between these data.

Among all maps from $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$, we have a particularly useful one: the universal formal group law $L = MU_* \to \mathcal{M}_{FG}$.

$$\begin{tikzcd} \mathrm{Spec}(MU_*) \arrow[d] \\ \mathcal{M}_{FG} \end{tikzcd}$$

We can then form a quasi-coherent sheaf on this

$$\begin{tikzcd} \mathrm{Spec}(MU_* MU) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}(MU_*) \arrow[d] \\ \mathrm{Spec}(MU_*) \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

which consists of the data of

- an MU_*-module M_0 and,
- an MU_*-module M_1 and,

- an isomorphism of $\mathrm{MU}_* \mathrm{MU}$-modules

$$M_0 \otimes_{\mathrm{MU}_*} \mathrm{MU}_* \mathrm{MU} \cong \mathrm{MU}_* \mathrm{MU} \otimes_{\mathrm{MU}_*} M_1.$$

It turns out that this is exactly equivalent to the data of an $\mathrm{MU}_* \mathrm{MU}$-comodule (exercise). $\square$

Corollary 4.6. Comod$(\mathrm{BP}_* \mathrm{BP}) \cong$ **Comod** $\left((\mathrm{MU}_* \mathrm{MU})_{(p)}\right)$

Hence, we can compute Ext in either of these categories. In particular,

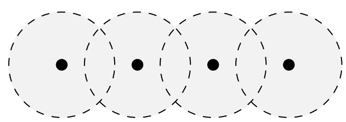
$$\mathrm{Ext}_{\mathrm{BP}_* \mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \cong \mathrm{Ext}_{(\mathrm{MU}_* \mathrm{MU})_{(p)}}((\mathrm{MU}_*)_{(p)}, (\mathrm{MU}_*)_{(p)})$$

BP_* is the tensor unit in $\mathrm{BP}_* \mathrm{BP}$-comodules. In the land of quasi-coherent sheaves, this is the structure sheaf $\mathcal{O}_{(\mathcal{M}_{\mathrm{FG}})_{(p)}}$.

We reinterpret $\mathrm{Ext}^*_{\mathrm{MU}_* \mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$ as sheaf cohomology as

$$\mathrm{Ext}^*_{\mathrm{MU}_* \mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*) = R\Gamma^*(\mathcal{O}_{\mathcal{M}_{\mathrm{FG}}}) = H^*(\mathcal{M}_{\mathrm{FG}}).$$

Let's compute the cohomology of $\mathcal{M}_{\mathrm{FG}}$ by finding an open cover and using Mayer-Vietoris: try taking a small neighborhood around each point in $\mathcal{M}_{\mathrm{FG}}$ and compute the cohomology there.

$(\mathcal{M}_{\mathrm{FG}})_{(p)} =$  $\cdots$

What is an infinitesimal neighborhood of a point in $\mathcal{M}_{\mathrm{FG}}$? Well, functions on an infinitesimal neighborhood should be like power series, e.g. $\mathrm{Spec}(\mathbb{F}_p[\![t]\!])$. More generally, a neighborhood should be $\mathrm{Spec}(R)$, where R is a complete, Noetherian, local ring.

What is an infinitesimal neighborhood of a height n formal group law over a field k, inside $\mathcal{M}_{\mathrm{FG}}$?

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{FG}} \\
\mathrm{Spec}(R) \arrow[ur, hook] &
\end{tikzcd}$$

We search for the universal example of a Noetherian complete local ring R with a diagram as above.

Let **C** be the category of tuples (R, M, k, ϕ) where R is a complete Noetherian local ring with maximal ideal M, k is a field such that $R/M \cong k$, and $\phi\colon R \to k$ is a homomorphism.

Definition 4.7. Fix a formal group law Γ over a perfect field k of height n. Define a functor $\mathrm{Def}(\Gamma)\colon \mathbf{C} \to \mathbf{Groupoid}$ that takes (R, M, k, ϕ) to the groupoid G whose objects are formal group laws F over R such that $F \otimes_R {}^R/_M = \Gamma$ and whose morphisms are identities mod η.

Theorem 4.8 (Lubin–Tate)**.** $\mathrm{Def}(\Gamma)$ *takes values in the category of sets, and is corepresented by a ring*

$$E(k, \Gamma)_0 \cong W(k)[\![u_1, \ldots, u_{n-1}]\!]$$

where $W(k)$ *is the ring of Witt vectors over* k*.*

The upshot of this is that $E(k, \Gamma)_0$ carries the universal such deformation of Γ; i.e. a formal group law

$$MU_* \to E(k, \Gamma)_* := E(k, \Gamma)[u^{\pm 1}].$$

This is Landweber exact, so we get a cohomology theory $E(k, \Gamma)$ called **Lubin–Tate theory** or **Morava E-theory**.

Automorphisms of Γ over k produce automorphisms of the universal deformation by naturality. The automorphism group $\mathrm{Aut}(k, \Gamma) = \mathbb{G}_n$ is called the **Morava stabilizer group**, which acts on $E(k, \Gamma)_*$.

$$H^*(\text{formal nbhd of } (k, \Gamma)) = H^*(\mathbb{G}_n; E(k, \Gamma)_*)$$

$$\begin{tikzcd}
H^*(\mathbb{G}_n; E(k, \Gamma)_*) \arrow[r, Rightarrow] & \pi_*(L_{K(n)} S^0) \\
H^*(\mathcal{M}_{FG}) \arrow[u] \arrow[r, Rightarrow] & \pi_*(S^0) \arrow[u]
\end{tikzcd}$$

Remark 4.9. So we have a procedure to solve any problem in homotopy theory: we pick a prime (hope it's not 2), and pick a height (hope it's 1), and then use the diagram as above together with the homotopy pullback square in the previous talk.

5 Some $K(1)$-local computations

We ended the last talk with $H_c^*(\mathbb{G}_n, (E_n)_*)$, where $\mathbb{G}_n = \mathrm{Aut}(\Gamma, \mathbb{F}_{p^n})$ and $E_n = E(\Gamma, \mathbb{F}_{p^n})$. This is trying to compute the Bousfield localization of the sphere at Morava K-theory, via a spectral sequence:

$$H_c^s(\mathbb{G}_n, (E_n)_t) \implies \pi_{t-s} L_{K(n)} S^0.$$

All of the occurrences of n are the same, and are the height of the formal group law Γ.

If Γ is the **Honda formal group law**, then $[p]_\Gamma(x) = x^{p^n}$.

$$(E_n)_* = \mathbb{Z}_p(\zeta_{p^n-1})[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}].$$

Theorem 5.1 (Derihatz–Hopkins). $E_n^{h\mathbb{G}_n} \simeq L_{K(n)} S^0$.

Now take $n = 1$. We will think about K-theory.

$$KU_* \cong \mathbb{Z}[\beta^{\pm 1}]$$

with $\beta \in KU_2$.

$$KU^*(\mathbb{CP}) \cong \mathbb{Z}[\beta^{\pm 1}][x] / \langle x^{n+1} \rangle.$$

When we complete, we have

$$KU^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[\beta^{\pm 1}][\![x]\!],$$

with $x \in KU^0(\mathbb{CP}^\infty)$. The group law in question is the multiplicative formal group law

$$F_{KU}(x, y) = x + y + xy.$$

The p-series is

$$[p]_{KU}(x) \equiv x^p \pmod p$$

KU/p is the cohomology theory with height 1 formal group law.

The Morava K-theory $K(1)$ is K/p with formal group law

$$\Gamma(x, y) = x + y + xy$$

over $\mathbb{F}_p[u^{\pm 1}]$.

The Morava E-theory E_1 is $K_p = \lim_i KU/p^i$, which is p-completed K-theory.

Exercise 5.2. $(E_1)_* = \mathbb{Z}_p[\beta^{\pm 1}]$, where $\beta = u$.

What is $\mathbb{G}_1 = \mathrm{Aut}(\Gamma, \mathbb{F}_p)$?

Proposition 5.3. *Let k be a field of characteristic p and let Γ be the height $n = 1$ Honda formal group law viewed over k. Let $f(x) \in k[\![x]\!]$ be an endomorphism. Then*

$$f(x) = \sum_{i \geq 0}^{\Gamma} a_i x^{p^i},$$

for $a_i \in \mathbb{F}_p$, where $\sum^{\Gamma}$ means that we are using the formal group law $+_{\Gamma}$ to add elements.

Furthermore, the map below is an isomorphism:

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & \mathrm{End}(\Gamma) \\ \alpha = a_0 + pa_1 + p^2 a_2 + \dots & \longmapsto & \sum_{i\geq 0}^{\Gamma} a_i [p^i]_{\Gamma}(x) = \sum_{i \geq 0}^{\Gamma} ax^{p^i} \end{array}$$

We have $\mathrm{End}(\Gamma) \cong \mathbb{Z}_p$, and $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_p^{\times}$. What does $\mathbb{Z}_p^{\times}$ look like?

We have an isomorphism

$$\mathbb{Z}_p^{\times} \cong \mu \times U$$

If p is odd, then $\mu = C_{p-1}$ and $U = \overline{\langle 1+p \rangle} \cong (\mathbb{Z}_p, +)$. We will write $g = 1 + p$ for the additive generator of $\mathbb{Z}_p$.

If p is even, then $\mu = (\pm 1) = C_2$, and $U = \overline{\langle 1+4 \rangle} \cong \mathbb{Z}_2$. Therefore, $\mathbb{Z}_2^{\times} \cong C_2 \times \mathbb{Z}_2$. Hence,

$$\mathbb{Z}_p^{\times} \cong \begin{cases} C_{p-1} \times \mathbb{Z}_p & p \text{ odd} \\ C_2 \times \mathbb{Z}_2 & p \text{ even} \end{cases}$$

$(K_p)_* = \mathbb{Z}_p[\beta^{\pm 1}]$ has an action of $\mathbb{G}_1$.

The **Adams operations on K-theory** are for $k, \ell \geq 0$,

$$\psi^k \colon \mathrm{KU}^0(-) \to \mathrm{KU}^0(-)$$

such that when L is a line bundle,

$$\psi^k(L) = L^{\otimes k}$$

and moreover,

$$\psi^k \psi^{\ell} = \psi^{k\ell} = \psi^{\ell} \psi^k.$$

Exercise 5.4. $\psi^k(\beta^n) = k^n \beta^n$

However, the Adams operations are not stable. If instead we take K-theory with coefficients in R, e.g. $\mathbb{Z}[1/k]$, where $k \in R^{\times}$, then we can extend ψ^k to stable operations on the spectrum of K-theory with coefficients in R.

Consider $\mathbb{Z}_p^\times$ acting on K_p. For any $\alpha \in \mathbb{Z}_p^\times$,

$$\psi^\alpha(\beta^n) = \alpha^n \beta^n.$$

Then we have an action of $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$ acting on $E_1 = K_p$ via the p-completed Adams operations.

Let's recall what we're trying to do: compute

$$\mathrm{Ext}^{*,*}_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}(\mathbb{Z}_p, \mathbb{Z}_p[\beta^{\pm 1}]) = H^*_c(\mathbb{Z}_p^\times, \mathbb{Z}_p[\beta^{\pm 1}]) \implies \pi_* L_{K(1)} S^0.$$

Here,

$$\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] = \lim_{i,j} \mathbb{Z}/p^i \left[\mathbb{Z}_p^\times/(1+p^j)\mathbb{Z}_p^\times\right].$$

Likewise,

$$\mathbb{Z}_p[\![\mathbb{Z}_p]\!] = \lim_{i,j} \mathbb{Z}/p^i \left[\mathbb{Z}/p^j\right].$$

Now assume that p is odd. So

$$\mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$$

and $|C_{p-1}|$ is a unit in $\mathbb{Z}_p$. Then we learn from standard group cohomology facts (involving the Lyndon–Hochschild–Serre spectral sequence)

$$H^*(\mathbb{Z}_p^\times, M) \cong H^*(\mathbb{Z}_p, M^{C_{p-1}}).$$

In our case, let's figure out what the C_{p-1}-fixed points of M are. Given $\alpha \in C_{p-1} \subseteq \mathbb{Z}_p^\times$, we have $\alpha^{p-1} = 1$. We also know

$$\psi^\alpha(\beta^m) = \alpha^m \beta^m = \beta^m \iff (p-1) \mid m.$$

So

$$(\mathbb{Z}_p[\beta^{\pm 1}])^{C_{p-1}} = \mathbb{Z}_p[\beta^{\pm(p-1)}] = \mathbb{Z}_p[v_1^{\pm 1}].$$

So the new goal is to compute the group cohomology

$$H^*(\mathbb{Z}_p, \mathbb{Z}_p[v_1^{\pm 1}]),$$

where $\langle 1+p \rangle = \mathbb{Z}_p$ and

$$\psi^{1+p}(v_1^k) = (1+p)^{k(p-1)} v_i^k.$$

Exercise 5.5. Use the projective resolution

$$0 \to \mathbb{Z}_p[\![\mathbb{Z}_p]\!] \xrightarrow{g-1} \mathbb{Z}_p[\![\mathbb{Z}_p]\!] \to \mathbb{Z}_p \to 0.$$

where the first $\mathbb{Z}_p$ is $\langle g \rangle$ and the second one is $\overline{\langle g \rangle}$.

Another method to do these computations is to consider

$$L_{K(1)}S^0 \cong E_1^{h\mathbb{G}_1} \cong (K_p)^{h\mathbb{Z}_p^\times}$$

where $\mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$, as before. Hence, we have

$$L_{K(1)}S^0 \cong E_1^{h\mathbb{G}_1} \cong (K_p)^{h\mathbb{Z}_p^\times} \cong (K_p^{hC_{p-1}})^{h\mathbb{Z}_p} \cong (K_p^{C_{p-1}})^{h\mathbb{Z}_p}.$$

Earlier, we learned that $(K_p^{C_{p-1}})_* = \mathbb{Z}_p[v_1^{\pm 1}]$. This is the p**-complete Adams summand**.

There is a shadow of the short exact sequence from the previous exercise in homotopy theory: namely, a fiber sequence

$$L_{K(1)}S^0 \to K_p^{C_{p-1}} \xrightarrow{\psi^g - 1} K_p^{C_{p-1}} \to \Sigma L_{K(1)}S^0.$$

This gives a long exacts sequence in homotopy:

$$0 \longrightarrow \pi_{2(p-1)k}L_{K(1)}S^0 \longrightarrow \pi_{2(p-1)k}K^{C_{p-1}} \xrightarrow{\psi^q - 1} \pi_{2(p-1)k}K^{C_{p-1}} \longrightarrow \pi_{2(p-1)k-1}L_{K(1)}S^0 \longrightarrow 0$$

When $k \neq 0$, we have

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2(p-1)k}L_{K(1)}S^0 \arrow[r] \arrow[d, equal] & \pi_{2(p-1)k}K^{C_{p-1}} \arrow[r, "\psi^q - 1"] \arrow[d, "\cong"] & \pi_{2(p-1)k}K^{C_{p-1}} \arrow[r] \arrow[d, "\cong"] & \pi_{2(p-1)k-1}L_{K(1)}S^0 \arrow[r] \arrow[d, "\cong"] & 0 \\
& 0 \arrow[r] & \mathbb{Z}_p\{v_1^k\} \arrow[r, hook, "p^{\nu_p(k)+1}\text{ unit}"] & \mathbb{Z}_p\{v_1^k\} \arrow[r] & \mathbb{Z}/_{p^{\nu_p(k)+1}}\{v_1^k\} \arrow[r] & 0 \\
& & v_1^k \arrow[r, mapsto] & ((1+p)^{k(p-1)} - 1)v_1^k & &
\end{tikzcd}$$

And when $k = 0$, this becomes

$$0 \to \pi_0 L_{K(1)}S^0 \xrightarrow{\cong} \mathbb{Z}_p\{1\} \xrightarrow{0} \mathbb{Z}_p\{1\} \xrightarrow{\cong} \pi_{-1}L_{K(1)}S^0 \to 0$$

Exercise 5.6. $\nu_p((1+p)^{k(p-1)} - 1) = \nu_p(k) + 1$ where $\nu_p(x)$ is the highest power of p that divides x.

Theorem 5.7.

$$\pi_n L_{K(1)}S^0 = \begin{cases} \mathbb{Z}_p & n = 0, -1, \\ \mathbb{Z}/p^{\nu_p(k)+1} & n = 2k(p-1) - 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Consider the J-homomorphism $J\colon \pi_* O \to \pi_*^S$ where O is the infinite orthogonal group. This is given by the colimit of maps

$$\operatorname{colim} \left(\begin{array}{ccc} O(n) & \longrightarrow & \Omega^n S^n \\ g & \longmapsto & \overline{g}\colon S^{\mathbb{R}^n} \to S^{\mathbb{R}^n} \end{array} \right)$$

where $S^{\mathbb{R}^n}$ is the one-point compactification.

Since

$$\pi_i O \cong \begin{cases} \mathbb{Z}/2 & i \equiv 0, 1 \pmod 8 \\ \mathbb{Z} & i = 4k-1 \\ 0 & \text{otherwise.} \end{cases}$$

We have a theorem:

Theorem 5.8. *The composite below is an isomorphism*

$$\mathrm{im}_p(J) \to \pi_* S^0 \to \pi_* L_{K(1)} S^0.$$

6 THE GREAT CONJECTURES

Let V be a finite complex equipped with maps

$$S^{d+k} \xrightarrow{\Sigma^d i} \Sigma^d V \xrightarrow{v} V \xrightarrow{j} S^\ell.$$

with the following properties:

- $d > 0$ and all iterates of v are essential. We say that such a map v is **periodic**. We know that v has this property because it induces an isomorphism in $K(n)_*(-)$ for some $n > 0$ with $K(n)_* V \neq 0$.
- In the known examples, i is inclusion of the bottom cell into V, and j is projection onto the top cell.
- It was known that for each $t > 0$, the composite

  $$S^{td+k} \xrightarrow{\Sigma^{dt} i} \Sigma^{dt} V \xrightarrow{v^t} V \xrightarrow{j} S^\ell$$

  represents a nontrivial element in $\pi_{td+k} S^\ell$.

Such complexes can be used to find examples of periodic families of elements in the homotopy groups of spheres.

Example 6.1. Only three examples were known in 1973. Toda had constructed finite complexes he called $V(n)$ with

$$BP_* V(n) \cong BP_* / \langle p, v_1, \ldots, v_n \rangle$$

for $0 \leq n \leq 3$, and cofiber sequences

$$\Sigma^{2p^n-2} V(n-1) \xrightarrow{v_n} V(n-1) \to V(n)$$

for $1 \leq n \leq 3$.

To this day, nobody has constructed $V(4)$.

In each case there is a lower bound on the prime p: in 2010, Lee Nave showed that $V((p+1)/2)$ does not exist.

Question 6.2.

(a) Are there more maps like this? Can we use them to construct more periodic families in the homotopy groups of spheres?

(b) Are there any periodic maps that are not detected by BP-theory or MU-theory?

(c) What happens if we replace the prime ideal $I_n = \langle p, v_1, \ldots, v_{n-1} \rangle$ with a smaller invariant regular ideal with n generators?

Recall that

$$BP_* \cong \mathbb{Z}_p[v_1, v_2, \ldots]$$

where $|v_n| = 2(p^n - 1)$ and

$$\Gamma := BP_* BP \cong BP_*[t_1, t_2, \ldots]$$

with $|t_i| = 2(p^i - 1)$, which has Hopf algebroid structure.

The E_2-term of the Adams-Novikov spectral sequence converging to the p-local stable homotopy groups of spheres is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_* BP}(BP_*, BP_*)$$

So this object is of great interest. It can be studied using the LES $BP_*(BP)$-comodules

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to \cdots$$

called the **chromatic resolution**.

This leads to a trigraded **chromatic spectral sequence** converging to the bigraded Adams-Novikov E_2-term, with

$$E_1^{n,s,t} = \mathrm{Ext}^{s,t}_{BP_* BP}(BP_*, M^n) \implies E_2^{n+s,t}.$$

For fixed n, this group is related to the cohomology fo the n-th Morava stabilizer group, which is the automorphism group of a certain formal group law of height n. It is also related to the v_n-periodic phenomena in the stable homotopy groups of spheres.

We used the term **chromatic** because each column (value of n) displays periodic families of elements with varying frequencies, like the astronomical[1] spectrum of light.

The comodules M^n are defined inductively as follows:

- M^0 is obtained from BP_* by inverting p. This means there is a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & N^0 \arrow[r] \arrow[d, equal] & M^0 \arrow[r] \arrow[d, equal] & N^1 \arrow[r] \arrow[d, equal] & 0 \\
 & BP_* & BP_* \otimes \mathbb{Q} & BP_*/\langle p^\infty \rangle &
\end{tikzcd}$$

- For $n > 0$, M^n is obtained from N^n by inverting v_n.

$$\begin{tikzcd}
0 \arrow[r] & N^n \arrow[r, "v_n^{-1}"] \arrow[d, equal] & M^n \arrow[r] \arrow[d, equal] & N^{n+1} \arrow[r] & 0 \\
 & BP_*/\langle p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty \rangle & v_n^{-1} BP_*/\langle p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty \rangle & &
\end{tikzcd}$$

[1]Why is light astronomical? Because stars.

The chromatic resolution

$$0 \to BP_* \to M^0 \to M^1 \to \cdots$$

is obtained by splicing together these short exact sequences for all $n \geq 0$. This construction is purely algebraic. It takes place in the category of $BP_*(BP)$-comodules.

Question 6.3. Is there a similar construction, and the beautiful algebra that goes along with it, in the stable homotopy category?

6.1 Bousfield Localization

$$0 \to N^n \xrightarrow{v_n^{-1}} M^n \to N^{n+1} \to 0$$

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to \cdots$$

It would be nice if each short exact sequence above were the BP_*-homology of a cofiber sequence of spectra. Then we would have spectra M_n and N_n with $BP_*(M_n) \cong M^n$ and $BP_*(N_n) \cong N^n$. This was easy enough for $n = 0$. We know how to invert a prime p homotopically. The resulting N^1 is the Moore spectrum for the group $\mathbb{Q}/\mathbb{Z}_p$.

Question 6.4. But how would we invert v_1 to do the next step?

As luck would have it, Bousfield localization exists! This is the secret weapon.

Definition 6.5. Suppose we have a generalized homology theory represented by a spectrum E. We say a spectrum Z is E**-local**, if whenever $f \colon A \to B$ is an E_*-equivalence, then the induced map

$$f^* \colon [B, Z] \to [A, Z]$$

is an isomorphism.

The condition that Z is an E-local spectrum is the following: if C is an E_*-acyclic spectrum, meaning that $E_*C = 0$, then $[C, Z] = 0$.

Theorem 6.6 (Bousfield Localization)**.** *For a given E there is a coaugmented functor L_E such that for any spectrum X, $L_E(X)$ is E-local, and the map $X \to L_E(X)$ is an E_* equivalence.*

It turns out that when E and X are both connective, $L_E(X)$ can be described in purely algebraic terms. It is either obtained from X by inverting some set of primes, or it is the p-adic completion for a single prime p. Things are more complicated if either E or X fails to be connective.

Question 6.7. What if our hypothetical spectrum M_n could be obtained from the inductively constructed N_n by some form of Bousfield localization?

The logical choice for E appeared to be the Johnson-Wilson spectrum $E(n)$. It is a BP-module spectrum with

$$\pi_* E(n) \cong \mathbb{Z}_p[v_1, \dots, v_{n-1} v_n^{\pm 1}]$$

It is closely related to the fancier Morava spectrum E_n, but not exactly the same. However, they give the same localizations.

Definition 6.8. Two spectra E and E' are **Bousfield equivalent** if they have the same class of acyclic spectra, that is spectra C with $E_* C = 0$ if and only if $E' C = 0$.

The Bousfield equivalence class of E is denoted by $\langle E \rangle$. We say that $\langle E \rangle \geq \langle F \rangle$ if $E_* C = 0$ implies that $F_* C = 0$.

Writing $\langle E \rangle \geq \langle F \rangle$ means that the class of E-acyclic spectra is bigger than or equal to the class of F-acyclic spectra: the homology theory E_* gives at least as much information as F_*.

It follows that the maximal Bousfield class is that of the sphere spectrum S, and the minimal Bousfield class is that of a point $*$.

It is easy to check that wedges and smash products of Bousfield classes are well-defined, that is,

$$\langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle$$
$$\langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle$$

These two operations satisfy the expected distributive law. A collection with such operations is called a **lattice**, and this particular collection is called the **Bousfield lattice** $\mathbb{A}$.

Remark 6.9. For any spectrum E, $\langle E \rangle \vee \langle E \rangle = \langle E \rangle$, but it is not the case that $\langle E \rangle \wedge \langle E \rangle = \langle E \rangle$. There are spectra that become contractible when smashed with themselves.

Definition 6.10. The collection of classes $\langle E \rangle$ for which $\langle E \rangle \wedge \langle E \rangle = \langle E \rangle$ is called the **Bousfield distributive lattice** $\mathbb{DL}$. It includes all connective and ring spectra.

The **compliment** (if it exists) $\langle E \rangle^c$ of a class $\langle E \rangle$ is a class with

$$\langle E \rangle^c \vee \langle E \rangle = \langle S \rangle$$

and

$$\langle E \rangle^c \wedge \langle E \rangle = \langle * \rangle.$$

The collection of Bousfield equivalence classes with complements forms a Boolean algebra.

Theorem 6.11 (Formal properties of Bousfield classes)**.**

(a) If $W \to X \to Y \xrightarrow{f} \Sigma W$ *is a cofiber sequence, then* $\langle X\rangle \le \langle W\rangle \vee \langle Y\rangle$.

(b) If f *is* ***smash nilpotent****, i.e.* $f^{\wedge k}\colon Y^k \to (\Sigma W)^{\wedge k}$ *is null for some* k*, then* $\langle X\rangle = \langle W\rangle \vee \langle Y\rangle$.

(c) For a self-map $\Sigma^d X \xrightarrow{v} X$*, let* C_v *denote its cofiber and let* $\widehat{X}$ *denote the homotopy colimit (mapping telescope) of*

$$X \xrightarrow{v} \Sigma^{-d} X \xrightarrow{v} \cdots$$

Then $\langle X\rangle = \langle \widehat{X}\rangle \vee \langle C_V\rangle$ *and* $\langle X\rangle \wedge \langle \widehat{X}\rangle = \langle *\rangle$.

Theorem 6.12 (Some Bousfield equivalence classes)**.**

(a)

$$\langle S\rangle = \langle S\mathbb{Q}\rangle \vee \bigvee_{p \text{ prime}} \langle S/p\rangle$$

where $S\mathbb{Q}$ *is the rational Moore spectrum and* S/p *is the mod* p *Moore spectrum;*

(b)

$$\langle BP\rangle = \langle H/p\rangle \vee \bigvee_{n\ge 0} \langle K(n)\rangle$$

where H/p *is the mod* p *Eilenberg–MacLane-spectrum and* $K(n)$ *is Morava* K*-theory;*

(c)

$$\langle E(n)\rangle = \langle E_n\rangle = \bigvee_{0\le i\le n} \langle K(i)\rangle.$$

In each case, the smash of any two wedge summands on the right is contractible.

6.2 The Chromatic tower

The localization functor L_E is determined by the Bousfield class $\langle E\rangle$. When $\langle E\rangle \ge \langle F\rangle$, then there is a natural transformation $L_E \implies L_F$.

Definition 6.13. For a fixed prime p, let $L_n = L_{E(n)}$. Then for any spectrum X, we get a diagram called the **chromatic tower** of X:

$$X \to L_\infty X \to \cdots \to L_n(X) \to L_{n-1}X \to \cdots \to L_1 X \to L_0 X,$$

where $L_\infty X$ is localization with respect to $\bigvee_{n\ge 0}\langle K(n)\rangle$.

This raises some questions:

(1) When is the map $X \to L_\infty X$ is an equivalence? Then X is called **harmonic**. We call $L_\infty X$ **dissonant** if $L_\infty X \simeq *$. $L_\infty X$ is the **harmonic localization** of X.

(2) When the map $X \to \operatorname{holim} L_n X$ is an equivalence? This is the **chromatic convergence question**.

(3) Can we describe $BP_* L_n X$ in terms of $BP_* X$? This is the **localization question**.

If X is dissonant, then $L_\infty X \simeq *$, so $K(n)_* X = 0$ for all n. It follows from the definitions that there are no essential maps from a dissonant spectrum to a harmonic one.

Theorem 6.14 (Ravenel)**.**

(a) every p-local finite spectrum is harmonic

(b) a p-local connective spectrum X is harmonic when $BP_* X$ has finite projective dimension as a BP_*-module.

(c) The mod p Eilenberg MacLane spectrum H/p is dissonant. This is true for any spectrum whose homotopy group are torsion and bounded above.

A p-local spectrum X is **chromatically convergent** if it is equivalent to the homotopy limit of the diagram

$$\cdots \to L_n X \to L_{n-1} X \to \cdots \to L_1 X \to L_0 X.$$

Theorem 6.15 (Hopkins–Ravenel)**.** *All p-local finite spectra are chromatically convergent.*

Theorem 6.16 (Barthel)**.** *Any p-local connective spectrum X is chromatically convergent when $BP_* X$ has finite projective dimension as a BP_*-module.*

Recall that in question 6.3 we asked whether or not the chromatic resolution has a geometric underpinning. This is a special case fo the localization question – how can we describe $BP_* L_n X$ in terms of $BP_* X$?

It turns out that $L_n BP$ is easy to analyze, and this makes it easy to understand $X \wedge L_n BP$.

Theorem 6.17 (Localization conjecture)**.** *For any spectrum X,*

$$BP \wedge L_n X \simeq X \wedge L_n BP.$$

In particular, when $E(n-1)_* X = 0$, then $BP_* L_n X = v_n^{-1} BP_* X$.

It follows that the chromatic resolution can be realized as desired. Moreover, the functor L_n satisfies a stronger condition:

Theorem 6.18 (Smash product conjecture)**.** *For any spectrum X, $L_n X \cong X \wedge L_n S$.*

6.3 Some Conjectures

Ravenel's 1984 conjecture ends with a list, and all but one of them (the telescope conjecture) were proved before 2000 by Hopkins and collaborators.

Theorem 6.19 (Nilpotence Theorem, Devinatz–Hopkins–Smith)**.** *(a) For a finite spectrum X, a map $v\colon \Sigma^d X \to X$ is nilpotent if and only if $MU_*(v)$ is nilpotent.*

(b) For a finite spectrum X, a map $g\colon X \to Y$ is smash nilpotent if the map $MU \wedge g$ is null-homotopic

(c) Let R be a connective ring spectrum of finite type, and let $h\colon \pi_* R \to MU_* R$ be the Hurewicz map. Then $\alpha \in \pi_* R$ is nilpotent when $h(\alpha) = 0$.

(d) Let $W \to X \to Y \xrightarrow{f} \Sigma W$ be a cofiber sequence of finite spectra with $MU_*(f) = 0$. Then $\langle X \rangle = \langle W \rangle \vee \langle Y \rangle$

If it were the case that $\langle MU \rangle = \langle S \rangle$, or if $\langle BP \rangle = \langle S_{(p)} \rangle$, for each prime p, then the nilpotence theorem would follow. But $\langle BP \rangle < \langle S_{(p)} \rangle$, meaning that there are BP_*-acyclic p-local spectra that are not contractible. In other words, MU doesn't see everything!

In fact, there are connective p-local spectra $T(m)$ for $m \geq 0$ with

$$BP_* T(m) \cong BP_*[t_1, t_2, \ldots, t_m]$$

(so $T(0) = S_{(p)}$). and

$$\langle T(0) \rangle > \langle T(1) \rangle > \langle T(2) \rangle > \ldots > \langle BP \rangle.$$

So not only does $\langle BP \rangle = \langle S_{(p)} \rangle$ fail, but there's an infinite sequence of failures in between!

Now consider the first part of the nilpotence conjecture: For a finite spectrum X, a map $v\colon \Sigma^d X \to X$ is nilpotent if and only if $MU_*(v)$ is nilpotent. This means that such a map can be periodic (the opposite of nilpotent) only if it detected as such by MU-homology. In the p-local case, the internal properties of MU-theory imply that v must induce a nontrivial isomorphism in some Morava K-theory $K(n)_*$.

Definition 6.20. A p-local spectrum X has **chromatic type** n if $K(n-1)_* X = 0$ but $K(n)_* X \neq 0$.

Notice that $K(i)_* X = 0$ for all $i \leq n-1$ if X has chromatic type n, since $K(i)_* X = 0 \implies K(i-1)_* X = 0$.

Theorem 6.21 (Periodicity Theorem, Hopkins–Smith 1998)**.** *Let X be a p-local finite spectrum of chromatic type n. Then there is a map $v\colon \Sigma^d X \to X$ (a v_n**-self map**) with $K(n)_*(v)$ an isomorphism and $H_*(v; \mathbb{Z}/p) = 0$.*

We may in fact also relate the degree d of this v_n-self map to n. If $n = 0$ then $d = 0$, and if $n > 0$, d is a multiple of $2p^n - 2$.

Additionally, such a map satisfies an **asymptotic uniqueness** property: given another such map $w\colon \Sigma^e X \to X$, then there are positive integers i, j such that $id = je$ and $v^i = w^j$.

It follows that the cofiber of v (or of any of its iterates) is a p-local finite spectrum of chromatic type $n+1$. This means that there are finite complexes of all chromatic types. Finite complexes of arbitrary chromatic type were first constructed by Steve Mitchell in 1985. Consequently, there are lots of periodic families in $\pi_* \mathbb{S}$.

A pleasant consequence of the Nilpotence Theorem is the following:

Theorem 6.22 (The class invariant conjecture)**.** *The Bousfield class of a p-local finite spectrum X is determined by its chromatic type, i.e. the smallest n for which $K(n)_* X \neq 0$.*

In particular, if $H_* X$ is not all torsion, then $\langle X \rangle = \langle S_{(p)} \rangle$.

Suppose that X is a p-local finite spectrum of chromatic type n. The Periodicity theorem says that it has a v_n self-map $v\colon \Sigma^d X \to X$. Let $\widehat{X}$ be the associated mapping telescope, meaning the homotopy colimit of

$$X \xrightarrow{v} \Sigma^{-d} X \xrightarrow{v} \Sigma^{-2d} X \xrightarrow{v} \cdots$$

Note that it is independent of the choice of v. Since v is a $K(n)$-equivalence and therefore an $E(n)$-equivalence, we have maps

$$X \to \widehat{X} \xrightarrow{\lambda} L_n X$$

Conjecture 6.23 (The telescope conjecture)**.** *For any p-local spectrum X of chromatic type n, the map $\lambda\colon \widehat{X} \to L_n X$ is an equivalence.*

This is easy for $n = 0$, and true for $n = 1$.

Conjecture 6.24 (Ravenel)**.** *The telescope conjecture is false for $n \geq 2$.*

7 Chromatic splitting

There are two ideas going on today:

(1) **Chromatic assembly:** If X is a finite CW-spectrum, then we may compare X to the limit of its chromatic tower:

$$X \to \operatorname{holim} L_n X.$$

(2) **Chromatic fracture:** There is a pullback square:

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

after $\mathbb{Z}_p$-localization.

In this talk, we'll focus on the map $L_{n-1}X \to L_{n-1}L_{K(n)}X$. Let's recall some of the typical players:

- $E_n = E(\mathbb{F}_{p^n}, \Gamma_n)$ is the **Morava** E-theory, which we will start to call just E shortly;
$$(E_n)_* X = \pi_* L_{K(n)}(E_n \wedge X).$$
- The **Adams–Novikov spectral sequence**
$$H^s(\mathbb{G}_n, E_t X) \implies \pi_{t-s} L_{K(n)} X.$$
- $\mathbb{G}_n = \operatorname{Aut}(\mathbb{F}_{p^n}, \Gamma_n) \cong \operatorname{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \rtimes \operatorname{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$
- Goerss–Hopkins–Miller: $\mathbb{G}_n$ acts on E_n and various fixed point spectra E_n^{hH} for $H \subseteq \mathbb{G}_n$ closed.
- $\mathbb{S}_n = \operatorname{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \subseteq \operatorname{End}(\Gamma_n/\mathbb{F}_{p^n})$; the latter is a free module over the Witt vectors $W = W(\mathbb{F}_{p^n})$ of rank n.

$$\begin{tikzcd} \operatorname{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \arrow[r] \arrow[drr] & GL_n(W) \arrow[r, "\det"] & W^\times \\ & & \mathbb{Z}_p^\times \arrow[u] \end{tikzcd}$$

- $\zeta_n \in H^1(\mathbb{G}_n, \mathbb{Z}_p)$ is the homomorphism that is the composite below:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[r, equal] \arrow[rrrrr, bend left, "\zeta_n"] & \mathbb{S}_n \rtimes \mathrm{Gal} \arrow[r] \arrow[rrr, bend right, "\det"'] & \mathbb{Z}_p^\times \times \mathrm{Gal} \arrow[r] & \mathbb{Z}_p^\times \arrow[r] & \mathbb{Z}_p^\times / C \arrow[r, "\cong"] & \mathbb{Z}_p
\end{tikzcd}$$

Alternatively, we also say that $\zeta_n \in H^1(\mathbb{G}_n, E_0)$ under the map

$$H^1(\mathbb{G}_n, \mathbb{Z}_p) \to H^1(\mathbb{G}_n, E_0)$$

where $E_0 = W[\![\mu_1, \ldots, \mu_{n+1}]\!]$.

Theorem 7.1. *ζ_n is a permanent cycle in the Adams-Novikov spectral sequence detecting a homotopy class $\zeta_n \in \pi_{-1} L_{K(n)} S^0$.*

Proof. Let $\mathbb{G}_n^1 = \ker(\zeta_n)$. Then there is a fiber sequence

$$\begin{tikzcd}
E^{h\mathbb{G}_n} \arrow[r] \arrow[d, equal] & E^{h\mathbb{G}_n^1} \arrow[r, "\psi - 1"] & E^{h\mathbb{G}_n^1} \arrow[r] & \Sigma L_{K(n)} S^0 \\
L_{K(n)} S^0 & & S^0 \arrow[u, "1"] \arrow[ur, "\zeta_n"'] &
\end{tikzcd}$$

for $\psi \in \mathbb{G}_n$, $\zeta_n(\psi) \in \mathbb{Z}_p$ is a generator. □

$$\begin{tikzcd}
 & L_1 S^0 \arrow[r] \arrow[d] & L_{K(1)} S^0 \arrow[d] & \supseteq & \mathbb{Z}_p \oplus \mathbb{Z}_p \zeta_1 \arrow[r, equal] \arrow[ddll] & \pi_* L_{K(1)} S^0 / \mathrm{torsion} \\
L_{H\mathbb{Q}} S^0 \arrow[r, equal] & L_0 S^0 \arrow[r] \arrow[d, equal] & L_0 L_{K(1)} S^0 & & & \\
 & H\mathbb{Q} \arrow[r] & H\mathbb{Q}_p \vee \Sigma^{-1} H\mathbb{Q}_p & & &
\end{tikzcd}$$

When $n = 1$ and $p \geq 3$, then $H^*(\mathbb{G}_1, (E_1)_*/\mathrm{torsion}) \cong \Lambda(\zeta_1)$.
If $p = 2$,

$$H^*(\mathbb{G}_1, (E_1)_0) = H^*(\mathbb{Z}_2^\times, \mathbb{Z}_2) \cong \Lambda(\zeta_1) \otimes \mathbb{Z}_p[x]/(2x)$$

where $|x| = 2$. And here,

$$E_\infty^{s,0} \cong \Lambda(\zeta_1).$$

Lets consider $n = 2$. Our job is to calculate $L_1 L_{K(2)} S^0$. It fits into the following (homotopy) pullback square:

$$\begin{tikzcd} L_1 L_K(2) S^0 \arrow[r] \arrow[d] & L_{K(1)} L_{K(2)} S^0 \arrow[d] \\ L_0 L_{K(1)} S^0 \arrow[r] & L_0 L_{K(1)} L_{K(0)} S^0 \end{tikzcd}$$

Theorem 7.2 (Morava)**.**

$$H^*(\mathbb{S}_n, \mathbb{Q}) \cong \Lambda(x_1, x_3, \ldots, x_{2n-1})$$

with $|x_i| = i$. *The class* x_1 *is a rational version of* ζ_n.

Likewise, $H^*(\mathbb{S}_n, \mathbb{Z}_p)/\text{torsion}$ is an exterior algebra. It fits into the diagram below:

$$\Lambda(x_1, x_3, \ldots, x_{2n-1}) \xrightarrow{\cong} H^*(\mathbb{S}_n, \mathbb{Z}_p) \xrightarrow{\cong} H^*(\mathbb{G}_n, W) \longrightarrow H^*(\mathbb{G}_n, E_0)$$

This gives a bunch of classes in $H^*(\mathbb{G}_n, E_0)$.

Theorem 7.3 (Shimomura–Yabe $(p \geq 5)$, Henn–Goerss–Mahowald $(p = 3)$, Beaudry–Bobkova–Henn–Goerss $(p = 2)$)**.**

$$\pi_* L_{K(2)} S^0 /_{\text{torsion}} \cong \Lambda_{\mathbb{Z}_p}(\zeta_2, e)$$

where $|e| = 3$.

Let's calculate $L_{K(1)}X$. This is periodic:

$$v_1^{p^k} : \Sigma^d S/p^n \to S/p^n$$

for some d. Write

$$v_1^{-1} X \wedge S/p^n = X \widehat{\wedge S/p}^n.$$

Then

$$L_{K(1)}X = \operatorname{holim} v_1^{-1}\,(X \wedge S/p^n)\,;$$

this is a special fact for $n = 1$.

Theorem 7.4 (Shimamura–Yabe $(p \geq 5)$, Henn–Maramanov–Mahowald $(p = 3)$)**.** *For* $p \geq 3$,

$$v_1^{-1} \pi_* L_{K(2)}(S/p) \cong \Lambda(\zeta_1, \alpha_1) \otimes \mathbb{F}_p[v_1^{\pm 1}],$$

where $\alpha_1 \in \pi_{2p-3} S^0 \cong \mathbb{Z}/p$.

Note that
$$\pi_* L_{K(1)}(S^0/p) \cong \Lambda(\zeta_1) \otimes \mathbb{F}_p[v_1^{\pm 1}],$$
and $v_1\zeta_1 = \alpha_1$ up to sign. The theorem and this observation yield
$$L_{K(1)}L_{K(2)}(S/p) \simeq L_{K(1)}(S/p) \vee \Sigma^{-1} L_{K(1)}(S/p).$$
Therefore,
$$L_{K(1)}L_{K(2)}S^0 \simeq L_{K(1)}S^0 \vee \Sigma^{-1} L_{K(1)}S^0.$$
Back to the pullback square: we can fill in some of the equalities

$$\begin{tikzcd}
L_{K(1)}S^0 \vee \Sigma^{-1}L_1S^0 \vee \Sigma^{-3}L_0S^0 \vee \Sigma^{-4}L_0S^0 \arrow[r, "\simeq"] & L_1L_K(2)S^0 \arrow[r] \arrow[d] & L_{K(1)}L_{K(2)}S^0 \arrow[d] \\
& L_0L_{K(1)}S^0 \arrow[r] \arrow[d, equal] & L_0L_{K(1)}L_{K(0)}S^0 \\
& H\mathbb{Q}_p \vee \Sigma^{-1}H\mathbb{Q}_p \vee \Sigma^{-3}H\mathbb{Q}_p \vee \Sigma^{-1}H\mathbb{Q}_p &
\end{tikzcd}$$

Recall also that $L_0S^0 = H\mathbb{Q}$.

Theorem 7.5 (Hopkins' Chromatic Splitting Conjecture)**.**

(a) The image of any element under the inclusion
$$\Lambda(\zeta_n, x_3, \ldots, x_{2n-1}) \hookrightarrow H^*(\mathbb{G}_n, E_0)$$
is a non-zero infinite cycle.

(b) The map
$$\bigvee_{i_1<\ldots<i_k} L_{n-i_k}S^r \xrightarrow{\simeq} L_{n-1}L_{K(n)}S^0$$
is an equivalence, where $r = \sum_t |x_{2i_t} - 1|$

(c) $H^*(\mathbb{S}_n, \mathbb{Z}_p) \cong H^*(\mathbb{G}_n, W) \xrightarrow{\cong} H^*(\mathbb{G}_n, E_0)$ *for all* n *and all* p.

Question 7.6. The right-hand-side of the last part of the conjecture is part of the second page of a spectral sequence with $E_2^{*,0} = H^*(\mathbb{G}_n, E_0)$. Is $E_\infty^{s,0} = 0$ for $s > n$.

Let's consider the case $n = p = 2$. What is $L_{K(1)}L_{K(2)}S^0$? We have a diagram

$$\begin{tikzcd}
& & \mathbb{Z}_2^\times/C_2 \cong \mathbb{Z}_2 \\
\mathbb{G}_n \arrow[r, "\det"] \arrow[urr, bend left, "\zeta_2"] \arrow[drr, bend right, "\chi"'] & \mathbb{Z}_2^\times \arrow[ur] \arrow[dr] & \\
& & C_2 = \mathbb{Z}/2
\end{tikzcd}$$

In this case, we have a new cohomology class

$$\chi \in H^1(\mathbb{G}_2, \mathbb{Z}/2).$$

$$\begin{tikzcd}
H_1(\mathbb{G}_2, \mathbb{Z}/2) \arrow[r] \arrow[d, "\beta"] & H^1(\mathbb{G}_2, E_0/2) \arrow[d, "\beta"] & \\
H^2(\mathbb{G}_2, \mathbb{Z}/2) \arrow[r] & H^2(\mathbb{G}_2, E_0/2) \arrow[r, Rightarrow] & \pi_* L_{K(2)} S/p
\end{tikzcd}$$

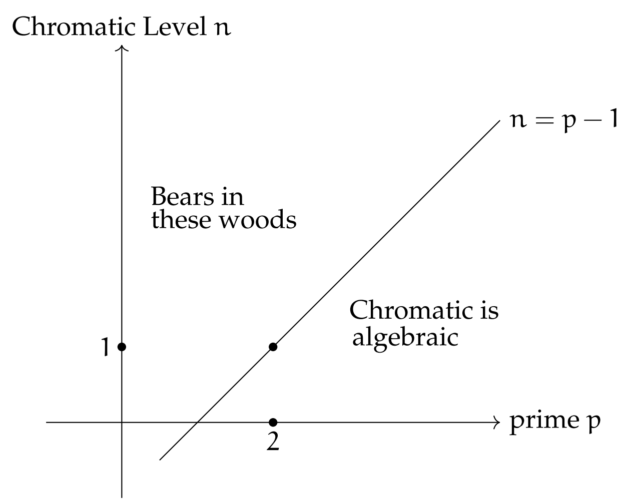
Theorem 7.7 (Beaudry–Goerss–Henn)**.**

(a) $H^*(\mathbb{G}_0, W) \xrightarrow{\cong} H^*(\mathbb{G}_2, E_0)$

(b) $E_\infty^{s,0} = 0$ *for* $s > n^2 = 4$.

(c) $E_\infty^{*,0} \cong \Lambda(\mathbb{Z}_2, e) \oplus \mathbb{Z}/2 \cdot \widetilde{x} \oplus \mathbb{Z}/2\widetilde{x} \cdots \zeta_2$ *with* $|e| = 3$.

(d) $L_{K(1)} L_{K(2)} S^0 \simeq L_{K(1)} S^0 \vee L_{K(1)} S^{-1} \vee L_{K(1)} \Sigma^{-2} S/p \vee L_{K(1)} \Sigma^{-3} S/p$



8 PICARD GROUPS AND DUALITY

Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category.

Definition 8.1. On object M in $\mathbf{C}$ is **invertible** if there is some other object N such that $M \otimes N = I$.

Definition 8.2. The **Picard group** of $\mathbf{C}$ is the group $\mathrm{Pic}(\mathbf{C})$ of isomorphism classes of invertible objects in $\mathbf{C}$ with group operation $\otimes$ and unit $[I]$.

If $\mathbf{C}$ is such that this is genuinely a group (the isomorphism classes of objects form a set), then this is an abelian group.

Example 8.3. In the category $(\mathbf{Vect}_k, \otimes_k, k)$, an object is invertible if and only if it is one-dimensional and so isomorphic to k. Therefore, $\mathrm{Pic}(\mathbf{Vect}_k)$ is trivial.

Example 8.4. If $\mathbf{C}$ is the category of graded k-vector spaces, then an object is invertible if and only if it has dimension 1, but it may live in any degree. In this case, $\mathrm{Pic}(\mathbf{C}) = \{\Sigma^m k \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$.

Example 8.5. If $\mathbf{C}$ is the category of vector bundles over a space X, then invertible objects are line bundles on X. If we take $X = \mathbb{CP}^1$, the Picard group is cyclic, generated by the tautological line bundle.

8.1 PICARD GROUPS IN STABLE HOMOTOPY

Fact 8.6. *In the category* $(\mathbf{Sp}, \wedge, S^0)$*,*

$$\mathrm{Pic}(\mathbf{Sp}) = \{S^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Proof. By passing from spaces to spectra, we have inverted the spheres, so we know that S^n are invertible for all $n \in \mathbb{Z}$. On the other hand, let $X \in \mathrm{Pic}(\mathbf{Sp})$. Then there is some Y such that $X \wedge Y \simeq S^0$. If k is a field, there is a Künneth isomorphism

$$Hk_*X \otimes_k Hk_*Y \cong Hk_*(X \wedge Y) \simeq Hk_*(S^0) = k.$$

Therefore Hk_*X is an invertible graded k-vector space, for all fields k. Then the universal coefficient theorem shows that

$$H\mathbb{Z}_*X \cong \Sigma^m \mathbb{Z}.$$

To prove that $\Sigma^m\mathbb{Z} \simeq S^0$, consider the Postnikov tower of S^0:

$$\begin{tikzcd}
& & \vdots \arrow[d] & \\
& & \tau_{\leq 2}S^0 \arrow[d] & \\
& & \tau_{\leq 1}S^0 \arrow[d] & \\
\Sigma^{-m}X \arrow[r] \arrow[uurr, dashed] \arrow[urr, dashed] & H\mathbb{Z} \arrow[r, equal] & \tau_{\leq 0}S^0 & S^0 \arrow[l] \arrow[ul] \arrow[uul]
\end{tikzcd}$$

The obstructions to lifting the map vanish, and therefore $\Sigma^m X \to S^0$ is an equivalence. ☐

So this isn't that interesting, but we don't know a lot about the category of spectra. Instead, we will consider the chromatic picture. Much of this appears in a paper of Hopkins–Mahowald–Sadofsky.

The Picard groups we look at are the Picard groups of $K(n)$-local spectra

Definition 8.7.

$$\mathrm{Pic}_n := \mathrm{Pic}(\mathbf{Sp}_{K(n)}) = \{Z \in \mathbf{Sp}_{K(n)} \mid \exists\, Y \in \mathbf{Sp}_{K(n)}, L_{K(n)}(Z \wedge Y) \simeq S^0_{K(n)}\}/\simeq_{K(n)} .$$

where $\mathbf{Sp}_{K(n)} = L_{K(N)}\mathbf{Sp}$.

We might also consider

$$\mathrm{Pic}(\mathbf{Sp}_n) = \mathrm{Pic}(L_{E_n}\mathbf{Sp}).$$

Theorem 8.8 (Hopkins–Mahowald–Sadofsky)**.** *The following are equivalent:*

(a) $Z \in \mathrm{Pic}_n$,

(b) $\dim_{K(n)_*} K(n)_*Z = 1$,

(c) $K(n)_*Z \in \mathrm{Pic}(\mathbf{Sp}_{K(n)})$.

Proof. For the forward direction, use the Künneth isomorphism,

$$K(n)_* = K(n)_*S^0 \simeq K(n)_*Z \otimes_{K(n)_*} K(n)_*(Z^{-1})$$

Hence, $K(n)_*Z$ is an invertible graded $K(n)_*$-module. Notice that $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ is a graded field, so its invertible modules are one-dimensional.

Conversely, claim that the inverse of Z is the **Spanier–Whitehead dual** of Z, $D_n(Z) = F(Z, S^0_{K(n)})$, where $F(A,B)$ is the function spectrum representing the cohomology theory

$$X \mapsto [X \wedge A, B].$$

Consider the evaluation map

$$\mathrm{ev}\colon Z \wedge F(Z, S^0_{K(n)}) \to S^0_{K(n)}.$$

Claim that it is a $K(n)_*$ isomorphism, and therefore a $K(n)$-equivalence. To see this, notice that

$$K(n)_*(F(Z, S^0_{K(n)})) = \mathrm{Hom}_{K(n)_*}(K(n)_*Z, K(n)_*),$$

and then $K(n)_*\mathrm{ev}$ is the same as $\mathrm{ev}(K(n)_*Z)$, which is an isomorphism by the assumption on Z. □

8.2 The Picard group of spectra localized at E_n

Theorem 8.9 (Hopkins–Mahowald–Sadofsky)**.** *The following are equivalent:*

1. $Z \in \mathrm{Pic}_n$,
2. $(E_n)_*Z \in \mathrm{Pic}((E_n)_*) = \mathbb{Z}/2$,
3. $(E_n)_*Z$ *lies in the Picard group of continuous, graded* $(\mathbb{G}, E_*)$*-bimodules.*

Recall that $(E_n)_* = E_0[u^{\pm 1}]$, where $E_0 = W[\![u_1, \ldots, u_{n-1}]\!]$. This has a maximal ideal $M = \langle p, u_1, \ldots, u_{n-1}\rangle$ and $E_0/M = \mathbb{F}_{p^n}$.

The proof of this theorem, using the previous, is sort of like a Hensel's lemma argument. Importantly, E_0 is a complete local ring, and the Picard group of such a ring is trivial. There is a short exact sequence

$$0 \to \mathrm{Pic}(E_0) \to \mathrm{Pic}((E_n)_*) \to \mathbb{Z}/2 \to 0$$

Remark 8.10. Because it is cumbersome, we will drop the subscript n and write E instead of E_n.

Notice that $\pi_*(L_{K(n)}(E \wedge Z)) = (E_n)_*Z$, and the homotopy group $\pi_*(L_{K(n)}(E \wedge Z))$ has an action of $\mathbb{G}_n$. This is the idea behind second equivalence in the statement of the theorem.

Definition 8.11. Let $\mathrm{Pic}^{\mathrm{alg}}_n$ be the Picard group of continuous, graded $(\mathbb{G}, E_*)$-bimodules.

There is a map $\xi_n \operatorname{Pic}_n \to \operatorname{Pic}_n^{alg}$ given by $\xi_n(Z) = E_*Z$.

$$\begin{tikzcd}
& & \operatorname{Pic}_n \arrow[r, "\xi_n"] & \operatorname{Pic}_n^{alg} & \\
1 \arrow[r] & \kappa_n \arrow[r] & \operatorname{Pic}_n^0 \arrow[u] \arrow[r, "\xi_n^0"] & \operatorname{Pic}_n^{alg,0} \arrow[u] \arrow[r, equal] & H_c^1(\mathbb{G}, E_0^\times)
\end{tikzcd} \tag{8.1}$$

where $\kappa_n = \ker(\xi_n^0)$ is the **exotic Picard group**. In general, ξ_n^0 is not known to be surjective.

Theorem 8.12 (Hopkins–Mahowald–Sadofsky). *$\kappa_n = 0$ if $(p-1)$ does not divide n and $n^2 \leq 2(p-1)$.*

Let's investigate the diagram (8.1). First assume $n = 1$. In this case, $E = E_n = K_p^\wedge$, and $\mathbb{G} = \mathbb{Z}_p^\times$ acts by the Adams operations. Moreover, $E_0 \cong \mathbb{Z}_p$, so

$$H_c^1(\mathbb{Z}_p^\times, E_0^\times) = \operatorname{Hom}^c(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

If p is odd, then $\mathbb{Z}_p^\times = \overline{\langle l \rangle}$, and

$$\operatorname{Pic}_1^{alg,0} = \mathbb{Z}_p^\times = \mathbb{Z}/\langle p-1 \rangle \times \mathbb{Z}_p.$$

If $p = 2$, then $\mathbb{Z}_p^\times = \mathbb{Z}/2 \times \overline{\langle \ell \rangle}$, and

$$\operatorname{Pic}_1^{alg,0} = \mathbb{Z}_2^\times \times \mathbb{Z}/2 = \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

In both cases, ξ_n^0 is surjective. We search for a splitting of ξ_0^1. If p is odd:

$$S^0_{K(1)} \to K_p^\wedge \xrightarrow{\psi^\ell - 1} K_p^\wedge$$

$$X_\lambda \to K_p \xrightarrow{\psi^\ell - 1} K_p$$

X_λ is the fiber of the map $\psi^\ell - 1$ for $\lambda \in \mathbb{Z}_p = \{\text{units} \equiv_{(p)} 1\}$.

So when p is odd, $\kappa_1 = 0$. When $p = 2$, $\kappa_1 = \mathbb{Z}/2$, which is generated by $L_{K(1)}DQ$, where Q is the **question mark complex** and D is the Spanier–Whitehead dual.

If instead $n > 1$, things become more complicated. When $n = 2$, $\operatorname{Pic}_2^{alg,0}$ has been computed by Hopkins for $p > 3$, Karamonov for $p = 3$, and Henn for $p = 2$. When $p \geq 3$, ε_2 is onto, and $\kappa_2 = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $p = 3$ (due to Goerss–Hopkins–Miller–Ravenel) and for $p = 2$, $\kappa_2 \supseteq \mathbb{Z}/8$ (due to Beaudry–Bobkova–Goerss–Hopkins).

Conjecturally, the exotic Picard group κ_n is related to the Brown–Commenetz duals of M_n.

Definition 8.13. The **Brown–Comenetz spectrum** $I_{\mathbb{Q}/\mathbb{Z}}$ is the spectrum representing the cohomology theory $X \mapsto \mathrm{Hom}_{\mathbb{Z}}(\pi_* X, \mathbb{Q}/\mathbb{Z})$.

Fact 8.14.

$$\pi_n I_{\mathbb{Q}/\mathbb{Z}} = \begin{cases} \mathbb{Q}/\mathbb{Z} & n = 0, \\ (\pi_{-n} S^0)^\vee & n < 0. \end{cases}$$

Theorem 8.15 (Goerss–Hopkins)**.** $I_n = F(M_n S^0, I_{\mathbb{Q}/\mathbb{Z}}) \in \mathrm{Pic}_n$.

Conjecture 8.16. *If κ_n is nontrivial, then a suitable shift of I_n is a nontrivial element of κ_n.*