Homotopy Theory Summer Berlin: Equivariant Homotopy Theory and K-theory

Summer School Notes

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Abstract

This document contains live-TeX-ed notes from a series of three summer-school lectures delivered during the first week of the Homotopy Theory Summer Berlin from 18-22 June 2018. The original abstracts for the lecture series are repeated below.

These notes were lightly edited for grammar, spelling, and some of the more obvious mathematical errors, but I'm certain that errors and omissions remain. If you spot any, I would be grateful if you could send me an email at dmehrle@math.cornell.edu.

Applications of TC **and cyclotomic spectra**
Akhil Mathew

The cyclotomic trace $\mathrm{K} \to \mathrm{TC}$ plays an important role in numerous important computations and structural features of algebraic K-theory. The use of such trace methods arises largely from the theorem of Dundas–Goodwillie–McCarthy, which states that relative K-theory and relative TC agree for a nilpotent ideal. In practice, while the definition of TC is more complicated than that of K-theory (at least in the p-adic case), the theory has simpler formal properties and is often easier to compute. In these lectures, I'll review some of the landscape surrounding these ideas (e.g., aspects of the theory of cyclotomic spectra), and describe an extension of the Dundas–Goodwillie–McCarthy theorem to the setting of Henselian pairs. A consequence is that for reasonably finite p-adic rings, the cyclotomic trace is always a p-adic equivalence in large enough degrees. The new results here are joint with Dustin Clausen and Matthew Morrow.

Global Homotopy Theory
Stefan Schwede

Global homotopy theory studies equivariant phenomena that exist for all compact Lie groups in a uniform way. In this series of talks I present a rigorous formalism for this and discuss examples of global homotopy types. The emphasis will be on stable global homotopy theory, and the precise implementation proceeds via a new model structure on the category of orthogonal spectra, with ″global equivalences″ as weak equivalences. Looking at orthogonal spectra through the eyes of global equivalences leads to a rich algebraic structure on equivariant homotopy groups, including restriction maps, inflation maps and transfer maps. Many interesting global homotopy types support additional ultra-commutative multiplications, and these gives rise to power operations that interact nicely with the other structure. The localization of orthogonal spectra at the class of global equivalences gives a tensor triangulated category much finer than the traditional stable homotopy category of algebraic topology. Some examples of global homotopy types that I plan to discuss are:

- global ′Borel type′ cohomology theories,
- Eilenberg–MacLane spectra of global Mackey functors,
- global Thom spectra that represent bordism of G-manifolds, respectively a localized ′stable′ version thereof,
- global equivariant forms of K-theory.

Assembly maps and trace methods
Marco Varisco

Assembly maps are important tools in the study of algebraic K-theory of group rings, and a seminal conjecture of Farrell and Jones predicts that a certain assembly map is a weak equivalence. Trace maps from algebraic K-theory to related theories such as topological Hochschild homology (THH) or topological cyclic homology (TC) can be successfully used to prove injectivity results about assembly maps. In fact, TC and the cyclotomic trace were invented by Bökstedt, Hsiang, and Madsen precisely to prove the algebraic K-theory Novikov conjecture, i.e., the rational injectivity of the classical assembly map. In this series of lectures, I will introduce assembly maps and the Farrell–Jones conjecture, and briefly survey the main applications and the current status of this conjecture. Then I will explain the proof of the Bökstedt–Hsiang–Madsen theorem and of its generalization to the Farrell–Jones assembly map obtained in joint work with Lück, Reich, and Rognes, which relies on a quite complete picture of the behavior of assembly maps in THH, TC, and related theories.

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1 APPLICATIONS OF TC AND CYCLOTOMIC SPECTRA

If R is a ring, we associate to R the connective algebraic K-theory spectrum $K(R)$ where $\pi_0 K(R) = K_0(R)$ is the Grothendieck group of finitely generated projective R-modules. In particular,

$$\Omega^\infty K(R)_{\geq 1} \simeq B\,GL_\infty^+(R)$$

is the Quillen plus-construction of algebraic K-theory. Despite the fact that this has been around since the 1970's, algebraic K-theory is hard to compute. For example, we still don't know $K_n(\mathbb{Z})$ completely: it is conjectured that $K_{4n}(\mathbb{Z}) = 0$ but not known.

Why is algebraic K-theory hard?

(1) It's not easy to access $GL_\infty(R)$; $R \mapsto K(R)$ doesn't commute with geometric realization.

(2) $K(R)$ doesn't satisfy étale descent, which many theories of cohomology on rings or schemes do.

There is another invariant of rings, **topological cyclic homology** $TC(R)$. This was constructed by Bökstedt–Hsiang–Madsen in 1993. Although $TC(R)$ is much harder to define than K-theory, it is much better to work with due to its formal properties.

One reason for interest in topological cyclic homology is due to Bhatt–Morrow–Scholze, who show that TC is built from p-adic cohomology theories.

These two invariants are related by a **cyclotomic trace map** $K(R) \to TC(R)$.

Remark 1.1. We know that

$$TC(R)_p^\wedge \simeq (TC(R_p^\wedge))_p^\wedge.$$

On the other hand,

$$K(\mathbb{Z})_p^\wedge \neq K(\mathbb{Z}_p)^\wedge.$$

However, if we restrict our attention to p-adic K-theory of p-complete rings, then $K_p^\wedge$ is close to $TC_p^\wedge$.

Theorem 1.2 (Dundas–Goodwillie–McCarthy). *If $I \subseteq R$ is a nilpotent ideal, then form the **relative** K**-theory***

$$K(R, I) = \mathrm{hofib}(K(R) \to K(R/I))$$

*and **relative topological cyclic homology***

$$TC(R, I) = \mathrm{hofib}(TC(R) \to TC(R/I))$$

then the trace map $\mathrm{tr}\colon K(R, I) \to TC(R, I)$ is a weak equivalence.

Example 1.3. We can use this to compute $K(\mathbb{Z}_p)_p^\wedge$ as a spectrum.

Example 1.4 (Hesselholt–Madsen)**.** We can use this theorem to compute $K(\mathcal{E})_p^\wedge$, where $\mathcal{E}$ is a finite extension of $\mathbb{Q}_p$.

In this lectures, we will describe the techniques used in these theorems and examples.

Theorem 1.5 (Clausen–Mathew–Morrow)**.** *If R is a (reasonably finite) p-complete ring, then the completion of the trace map*

$$K(R)^\wedge \to TC(R)^\wedge$$

is an isomorphism in large enough degree.

1.1 Hochschild homology and cyclic homology

Fix a field k and let A be a commutative k-algebra.

Definition 1.6. The **Hochschild complex** of A with respect to k is the derived tensor product

$$HH(A/k) = A \otimes^L_{A \otimes_k A} A.$$

The homology groups of this are $HH_*(A/k)$.

This definition agrees with the cyclic bar construction from Marco Varisco's talks.

Recall that $\Omega^1_{A/k}$ is the A-module generated by symbols dx for $x \in A$, modulo the relation $d(xy) = x\,dy + y\,dx$. $\Omega^n_{A/k}$ is the n-th exterior power of this module.

Theorem 1.7 (Hochschild–Kostant–Rosenberg)**.** *If A is a smooth commutative k-algebra, then there is an equivalence of graded vector spaces*

$$HH_*(A/k) \cong \Omega^*_{A/k}.$$

Exercise 1.8. Prove the Hochschild–Kostant–Rosenberg theorem when $A = k[x_1, \dots, x_n]$ is a polynomial ring.

$HH(A/k)$ carries an action of S^1. Explicitly, we have homomorphisms

$$C_*(S^1) \otimes_k HH(A/k) \to HH(A/k).$$

Taking the cup product with the fundamental class $\varepsilon \in H_1(S^1)$, we have a map

$$B\colon\ HH_n(A/k) \to HH_{n+1}(A/k)$$

for each n. This map B is a differential, going in the opposite direction of the Hochschild differential. This is called the **Connes–Tsygan differential**.

Corollary 1.9. *The chain complex* $(\mathrm{HH}_*(A/k), B)$ *is isomorphic to the algebraic de Rham complex* $(\Omega^*_{A/k}, d)$.

Since A is commutative, $A \otimes^L_{A \otimes_k A} A$ is naturally an E_∞-k-algebra; it's the homotopy pushout fitting into the diagram

$$\begin{tikzcd} A \otimes_k A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & \mathrm{HH}(A/k) \end{tikzcd}$$

In fact, there is an S^1-action on $\mathrm{HH}(A/k)$ in the world of E_∞-k-algebras.

Theorem 1.10 (McClure–Schwänzel–Vogt)**.** *The* S^1*-action on* $\mathrm{HH}(A/k)$ *is universal in the sense that if* E *is any* E_∞-k*-algebra with* S^1 *action, then*

(a) $\mathrm{Hom}_{S^1}(\mathrm{HH}(A/k), E) \cong \mathrm{Hom}(A, E)$,

(b) $\mathrm{Hom}_{E_\infty/k}(\mathrm{HH}(A/k), E) \simeq \mathrm{Hom}(A, E)^{S^1}$.

The category of commutative dg-k-algebras is tensored over simplicial sets in such a way that the Hochschild complex is the tensoring of a commutative algebra with the simplicial circle; we write $\mathrm{HH}(A/k) = S^1 \otimes A$. In this way, $\mathrm{HH}(A/k)$ carries an action of the simplicial circle.

Definition 1.11. Let A be a commutative k-algebra. The **negative cyclic homology** of A with respect to k is

$$\mathrm{HC}^-(A/k) = \mathrm{HH}(A/k)^{hS^1}.$$

The **periodic cyclic homology** of A with respect to k is

$$\mathrm{HP}(A/k) = \mathrm{HH}(A/k)^{tS^1}.$$

If the characteristic of k is zero, then HP is a form of 2-periodic de Rham cohomology.

$$\mathrm{HP}(A/k) \simeq \bigoplus_{i \in \mathbb{Z}} H_{dR}(A)[2i]$$

1.2 Topological Hochschild and topological cyclic homology

Let's remove the assumption that k is a field.

Definition 1.12. Let k be a commutative ring and let A be a commutative k-algebra, and define the **Shukla homology** of A with respect to k:

$$\mathrm{HH}(A/k) = A \otimes^L_{A \otimes^L_k A} A.$$

We may even remove the assumption that these objects are discrete rings, instead of spectra. If we take k to be the sphere spectrum $\mathbb{S}$, we get topological Hochschild homology.

Definition 1.13. Let A be a commutative ring. Then the **topological Hochschild homology** of A is
$$\mathrm{THH}(A) = HA \wedge_{HA \wedge HA} HA,$$
where HA is the Eilenberg–MacLane spectrum of A.

Remark 1.14. We also use the alternative notation $\mathrm{HH}(A/\mathbb{S})$ for $\mathrm{THH}(A)$.

Much of the previous discussion carries over. $\mathrm{THH}(A)$ is an E_∞-ring spectrum with S^1-action such that for any ring spectrum B,

$$\mathrm{Hom}_{E_\infty}(\mathrm{THH}(A), B) \simeq \mathrm{Hom}_{E_\infty}(A, B).$$

Theorem 1.15 (Bökstedt)**.** *If k is a perfect field of characteristic p, then*

$$\mathrm{THH}(k)_* = k[\sigma]$$

with $|\sigma| = 2$.

Example 1.16. When working over $\mathbb{Z}$ instead of $\mathbb{S}$, we get a divided power algebra:
$$\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \Gamma(\sigma),$$
with $|\sigma| = 2$.

Theorem 1.17 (Hesselholt)**.** *Let k be a perfect field of characteristic p and let R be a smooth k-algebra. Then*

$$\mathrm{THH}(R)_* = k[\sigma] \otimes_k \Omega^*_{R/k}.$$

In general if R' is any k-algebra, then $\mathrm{THH}(R')$ is a $\mathrm{THH}(k)$-module, so has an action of σ. If we quotient by this action, then we are left with ordinary Hochschild homology.

$$\mathrm{THH}(R')/\sigma \simeq \mathrm{HH}(R'/k)$$

Recall that $\mathrm{HH}(R/k)$ has an action of S^1, and so we can build HC^- and HP. Likewise, $\mathrm{THH}(R)$ has additional structure.

Bökstedt–Hsiang–Madsen: whenever $C \subseteq S^1$ is a finite subgroup, we can make sense of the C-fixed points of $\mathrm{THH}(R)$ in the sense of genuine equivariant homotopy theory. This construction is written $\mathrm{THH}(R)^C$.

We consider the cyclic subgroups C_{p^n} for a prime p. The set

$$\left\{\mathrm{THH}(R)^{C_{p^n}} \mid n \in \mathbb{N}, p \text{ prime}\right\}$$

is equipped with three maps, called restriction, Frobenius, and Verschiebung. The restriction and Frobenius maps make this into an inverse system, and we define **topological cyclic homology**

$$\mathrm{TC}(R) := \varprojlim_{\text{res, Frob}} \mathrm{THH}(R)^{C_{p^n}}.$$

Theorem 1.18 (Blumberg–Mandell)**.** *There is a good category* **CycSp** *of **cyclotomic spectra** containing* THH(R)*; we may define*

$$\mathrm{TC}(R) = \mathrm{Hom}_{\mathbf{CycSp}}(1, \mathrm{THH}(R)).$$

If $\mathbf{C}$ is an ∞-category and G is a group, we can form the category $\mathbf{Fun}(BG, \mathbf{C})$. Objects of this category are considered objects of $\mathbf{C}$ with G-action.

Definition 1.19. Given a spectrum X with an action of S^1, the **Tate construction** is

$$X^{tC_p} = \mathrm{hocofib}(X_{hC_p} \to X^{hC_p}).$$

This carries an action of $S^1/C_p \simeq S^1$.

Definition 1.20 (Nikolaus–Scholze)**.** A (p-complete) **cyclotomic spectrum** is a spectrum X with an action of S^1 together with maps $X \to X^{tC_p}$, equivariant with respect to identification of S^1 with S^1/C_p.

Theorem 1.21 (Nikolaus–Scholze)**.** *The category of cyclotomic spectra is a lax equalizer*

$$\mathbf{CycSp} \simeq \mathrm{LEq}\left(\mathbf{Sp}^{BS^1} \underset{(-)^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbf{Sp}^{BS^1} \right)$$

Definition 1.22. Let R be a commutative ring. Then define the **negative topological cyclic homology**

$$\mathrm{TC}^-(R) = \mathrm{THH}(R)^{hS^1},$$

and the **topological periodic homology**

$$\mathrm{TP}(R) = \mathrm{THH}(R)^{tS^1}.$$

In general, there is a canonical map $\mathrm{can}\colon \mathrm{TC}^-(R) \to \mathrm{TP}(R)$. There is another map $\phi\colon \mathrm{TC}^-(R) \to \mathrm{TP}(R)$ defined as follows: the cyclotomic structure on THH gives a map

$$\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}.$$

Taking S^1-homotopy invariants, we get

$$\mathrm{TC}^-(R) \to (\mathrm{THH}(R)^{tC_p})^{h(S^1/C_p)} \simeq_{(p)} \mathrm{TP}(R),$$

where the equivalence holds p-adically.

Theorem 1.23 (Nikolaus–Scholze). *The* p*-complete topological cyclic homology of a ring spectrum* R *is the equalizer of the two natural maps* $\mathrm{TC}^{-}(R) \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(R)$:

$$\mathrm{TC}(R) = \mathrm{eq}\left(\mathrm{TC}^{-}(R) \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(R)\right)$$

We may equivalently define this as the homotopy fiber of the difference of these two maps

$$\mathrm{TC}(R) = \mathrm{hofib}\left(\mathrm{can} - \phi\colon\ \mathrm{TC}^{-}(R) \to \mathrm{TP}(R)\right).$$

Remark 1.24. The Nikolaus–Scholze definition also agrees with the approach via G-spectra in the case of bounded below objects, due to Ayala–Mazel-Gee–Rozenblyum.

Fact 1.25.

(a) TC *is a functor from rings to spectra, and may be extended to a functor from connective ring spectra to spectra.*

(b) In fact, $\mathrm{TC}(R)$ *naturally takes values in* (-1)*-connected spectra in the* p*-complete case.*

(c) TC *comutes with geometric realizations.*

(d) TC/p *commutes with filtered colimits.*

Note that (b) is not true for HC^{-}; $\mathrm{HC}^{-}_{n}(R)$ may be nonzero for $n \leq -1$.

1.3 The cyclotomic trace

Theorem 1.26 (Bökstedt–Hsiang–Madsen). *Let* R *be a commutative ring spectrum. There is a natural map*

$$K(R) \to \mathrm{TC}(R),$$

called the ***cyclotomic trace map****.*

The original motivation for this was to prove things about the assembly maps of K-theory of group rings.

Definition 1.27 (Notation). If $F\colon \mathbf{Ring} \to \mathbf{Sp}$ is a functor, and R is a ring with ideal I, then define $F(R, I) := \mathrm{hofib}(F(R) \to F(R/I))$.

Theorem 1.28 (Dundas–Goodwillie–McCarthy). *If* R *is a ring with nilpotent ideal* I*, then the cyclotomic trace map*

$$K(R, I) \to \mathrm{TC}(R, I)$$

is a weak equivalence.

Remark 1.29. Equivalently, there is a homotopy Cartesian square

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(R/I) \arrow[r] & TC(R/I), \end{tikzcd}$$

in which case we may extend the theorem to include all connective commutative ring spectra R.

An application of this theorem is the following.

Theorem 1.30 (Hesselholt–Madsen)**.** *Let* k *be a perfect field. Then*

$$K_n^{\wedge}\left({}^{k[x]}/_{\langle x^n\rangle}\right) \cong \begin{cases} \mathbb{Z}_p & (n=0), \\ 0 & (n=2k, k>1), \\ {}^{W_{nj}(k)}/_{V_n W_j(k)} & (n=2k+1). \end{cases}$$

We won't define the Witt vectors $W_i(k)$ or the **Verschiebung maps** V_n, but suffice to say that the above theorem is an explicit computation of the K-groups of this truncated polynomial algebra.

Recall that we defined $F(R, I)$ for R a ring, I an ideal of R, and $F\colon \mathbf{Ring} \to \mathbf{Sp}$. We might ask: does $F(R, I)$ only depend on I as a non-unital ring? If $(R, I) \to (S, J)$ such that $I \cong J$, then do we have $F(R, I) \cong F(S, J)$. This is the question of excision.

Note that K-theory does *not* satisfy excision.

Theorem 1.31 (Cortiñas, Geisser–Hesselholt, Dundas–Kittang)**.** *The functor* $\mathrm{hofib}(K \to TC)$ *satisfies excision, and so defines a functor from non-commutative rings to spectra.*

1.4 K-THEORY AND THH OF $\mathbb{F}_p$-ALGEBRAS

A lot of the results in this section are due to Geisser–Levine and Geisser–Hesselholt.

Let k be a perfect field of characteristic p.

Theorem 1.32 (Quillen, Hiller, Ktratzer)**.**

$$K_n^{\wedge}(k) = \begin{cases} \mathbb{Z}_p & (n=0), \\ 0 & (n>0). \end{cases}$$

This theorem is an application of the Adams operations ψ^p.

Let's now compute $TC(k)$ when $k = \mathbb{F}_p$. Recall that

$$TC(k) = \mathrm{eq}\left(TC^-(k) \rightrightarrows TP(k)\right),$$

and recall that $THH(k)_* = k[\sigma]$ with $|\sigma| = 2$.

Example 1.33. In the case that $k = \mathbb{F}_p$, we can actually describe the spectrum $THH(\mathbb{F}_p)$.

$$THH(\mathbb{F}_p) \simeq \tau_{\geq 0}(\mathbb{Z}^{tC_p})$$

where the right hand side carries an action of $S^1/C_p \simeq S^1$.

$$TC^-(\mathbb{F}_p)_* = {}^{\mathbb{Z}_p[x,\sigma]}\!\big/_{x\sigma = p}$$

$$TP(\mathbb{F}_p)_* = \mathbb{Z}_p[x^{\pm 1}]$$

The two maps here are given as follows:

$$\begin{aligned} TC^-(\mathbb{F}_p) &\xrightarrow{\mathrm{can}} TP(\mathbb{F}_p) \\ x &\longmapsto x \\ \sigma &\longmapsto px^{-1} \end{aligned}$$

$$\begin{aligned} TC^-(\mathbb{F}_p) &\xrightarrow{\phi} TP(\mathbb{F}_p) \\ x &\longmapsto px \\ \sigma &\longmapsto x^{-1} \end{aligned}$$

Then $TC(\mathbb{F}_p)$ is the equalizer of these two maps, so

$$TC_n(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p & (n = 0, -1), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the cyclotomic trace map

$$K_*^{\wedge}(\mathbb{F}_p) \to TC_*^{\wedge}(\mathbb{F}_p)$$

is an isomorphism in positive degrees.

The failure of the trace map to be a p-adic equivalence is because $\mathbb{F}_p$ is not algebraically closed. When we pass to the algebraic closure, this oddity vanishes (p-adically).

Fact 1.34.

$$TC_n(k) = \begin{cases} \mathbb{Z}_p & n = 0, \\ \mathrm{coker}(F-1)\colon W(k) \to W(k) & n = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Therefore,

$$\mathrm{TC}_n(\overline{\mathbb{F}}_p)_p^\wedge = \begin{cases} \mathbb{Z}_p & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now let R be a k-algebra. We may form the algebraic de Rham complex $(\Omega^*_{R/k}, d)$.

Fact 1.35 (Grothendieck)**.** *When* $k = \mathbb{C}$*,* R *is a smooth* $\mathbb{C}$*-algebra, then*

$$H^*(\Omega^*_{R/\mathbb{C}}) \cong H^*_{sing}(\mathrm{Spec}(R)(\mathbb{C}); \mathbb{C})$$

When working over a finite field, the de Rham cohomology is much larger because of the Frobenius homomorphism.

Example 1.36. When $R = \mathbb{F}_p[t]$, then

$$H^0_{dR}(R) = \mathbb{F}_p[t^p]$$

$$H^1_{dR}(R) = \mathbb{F}_p[t^p]t^{p-1}\,dt$$

For $R/\mathbb{F}_p$ smooth, we can completely describe the de Rham cohomology of R:

Definition 1.37. The **Cartier operator** is a homomorphism

$$c^{-1} \colon \Omega^*_{R/\mathbb{F}_p} \to \Omega^*_{R/\mathbb{F}_p}/d\Omega^*_{R/\mathbb{F}_p}$$

given by

$$c^{-1}(a) = a^p$$
$$c^{-1}(db) = b^{p-1}\,db$$
$$c^{-1}(a\,db_1\,db_2) = a^p(b_1b_2)^{p-1}\,db_1\,db_2$$

Fact 1.38. *The image of the Cartier operator lies in the cycles, so it is a map* $\Omega^*_{R/\mathbb{F}_p} \to H^*(\Omega^*_{R/\mathbb{F}_p})$.

Theorem 1.39 (Cartier Isomorphism)**.** *If* R *is smooth, then* $c^{-1} \colon \Omega^*_R \to H^*(\Omega^*_R)$ *is an isomorphism.*

Definition 1.40. If R is an $\mathbb{F}_p$-algebra, then we may define the **logarithmic differential forms**

$$\Omega^n_{R/\mathbb{F}_p,\log} = \ker(c^{-1} - 1) \colon \Omega^n_{R/\mathbb{F}_p} \to {}^{\Omega^n_{R/\mathbb{F}_p}}\!/\!_{d\Omega^{n-1}_{R/\mathbb{F}_p}}.$$

Example 1.41. If $x \in R^\times$, then $\frac{dx}{x}$ is an example of a logarithmic form.

Fact 1.42. *If R is a regular local ring, then $\Omega^n_{R/\mathbb{F}_p,\log}$ is the submodule of Ω^n_R generated by*

$$\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

Examples of regular local rings include the localization of $\mathbb{F}_p[t]$ at zero, or the power series ring $\mathbb{F}_p[[t]]$.

Fact 1.43 (Néron–Popescu). *Any regular local $\mathbb{F}_p$-algebra is a filtered colimit of smooth $\mathbb{F}_p$-algebras.*

Corollary 1.44. *The Cartier isomorphism works for any regular $\mathbb{F}_p$-algebra, such as a power series ring.*

Theorem 1.45 (Geisser–Levine). *If R is a regular local $\mathbb{F}_p$-algebra, then*

(a) $K_n(R)$ *has no p-torsion for any n*

(b) $K_*(R; \mathbb{Z}/p) \cong \Omega^*_{R/\mathbb{F}_p,\log}$

The proof of this follows from the key case when R is a field of characteristic p. In this case, $K_*(R)/p \cong K^{\text{Milnor}}_*(R)/p$, and the relation between Milnor K-theory and logarithmic forms follows from results of Bloch–Kato and Gabber. The isomorphism here is proved using the motivic spectral sequence.

If R is the localization of a smooth $\mathbb{F}_p$-algebra, then $K_*(R)/p$ is bounded, vanishing above the dimension of R.

Let's now describe TC for $\mathbb{F}_p$-algebras.

Definition 1.46. Define $\widetilde{\nu}^n(R) := \operatorname{coker}(1 - c^{-1})\colon \Omega^n_{R/\mathbb{F}_p} \to \Omega^n_{R/\mathbb{F}_p}/d\Omega^{n-1}_{R/\mathbb{F}_p}$.

Theorem 1.47 (Geisser–Hesselholt). *If R is any regular $\mathbb{F}_p$-algebra, then there is a short exact sequence*

$$0 \to \widetilde{\nu}^{n+1}(R) \to \pi_n(\mathrm{TC}(R)/p) \to \Omega^n_{R/\mathbb{F}_p,\log} \to 0.$$

Furthermore, if R is local, then the cyclotomic trace $K(R) \to \mathrm{TC}(R)$ induces a splitting on homotopy groups with mod p coefficients.

In this situation, can we see the Dundas–Goodwillie–McCarthy theorem? We should have an equivalence $K(R, I) \simeq \mathrm{TC}(R, I)$.

Let's try $R = \mathbb{F}_p[[t]]$ with $I = \langle t \rangle$.

Theorem 1.48 (Geisser–Hesselholt). *There is a homotopy Cartesian square*

$$\begin{tikzcd}
K(\mathbb{F}_p[[t]])^\wedge_p \arrow[r] \arrow[d] & \mathrm{TC}(\mathbb{F}_p[[t]])^\wedge_p \arrow[d] \\
K(\mathbb{F}_p)^\wedge \arrow[r] & \mathrm{TC}(\mathbb{F}_p)^\wedge
\end{tikzcd}$$

The reason that this square exists is kind of silly: we can compute everything in sight. Let's nevertheless do this computation and see the Dundas–Goodwillie–McCarthy theorem.

Consider differential forms on $\mathbb{F}_p[[t]]$.

Fact 1.49.

$$\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} \cong \mathbb{F}_p[[t]]\, dt$$

This fact is true because $\mathbb{F}_p[[t]]$ is something called "F-finite," meaning that it is finitely generated over its p-th powers. Hence, $\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p}$ is finitely generated.

Now let's compute

$$\widetilde{\nu}^1(R) = \operatorname{coker}\left(1 - c^{-1} \colon \Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} \to \Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} \big/ d\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p}\right)$$

Here, we can compute

$$(1 - c^{-1})(f(t)\, dt) = f(t)\, dt - f^p(t) t^{p-1}\, dt = (f - f^p t^{p-1})\, dt.$$

Hence, c^{-1} is topologically nilpotent on $\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p}$. The upshot is that

(1) $\pi_n(\mathrm{hofib}(K \to TC)/p) \simeq \widetilde{\nu}^{n+2}(R)$

(2) $R = \mathbb{F}_p[[t]]$, $\widetilde{\nu}^{n+2}(R) \cong \widetilde{\nu}^{n+2}(\mathbb{F}_p)$

(3) $\mathrm{hofib}(K \to TC)/p$ is the same for $\mathbb{F}_p$, $\mathbb{F}_p[[t]]$.

This argument generalizes to $\mathbb{F}_p[[t_1, \dots, t_n]]$.

Corollary 1.50. *Let $R = \mathbb{F}_p[[t_1, \dots, t_n]]/I$ for some ideal I of the power series ring. Then there is a commutative square which is homotopy Cartesian after p-completion:*

$$\begin{array}{ccc} K(R) & \longrightarrow & TC(R) \\ \downarrow & & \downarrow \\ K(\mathbb{F}_p) & \longrightarrow & TC(\mathbb{F}_p) \end{array}$$

So we can push the argument to obtain a version of Dundas–Goodwillie–McCarthy for any quotient of a power series ring, and therefore we can say things about non-local rings.

Remark 1.51.

(a) K-theory does commute with simplicial resolution for local rings; you can prove this explicitly using the Q-construction.

(b) TC always commutes with simplicial resolutions.

Proposition 1.52. *Let $R_\bullet$ be a simplicial connective E_∞-ring such that $\pi_0(R_i)$ is local for all i, and $\pi_0(-)$ applied to the simplicial maps yields local maps. Then*

$$|K(R_\bullet)| \simeq K(|R_\bullet|).$$

Proof of Corollary 1.50. Write $R = \mathbb{F}_p[[t_1, \ldots, t_n]]/I$ by generators and relations. We may choose a simplicial resolution $X_\bullet$ of R such that X_i is a formal power series ring over $\mathbb{F}_p$. Then we have a homotopy cartesian square (after p-completion) for each X_i, which in turn yields the homotopy cartesian square we desire.

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(\mathbb{F}_p) \arrow[r] & TC(\mathbb{F}_p) \end{tikzcd}$$

□

Hence, both K and TC commute with the geometric realization in this case.

Recall (Geisser–Hesselholt) that if R is a regular local ring, then

$$\pi_n(\mathrm{hofib}(K \to TC)/p) \simeq \widetilde{\nu}^{n+2}(R)$$

where

$$\widetilde{\nu}^n(R) = \mathrm{coker}\left(1 - c^{-1} \colon \Omega^n_{R/\mathbb{F}_p} \to {}^{\Omega^n_{R/\mathbb{F}_p}}/_{d\Omega^n_{R/\mathbb{F}_p}}\right).$$

Here, $1 - c^{-1}$ is étale locally surjective. Then

$$c^{-1}(a\, db) = a^p b^{p-1}\, db$$

$$1 - c^{-1} = db \to a \text{ s.t. } a - a^p b^{p-1} = 1.$$

We can solve that in the étale local topology. For $\mathbb{F}_p$-algebras, $\widehat{K}^{\text{ét}} \simeq$ TC.

1.5 K-theory and TC of Henselian pairs

Recall the Dundas–Goodwillie–McCarthy theorem.

Theorem 1.53 (Dundas–Goodwillie–McCarthy)**.** *If R is a ring and I is a nilpotent ideal of R, then there is a homotopy cartesian square*

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(R/I) \arrow[r] & TC(R/I) \end{tikzcd}$$

Equivalently $K(R, I) \simeq TC(R, I)$*, where* $F(R, I) = \mathrm{fib}(F(R) \to F(R/I))$.

Proposition 1.54. *Suppose that* $\ell \in R^\times$. *Then* $K(R)/\ell \simeq K(R/I)/\ell$.

Proof. We can see this using the Hochschild–Serre spectral sequence

$$1 \to GL_n(I) \to GL_n(R) \to GL_n(R/I) \to 1.$$

Suppose that $I^2 = 0$. Then $GL_n(I) \cong I^{n^2}$ has no mod ℓ homology. Finally,

$$H_*(GL_n(R); \mathbb{Z}/\ell) \cong H_*(GL_n(R/I); \mathbb{Z}/\ell). \qquad \square$$

Henceforth, assume that all rings are commutative.

Definition 1.55. A pair (R, I) of a ring R and an ideal $I \leq R$ is a **Henselian pair** if for all $f(x) \in R[x]$, and all $\overline{\alpha} \in R/I$ such that $f(\overline{\alpha}) = 0$ and $f'(\overline{\alpha})$ is a unit in R/I, then there is $\alpha \in R$ which lifts $\overline{\alpha}$ and $f(\alpha) = 0$.

Remark 1.56.

(a) If (R, I) is a Henselian pair, then I is necessarily contained in the Jacobson radical of R.

(b) If R is local, and I is the unique maximal ideal, then R is called a **Henselian local ring**.

(c) The lift α is necessarily unique.

(d) If (R, I) is any pair, then I nilpotent implies that the pair is Henselian.

(e) If (R, I) is any pair, and R is I-adically complete, then (R, I) is Henselian.

Example 1.57.

(a) $R = \mathbb{C}[[x]]$ and $I = \langle x \rangle$

(b) Let $R_1 \subseteq R$ be the subring given by power series which converge near zero, and let $R_2 \subseteq R_1$ be the subring of R_1 given by algebraic power series $\mathbb{C}[x]$. Then (R_1, I) and (R_2, I) are both Henselian pairs.

A statement we made earlier is true more generally for Henselian pairs.

Theorem 1.58 (Gabber, Gillet–Thomason, Suslin)**.** *Given a Henselian pair* (R, I) *and a prime* $\ell \in R^\times$, *then* $K(R)/\ell \simeq K(R/I)/\ell$.

Example 1.59. Consider the pair $(\mathbb{Z}_p, (p))$ and choose any prime $\ell \neq p$. Then

$$K(\mathbb{Z}_p)/\ell \simeq K(\mathbb{F}_p)/\ell.$$

You could imagine trying to prove this using similar group cohomology as the previous theorem,

$$1 \to GL_n(p\mathbb{Z}_p) \to GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{F}_p) \to 1.$$

One would hope that

$$H_*(GL_n(p\mathbb{Z}_p); \mathbb{Z}_\ell) = 0$$

because it is a pro-p-group, but we want the cohomology as a discrete group, not as a topological group. It's not clear if this works.

Theorem 1.60 (Suslin). *Let ℓ be a prime. If E is any algebraically closed field with characteristic not ℓ, then $(K(E)/\ell)_* = \mathbb{Z}/\ell[\beta]$ with $|\beta| = 2$.*

This somehow says that the mod ℓ K-theory of the field E doesn't depend on its topology.

We have the following generalization of the Dundas–Goodwillie–McCarthy theorem.

Theorem 1.61 (Clausen–Mathew–Morrow)**.** *If (R, I) is a Henselian pair and p a prime, then the trace map is an equivalence*

$$K(R, I)/p \xrightarrow{\simeq} TC(R, I)/p.$$

Proof outline.

(1) If R is an $\mathbb{F}_p$-algebra, then K/p and TC/p can be written explicitly in terms of logarithmic de Rham–Witt complexes.

(2) If R is a k-algebra when the characteristic of k is different from p, then this theorem follows from Gabber rigidity.

(3) Apply results and techniques of Gabber to the mod p fiber of the trace map, $\mathrm{fib}(K \to TC)/p$. □

If R is a ring, then we may consider K and TC for the entire categories of R-algebras at the same time. These form sheaves for the Nisnevich topology. To learn about these sheaves, we look at their stalks; stalks in the Nisnevich topology are Henselian local rings. This allows us to reduce to the case of fields, where we can calculate the result.

Definition 1.62. If k is a field of characteristic $p > 0$, define

$$\dim_p(k) = \log_p[k\colon k^p] = \dim_k \Omega^1_{k/\mathbb{F}_p}.$$

It is possibly infinite.

Theorem 1.63 (Clausen–Mathew–Morrow)**.** *Let R be a p-Henselian (p-complete) ring such that R/p has finite Krull dimension. Define*

$$d = \sup_{x \in \mathrm{Spec}(R/p)} \dim_p(k(x)).$$

Then $K^\wedge(R) \to TC^\wedge(R)$ is an equivalence in degrees larger than $\max(1, d)$.

By deep results of Geisser–Levine and Geisser–Hesselholt, we know that if k is a field of characteristic p, then $K^\wedge(k)_n = 0$ for $n > \dim_p(k)$ and similarly for $TC_n^\wedge(k)$.

Theorem 1.64 (Hesselholt–Madsen)**.** *If k is a perfect field and R is a (not necessarily commutative) ring which is finite as a $W(k)$-module, then*

$$K^\wedge(R) \to TC^\wedge(R)$$

is an isomorphism in nonnegative degrees.

Consider the cyclotomic trace map $K(R) \to TC(R)$. The functor $R \mapsto TC(R)$ satisfies étale descent, but $R \mapsto K(R)$ only satisfies Nisnevich descent. What is the minimal approximation to K that satisfies étale descent?

Define **étale K-theory** $K^{ét}(R)$ to be the étale Posnikov sheafification of $K(R)$. This has a local to global spectral sequence in the étale topology.

Theorem 1.65 (Clausen–Mathew–Morrow)**.** *If R is a p-Henselian ring, then $\widehat{K}^{ét}(R) \simeq TC(R)$.*

Theorem 1.66. *If $(A, \mathfrak{m})$ is Henselian and $k = A/\mathfrak{m}$ is a separated closed field of characteristic p, then*

$$K^\wedge(A) \simeq TC^\wedge(A).$$

Proof sketch. There is a homotopy Cartesian square

$$\begin{tikzcd} K^\wedge(A) \arrow[r] \arrow[d] & TC^\wedge(A) \arrow[d] \\ K^\wedge(k) \arrow[r] & TC^\wedge(k) \end{tikzcd}$$

When A is a smooth algebra over a perfect field, this reduces to a theorem to Geisser–Hesselholt. □

So far we've been thinking about the relationship between K-theory and TC, but now we'll turn to a question purely about K-theory.

Question 1.67. Let R be a ring which is I-adically complete. How close is the map $K(R) \to \varprojlim_n K(R/I^n)$ to an equivlanece?

Since rationalization is difficult to pass through an inverse limit, we will primarily consider finite coefficients.

Example 1.68. If ℓ is invertible in R, then $K(R)/\ell \simeq K(R/I)/\ell$, so the tower is constant and the map is an equivalence.

Consider instead the case of p-adic coefficients.

Definition 1.69. If A is a Noetherian $\mathbb{F}_p$-algebra, we say that A is **F-finite** if A is finitely generated as a module over its p-th powers.

Example 1.70. Any finitely generated algebra over a perfect field is F-finite. Moreover, the class of F-finite rings is closed under completions, so $\mathbb{F}_p[[t]]$ is also F-finite.

If A is F-finite, then $\Omega^1_{A/\mathbb{F}_p}$ is a finitely generated A-module. Therefore, if A is F-finite, then

$$\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} \cong \mathbb{F}_p[[t]]\, dt.$$

Theorem 1.71 (Dundas–Morrow)**.** *If A is an F-finite (Noetherian) $\mathbb{F}_p$-algebra, then*

(a) $\mathrm{HH}_i(A/\mathbb{F}_p)$ *are finitely generated A-modules.*

(b) $\mathrm{THH}_i(A/\mathbb{F}_p)$ *are finitely generated A-modules, and likewise for* TR^n.

(c) The André–Quillen cohomology rings $H_*(L_{A/\mathbb{F}_p})$ are finitely generated.

If A is I-adically complete, then

$$\mathrm{HH}(A/\mathbb{F}_p) \cong \varprojlim \mathrm{HH}({}^{A}/_{I^n}/\mathbb{F}_p)$$

and similarly for THH, TC, *and* TR^n. *In particular*

$$\mathrm{TC}^{\wedge}(A) \cong \varprojlim \mathrm{TC}({}^{A}/_{I^n}).$$

Theorem 1.72 (Clausen–Mathew–Morrow)**.** *Let R be an I-adically complete Noetherian ring, and suppose R/p is F-finite. Then*

$$K(R)/p \cong \varprojlim K(R/I^n)/p.$$

This follows from rigidity and the work of Dundas–Morrow. This is essentially equivalent to rigidity.

Remark 1.73. These theorems fail for $k[[t]]$ if $[k\colon k^p] = \infty$, so F-finiteness is important for these results.

Question 1.74. What is the analog for noncommutative rings?

Theorem 1.75. *The functor $R \mapsto \mathrm{TC}(R)/p$ from rings to (-1)-connected spectra commutes with filtered colimits.*

Remark 1.76. We can check this theorem directly in many cases, such as spherical group rings or $\mathbb{F}_p$-algebras. In fact, the functor $X \mapsto \mathrm{TC}(X)/p$ from connective cyclotomic spectra to spectra commutes with filtered colimits. To prove this more general theorem, we can approximate by functors that look like X_{hS^1}.

2 Global homotopy theory

2.1 Introduction

A slogan is that "global homotopy theory is the homotopy theory of universally equivariant phenomena." Alternatively, it is the homotopy theory that is uniform and consistent for all groups. The idea is that all groups act on geometric objects at the same time. These slogans need to be made more rigorous; that is the purpose of these talks.

We will:

- describe a rigorous foundation for stable global homotopy theory (orthogonal spectra);
- indicate some of the theory (global model structure, global stable homotopy category);
- dwell on the relevant algebra (global Mackey functors);
- give interesting examples (global sphere spectrum, global classifying spaces, global Thom spectra, global K-theory spectra, global Eilenberg-MacLane spectra);
- calculate the zeroth global homotopy groups of symmetric products of spheres ($\pi_0^G(\mathrm{Sym}^n(\mathbb{S}))$).

For the purpose of the theory, when we say "all groups" we mean compact Lie groups. That's where the theory exists. For the most part, you can think finite groups – most of the phenomena appear already for finite groups, although there are a few places where unitary and orthogonal groups are required.

You may think of spectra as representing cohomology theories on spaces. Then naïve G-spectra represent $\mathbb{Z}$-graded cohomology theories on G-spaces, and genuine G-spectra represent $RO(G)$-graded cohomology theories on G-spaces. Global spectra represent genuine cohomology theories on stacks (orbispaces).

We won't see much more of stacks except by way of motivation. A stack (or orbifold/orbispace) is roughly the quotient of a manifold by the action of a group. Gepner–Henriques define a homotopy theory of these objects, where BG is the classifying stack for principal G-bundles, $M /\!/ G$ is the global quotient of G acting on M.

This homotopy theory is Quillen equivalent to a category called orthogonal spaces or $\mathcal{I}$-spaces, which we won't talk about very much. This, in turn, is Quillen equivalent to the category of orthogonal spectra via the $\Sigma^\infty_+ \dashv \Omega^\infty$

adjunction. Finally, this is Quillen equivalent to the global stable homotopy category, given by inverting the global equivalences in the category of orthogonal spectra.

For a stack X, tracing this chain of equivalences sends a stack X to the $\mathbb{Z}$-graded cohomology theory defined by an E in the stable homotopy category.

$$X \mapsto E^n(X) := \left[\Sigma^\infty_+(X), E[n]\right].$$

We may use this to define a cohomology theory of stacks.

$$E^n(BG) = \pi^G_{-n}(E)$$
$$E^n(M /\!/ G) = E^n_G(M)$$

2.2 Orthogonal Spectra

We will define the objects and equivalences in a category of spectra that will be used for global homotopy theory.

Definition 2.1. An **inner product space** is a finite-dimensional $\mathbb{R}$-vector space V with a scalar product. Write $O(V)$ for the orthogonal group of V.

This is isometrically isomorphic to $\mathbb{R}^n$ with the standard inner product, but it is better not to fix the basis.

Definition 2.2. An **orthogonal spectrum** X consists of:

- a based space $X(V)$ for each inner product space V;
- a continuous based map $L(V, W) \times X(V) \to X(W)$ whenever $\dim(V) = \dim(W)$, where $L(V, W)$ is the space of linear isometries between V and W;
- structure maps

  $$\sigma_{V,W} \to S^V \wedge X(W) \to X(V \oplus W)$$

  where S^V is the one-point compactification of V: $V \cup \{\infty\}$ as a set.

These data must be natural, associative, and unital.

Remark 2.3. Sometimes, in the literature, people define orthogonal spectra using Thom spaces; this is equivalent to the definition given above, in the sense that the categories of orthogonal spectra defined in these two ways are equivalent.

If $V = W$, then we have a map

$$O(V) \times X(V) \to X(V)$$

giving an $O(V)$-action on $X(V)$.

A morphism of orthogonal spectra is a set of maps $f(V)\colon X(V) \to Y(V)$ commuting with all the data. Let $\mathbf{Sp}^O$ denote the category of orthogonal spectra.

Despite the fact that there is no G explicitly acting on X, it is secretly there inside the orthogonal groups $O(V)$ via the representations of G.

Definition 2.4. Let G be a compact Lie group. A G-**representation** is an inner product space V with a continuous homomorphism $G \to O(V)$.

If X is an orthogonal spectrum, $X \in \mathbf{Sp}^O$, and V is a G-representation, $X(V)$ becomes a G-space.

Definition 2.5. A **complete** G-**universe** is a G-representation of countably infinite dimension $\mathcal{U}_G$ such that every finite-dimensional G-representation embeds into $\mathcal{U}_G$.

Such a complete G-universe always exists. We may always take

$$\mathcal{U}_G = \bigoplus_{[\lambda] \text{ irrep}} \bigoplus_{\mathbb{N}} \lambda.$$

If G is finite, we may in fact take

$$\mathcal{U}_G = \bigoplus_{\mathbb{N}} S_G$$

where S_G is the regular representation.

Fact 2.6. *Any two complete G-universes are isomorphic, but not uniquely, not even up to a preferred choice. However, the space of linear complete G-embeddings is contractible.*

Definition 2.7. Let X be an orthogonal spectrum and G a compact Lie group. Then

$$\pi_0^G(X) = \operatorname*{colim}_{V \leq \mathcal{U}_G} [S^V, X(V)]_*^G,$$

where the colimit is taken over the poset of finite-dimensional G-subrepresentations of $\mathcal{U}_G$; given $V \subseteq W$, we have a map

$$[S^V, X(V)]_*^G \longrightarrow [S^W, X(W)]_*^G$$

sending $f\colon S^V \to X(V)$ to

$$\begin{tikzcd}
S^W \arrow[r, "\cong"] \arrow[dr] & S^{W\setminus V} \wedge S^V \arrow[r, "\mathrm{id}\wedge f"] & S^{W\setminus V} \wedge X(V) \arrow[d, "\sigma_{W\setminus V, V}"] \\
& X(W) \arrow[r, equal] & X(W \setminus V \oplus V)
\end{tikzcd}$$

Fact 2.8.

- *The sets* $\pi_0^G(X)$ *have natural abelian group structure.*
- $\pi_k^G(X)$ *is defined for* $k \in \mathbb{Z}$ *by replacing* S^V *by* $S^{V \oplus \mathbb{R}^k}$ *when* $k > 0$ *or replacing* $X(V)$ *by* $X(V \oplus \mathbb{R}^{-k})$ *when* $k < 0$.

Definition 2.9. A morphism $f\colon X \to Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f)\colon \pi_k^G(X) \to \pi_k^G(Y)$ is an isomorphism for all $k \in \mathbb{Z}$ and all G.

Remark 2.10. We can be deliberately flexible about the interpretation of "all G" to get different notions of equivalence; an interesting one is finite groups. We will work with "all G" meaning all compact Lie groups.

Definition 2.11. The **global stable homotopy category** is the category of orthogonal spectra with the global equivalences inverted, denoted

$$\mathbf{GH} := \mathbf{Sp}^O[(\text{global equivalences})^{-1}]$$

Remark 2.12. If $f\colon X \to Y$ is a global homotopy equivalence, then it is also a (non-equivariant) stable equivalence. In fact, there is a triangulated forgetful functor $U\colon$ **GH** $\to$ **SHC** that has both left and right adjoints, where **SHC** is the stable homotopy categories.

We could embed **SHC** into **GH** in two ways – using either the left adjoint or the right adjoint. But because there are multiple ways to do this, it is better to think of **SHC** as a quotient category of **GH** rather than a subcategory.

Definition 2.13. For any $\alpha\colon K \to G$ a continuous homomorphism, there is a **restriction map**

$$\alpha^*\colon \pi_0^G(X) \to \pi_0^K(X)$$

given by sending $f\colon S^V \to X(V)$ to

$$\alpha^*(f)\colon \alpha^*(S^V) = S^{\alpha^*(V)} \to \alpha^*(X(V)) = X(\alpha^*(V)).$$

This is clearly contravariantly functorial: $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$ and $(\mathrm{id}_G^*) = \mathrm{id}_{\pi_0^G(X)}$.

The restriction maps only see the conjugacy classes of G.

Lemma 2.14. *Let* $Cg\colon G \to G$ *be conjugation by* g *as a homomorphism;* $Cg(h) = g^{-1}hg$. *Then* $(Cg)^* = \mathrm{id}_{\pi_0^G(X)}$.

So $G \mapsto \pi_0^G(X)$ is a contravariant functor on compact Lie groups and conjugacy classes of continuous homomorphisms.

Lemma 2.15. *Given a closed subgroup* $H \leq G$, *there is a transfer map*

$$\mathrm{tr}_H^G\colon \pi_0^H(X) \to \pi_0^G(X).$$

The definition of the transfer map involves an equivariant Thom–Pontryagin construction

Fact 2.16.

- $\mathrm{tr}_H^H = \mathrm{id}$;
- $\mathrm{tr}_K^G = \mathrm{tr}_H^G \circ \mathrm{tr}_K^H$ *if* $K \le H \le G$;
- $\mathrm{tr}_H^G = 0$ *if* H *has infinite index inside* $N_G(H)$;
- *transfer commutes with inflations: given a diagram*

$$\begin{tikzcd} K \arrow[rr, two heads, "\alpha"] & & G \\ L = \alpha^{-1}(H) \arrow[u, hook] \arrow[rr, two heads, "\alpha|_L"] & & H, \arrow[u, hook] \end{tikzcd}$$

we have

$$\mathrm{tr}_L^K \circ (\alpha|_L)^* = \alpha^* \circ \mathrm{tr}_H^G .$$

- *Double coset formula: for* $H, K \le G$,

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G = \sum_{[M]:\ KgH \subseteq M} \chi^\# \cdot \mathrm{tr}_{K \cap {}^g H}^K\, C_g^* \circ \mathrm{res}_{K^g \cap H}^H,$$

where the sum is over classes of orbit-type manifolds in $K \backslash G / H$ *and* $\chi^\#(M)$ *is the Euler number of* M*, and* $\mathrm{res}_H^G = (H \hookrightarrow G)^*$.

If G *is finite, or* $[G\colon H]$ *is finite, then* G/H *is finite and discrete,* $M = \{KgH\}$ *and* $\chi^\#(M) = 1$.

This kind of structure can be formulated into the concept of "global functors," which are the analog of abelian groups for stable homotopy type, or G-Mackey functors for G-stable homotopy type.

In algebra and representation theory, for finite groups, these global functors have been called different things: inflation functors, global Mackey functors, and biset functors. The upshot is that global equivariant homotopy groups give a **global functor** $\underline{\pi_0} = \{\pi_0^G, \alpha^*, \mathrm{tr}_H^G\}\colon \mathbf{Sp}^0 \to \mathbf{GF}$, where **GF** is the category of global functors.

Definition 2.17. The **global Burnside category A** is the category whose objects are compact Lie groups with morphisms

$$\mathbf{A}(G, K) = \mathbf{Nat}(\pi_0^G, \pi_0^K)$$

Then the **category of global functors GF** is the category of additive functors on $\mathbf{A}(G, K)$ for varying G and K.

Example 2.18 (The sphere spectrum)**.** The sphere spectrum $\mathbb{S}$ is the spectrum $\mathbb{S}(V) = S^V$ and $\sigma_{V,W} \colon S^V \wedge S^W \cong S^{V \oplus W}$. $\underline{\pi_0}(\mathbb{S})$ is the representable global functor represented by the trivial group. This gives it the property that,

$$\mathbf{GF}(\underline{\pi_0}(\mathbb{S}), F) \cong F(\{e\}).$$

Concretely,

$$\pi_0^G = \mathbb{Z}\{ \operatorname{tr}_H^G(1) \mid H \leq G, |W_G(H)| < \infty \}$$

due to Segal and tom Dieck. If G is finite, then $\pi_0^G(\mathbb{S}) \cong A(G)$, where $A(G)$ is the Burnside (representation) ring of G.

2.3 Some examples

Example 2.19 (Suspension spectra of global classifying spaces)**.** Let G be a compact Lie group and choose a faithful G-representation W. Then define an orthogonal spectrum $\Sigma_+^\infty B_{gl}G$ by

$$(\Sigma_+^\infty B_{gl}G)(V) = S^V \wedge (L(W, V)/G)_+.$$

This object deserves its name by analogy to other versions of classifying spaces. If K is another compact Lie group, then $L(W, \mathcal{U}_K)/G$ is a classifying K-space for principal G-bundles. And if $K = \{e\}$, then $BG \simeq L(W, \mathbb{R}^\infty)/G$.

$\underline{\pi_0}(\Sigma_+^\infty B_{gl}G)$ is represented by G; it has the property that

$$\mathbf{GF}(\underline{\pi_0}(\Sigma_+^\infty B_{gl}G), F) \cong F(G).$$

More concretely,

$$\pi_0^K(\Sigma^\infty B_{gl}G) = \mathbb{Z}\left\{ (\operatorname{tr}_L^K \circ \alpha^*)(u_G) \;\middle|\; L \leq K, |W_K L| < \infty, \alpha \colon L \to G, u_G \in \pi_0^G\left(\Sigma_+^\infty B_{gl}G\right) \right\}$$

Example 2.20 (Connective Thom spectra)**.** Define the orthogonal spectrum mO by

$$mO(V) = \mathrm{Th}(\gamma)$$

where γ is the universal bundle over $\mathrm{Gr}_{\dim(V)}(V \oplus \mathbb{R}^\infty)$ and $\mathrm{Th}(\gamma)$ is its Thom space. The structure maps are the obvious ones.

The Thom spectrum is globally connective, i.e. $\pi_0^G(mO) = 0$ for all G and all $k < 0$.

$$\underline{\pi_0}(mO) = {}^{A(-)}/_{\langle \operatorname{tr}_e^{C_2} \rangle}$$

where $A(-)$ is the Burnside ring global functor and $\operatorname{tr}_e^{C_2} \in A(C_2)$.

Exercise 2.21. Calculate $\pi_0^G(mO)$ using the presentation of $\underline{\pi_0}(mO)$ as ${}^{A(-)}/_{\langle \operatorname{tr}_e^{C_2} \rangle}$.

There is an equivariant Pontryagin–Thom construction: a homomorphism $\mathcal{N}_k^G \to \pi_k^G(\mathrm{mO})$, where $\mathcal{N}_k^G$ is the bordism group of closed smooth G-manifolds of dimension k. For certain G,

Theorem 2.22 (Wasserman)**.** *If* G *is a finite group or the product of a finite group and a torus, then there is an isomorphism*

$$\mathcal{N}_k^G \cong \pi_k^G(\mathrm{mO}).$$

However, this doesn't hold in general. If $N \leq SU(2)$ is the normalizer of a maximal torus in $SU(2)$, then the map is not surjective; in particular, the element $\mathrm{tr}_N^{SU(2)}(1) \in \pi_0^{SU(2)}(\mathrm{mO})$ is not in the image of this homomorphism.

Example 2.23 (Thom spectra)**.** Define the orthogonal spectrum MO by

$$\mathrm{MO}(V) = \mathrm{Th}(\gamma)$$

where γ is the universal bundle over $\mathrm{Gr}_{\dim(V)}(V \oplus V)$.

The spectra mO and MO can be non-equivariantly equivalent, but not equivariantly. For nontrivial G, $\pi_*^G(\mathrm{MO})$ has non-trivial groups in negative degrees. Moreover, MO is globally oriented, i.e. the cohomology theories represented by MO have Thom isomorphisms for equivariant vector bundles.

There is another Pontyragin–Thom homomorphism:

$$\mathfrak{N}_*^G \to \pi_*^G(\mathrm{MO}),$$

where $\mathfrak{N}_*^G$ is a localization of $\mathcal{N}_*^G$. This is an isomorphism for all G, unlike the previous one.

Example 2.24 (Connective global K-theory)**.** Define an orthogonal spectrum

$$\mathrm{ku}(V) = \mathrm{Conf}(S^V, \mathrm{Sym}(V_{\mathbb{C}})),$$

where $\mathrm{Conf}(S^V, \mathrm{Sym}(V_{\mathbb{C}}))$ is the space of finite unordered configurations of points in S^V labelled by finite-dimensional, pairwise orthogonal subspaces of

$$\mathrm{Sym}(V_{\mathbb{C}}) = \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}).$$

Non-equivariantly, this is connective topological K-thoery. There is a ring homomorphism

$$\mathrm{RU}(G) \to \pi_0^G(\mathrm{ku})$$

compatible with all the global structure, except the transfer maps tr_H^G for $\dim(G) > \dim(H)$. Hence, this is not a morphism of global functors.

Likewise, we may define connective real topological K-theory as an orthogonal spectrum

$$\mathrm{ko}(V) = \mathrm{Conf}(S^V, \mathrm{Sym}(V)).$$

In this case, the map

$$\mathrm{RO}(G) \to \pi_0^G(\mathrm{ko})$$

is given by

$$[V] \mapsto \left(\begin{array}{ccc} S^V & \longrightarrow & \mathrm{ko}(V) = \mathrm{Conf}(S^V, \mathrm{Sym}(V)) \\ x & \longmapsto & [x, V] \end{array} \right)$$

When G is finite, these maps are isomorphisms.

Example 2.25 (Periodic global K-theory, Joachim)**.** Define an orthogonal spectrum KU by

$$\mathrm{KU}(V) = C^*_{gr}(C_0(\mathbb{R}), \mathrm{Cl}(V) \otimes_{\mathbb{C}} \mathcal{K}(\widehat{\mathrm{Sym}}(V)))$$

where C^*_{gr} is the space of $*$-homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras, $C_0(\mathbb{R})$ is continuous functions from $\mathbb{R}$ to $\mathbb{C}$ that vanish at infinity, $\mathrm{Cl}(V)$ is the complexified Clifford algebra, $\widehat{\mathrm{Sym}}(V)$ is the Hilbert space completion of $\mathrm{Sym}(V)$, and $\mathcal{K}(-)$ is the space of compact operators on a Hilbert space.

This spectrum KU is Bott-periodic, with a class $\beta \in \pi_2^e(\mathrm{ku})$ mapping to a unit in $\pi_2^e(\mathrm{KU})$ under a specific map $j\colon \mathrm{ku} \to \mathrm{KU}$.

KU is globally oriented for equivariant Spin^c-vector bundles (Atiyah–Bott–Shapiro).

The composite

$$\mathrm{RU}(G) \to \pi_0^G(\mathrm{ku}) \xrightarrow{j_*} \pi_0^G(\mathrm{KU})$$

is an isomorphism for all G. In fact,

$$\underline{\pi_0}(\mathrm{KU}) \cong \underline{\mathrm{RU}}$$

as global power functors, and

$$\underline{\pi_1}(\mathrm{KU}) = 0.$$

2.4 The global stable homotopy category

Theorem 2.26. *The global equivalences of orthogonal spectra are part of a proper, cofibrantly generated, topological stable model structure.*

Corollary 2.27. *The global homotopy category* **GH** *is a triangulated category, and the smash product of orthogonal spectra can be left-derived with respect to global equivalences to a symmetric monoidal structure on* **GH**.

Definition 2.28. A triangulated category with a compatible monoidal structure is called a **tensor-triangulated category**.

Recall the suspension spectra

$$(\Sigma^\infty_+ B_{gl}G)(V) = S^V \wedge (L(W, V)/G)_+$$

where W is a faithful G-representation, and the elements

$$u_G = \left(\begin{array}{ccc} S^W & \longrightarrow & S^W \wedge (L(W,W)/G)_+ \\ w & \longmapsto & w \wedge \mathrm{id} \cdot G \end{array} \right) \in \pi_0^G(\Sigma^\infty_+ B_{gl}G).$$

The pair $(\Sigma^\infty_+ B_{gl}G, u_G)$ represents the functor $\pi_0^G \colon \mathbf{GH} \to \mathbf{Set}$.

Corollary 2.29. *The spectra* $\Sigma^\infty_+ B_{gl}G$ *form a set of compact weak generators for* **GH**. *So* **GH** *is a compactly generated tensor-triangular category.*

Remark 2.30 (Warning!)**.** For nontrivial G, $\Sigma^\infty_+ B_{gl}G$ is *not* dualizable!

The preferred t-structure on **GH** is given naturally by globally connective and globally coconnective spectra.

Theorem 2.31. *The functor* $\underline{\pi_0}$ *from the full subcategory of connective, coconnective global spectra to the category of global functors*

$$\underline{\pi_0} \colon \{X \in \mathbf{GH} \mid \underline{\pi}_k(X) = 0, k \neq 0\} \to \mathbf{GF}$$

is an equivalence of categories.

Corollary 2.32. *Every global functor* F *has an* ***Eilenberg–MacLane spectrum*** HF*, such that*

$$\underline{\pi}_k(HF) = \begin{cases} F & (k = 0), \\ 0 & (k \neq 0). \end{cases}$$

2.5 Filtering the global Hurewicz map

Definition 2.33. If X is a space, then the n**-th symmetric product** $\mathrm{Sym}^n(X)$ of X is the quotient of X^n by the action of the symmetric group Σ_n.

If X is a based space, then we have a map

$$\begin{array}{ccc} \mathrm{Sym}^n(X) & \longrightarrow & \mathrm{Sym}^{n+1}(X) \\ [x_1, \ldots, x_n] & \longmapsto & [x_1, \ldots, x_n, *] \end{array}$$

And we define the **infinite symmetric product** as the colimit over these maps

$$\mathrm{Sym}^\infty(X) := \bigcup_{n \geq 1} \mathrm{Sym}^n(X)$$

Theorem 2.34 (Dold–Thom)**.** *For $m \geq 1$, $\mathrm{Sym}^\infty(S^m)$ is an Eilenberg–MacLane space of type $K(\mathbb{Z}, m)$ and the inclusion $S^m = \mathrm{Sym}^1(S^m) \to \mathrm{Sym}^\infty(S^m)$ represents a generator of $\pi_m(\mathrm{Sym}^\infty(S^m))$.*

Write $\mathrm{Sym}^n = \{\mathrm{Sym}^n(S^V)\}_V$ for the orthogonal spectrum $V \mapsto \mathrm{Sym}^n(S^V)$. By the Dold–Thom theorem, $\mathrm{Sym}^\infty \simeq H\mathbb{Z}$. This has a filtration of subspectra

$$\mathbb{S} = \mathrm{Sym}^1 \hookrightarrow \mathrm{Sym}^2 \hookrightarrow \cdots \hookrightarrow \mathrm{Sym}^\infty \simeq H\mathbb{Z},$$

whose composite is the Hurewicz map. This has been much studied non-equivariantly:

- Each inclusion $\mathrm{Sym}^n \to \mathrm{Sym}^{n+1}$ is a rational stable equivalence.
- If $n \geq 2$ is not a prime power, then $\mathrm{Sym}^n / \mathrm{Sym}^{n-1} \simeq *$.
- If $n = p^k \geq 2$ is a prime power, then $\pi_*(\mathrm{Sym}^n / \mathrm{Sym}^{n-1})$ is annihilated by p.
- Nakauka calculated $H^*(\mathrm{Sym}^n; \mathbb{F}_p)$.
- The spectra $\mathrm{Sym}^n / \mathrm{Sym}^{n-1}$ appear in the stable splitting of classifying spaces (Mitchell–Priddy), the Whitehead conjecture (Kuhn), partition complexes and Tits buildings (Arome–Dwyer).

Equivariantly, only the extreme cases have been studied before. The spectrum $\mathbb{S} = \mathrm{Sym}^1$ is the global sphere spectrum, and $\pi_0^G(\mathbb{S}) \cong A(G)$ is the Burnside ring. For finite G, Sym^∞ is an Eilenberg–MacLane spectrum for the constant Mackey functor $\underline{\mathbb{Z}}$.

There are some key differences in the equivariant case. For G nontrivial, the map

$$A(G) = \pi_0^G(\mathrm{Sym}^1) \to \pi_0^G(\mathrm{Sym}^\infty) = \mathbb{Z}$$

is *not* a rational isomorphism; it is the augmentation map. So what happens in the intervening stages of the filtration?

Define the element

$$t_n = n \cdot 1 - \mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1) \in \pi_0^{\Sigma_n}(\mathbb{S}) = A(\Sigma_n).$$

Let $I_n = \langle t_n \rangle$ be the global subfunctor of the Burnside ring functor $\mathbb{A} = \underline{\pi_0}(\mathbb{S})$ generated by t_n. Observe that $I_n \leq I_{n+1}$, because

$$\mathrm{res}_{\Sigma_n}^{\Sigma_{n+1}}(t_{n+1}) = t_n.$$

Define $I_\infty = \bigcup_{n \geq 0} I_n$.

	1	2	3	4	5	6
Σ_2	2/0	1/0	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Σ_3	4/0	2/0	1/0	$\cdots$	$\cdots$	$\cdots$
Σ_4	11/0	3/0	1/3	1/0	$\cdots$	$\cdots$
A_5	9/0	5/0	3/3	3/0	1/5	1/0.

Figure 1: This table lists the rank and torsion of $\pi_0^G(\mathrm{Sym}^n)$ for varying G and n. The number before the slash is the rank, and the number following the slash is the torsion.

Theorem 2.35. *The inclusion* $S = \mathrm{Sym}^1 \hookrightarrow \mathrm{Sym}^n$ *induces an epimorphism on* $\underline{\pi}_0$ *that factors over an isomorphism*

$$\mathbb{A}/_{I_n} \cong \underline{\pi_0}(\mathrm{Sym}^n).$$

(Recall that $\mathbb{A} = \underline{\pi_0}(S)$*.)*

It follows from this theorem that

$$\pi_0^G(\mathrm{Sym}^n) \cong A(G)/I_n(G).$$

So we need to figure out what $I_n(G)$ is, algebraically.

Consider nested compact Lie groups $K \leq H \leq G$ such that $[H\colon K] < \infty$ and $|W_G H| < \infty$. Define

$$t_K^H = [H\colon K] \cdot \mathrm{tr}_H^G(1) - \mathrm{tr}_K^G(1) \in A(G).$$

Notice that $t_{\Sigma_{n-1}}^{\Sigma_n} = t_n$.

Proposition 2.36. *Let* $n \geq 2$ *and let* G *be any compact Lie group. Then* $I_n(G)$ *is the subgroup of* $A(G)$ *generated by* t_K^H *for* $[H\colon K] \leq n$ *and* $W_G H$ *finite.*

Proof sketch. Suppose that $t_K^H \in I_n(G)$ if $[H\colon K] \leq n$. Write $m = [H\colon K] \leq n$. Choose a bijection $H/K \cong \{1,\ldots,m\}$. Translate the left H-action on H/K into an action on $\{1,\ldots,m\}$, i.e. a homomorphism $\beta\colon H \to \Sigma_m$, with $H/K \cong \beta^*\{1,\ldots,m\}$.

$$\begin{aligned} t_K^H &= [H\colon K] \cdot \mathrm{tr}_H^G(1) - \mathrm{tr}_K^G(1) \\ &= \mathrm{tr}_H^G([H\colon K] \cdot 1 - \mathrm{tr}_K^H(1)) \\ &= \mathrm{tr}_H^G(\beta^*(1 - \mathrm{tr}_{\Sigma_{m-1}}^{\Sigma_m})) \in I_m \subseteq I_n. \end{aligned}$$

Conversely, we realize $I_n = \mathrm{coker}(\mathbb{A}(\Sigma_n, -) \xrightarrow{t_n} \mathbb{A})$. $\square$

2.5.1 Ingredients in the proof of Theorem 2.35

If V is an inner product space, define

$$S(V,n) = \left\{ (v_1, \ldots, v_n) \in V^n \;\middle|\; \sum_{i=1}^{n} v_i \text{ and } \sum_{i=1}^{n} |v_i|^2 = 1 \right\}.$$

This is the unit sphere in $V \otimes \overline{\rho_n}$, where ρ_n is the reduced regular representation of Σ_n on the vector space

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n\}.$$

This is functorial for linear isometric embeddings in V, so these form an orthogonal space.

Then $V \mapsto S(V,n)$ is a global universal space for the family of non-transitive subgroups of Σ_n, i.e. for any compact lie group K, the space $S(\mathcal{U}_K, n)$ with action of $K \times \Sigma_n$ is a universal space for the family of those $\Gamma \leq K \times \Sigma_n$ such that $\Gamma \cap 1 \times \Sigma_n$ is non-transitive.

Define $B_{gl}\mathcal{F} := S(-,n)/\Sigma_n$, for $\mathcal{F}$ any family of non-transitive subgroups of Σ_n. For example, $B_{gl}\mathcal{F}_2 = B_{gl}\Sigma_2$, but $B_{gl}\mathcal{F}_3 \not\simeq B_{gl}\Sigma_3$.

Proposition 2.37. *$\pi_0(\Sigma^\infty_+ B_{gl}\mathcal{F}_n)$ is generated as a global functor by a single element*

$$\pi_0^{\Sigma_n}(\Sigma^\infty_+ B_{gl}\mathcal{F}_n).$$

Theorem 2.38. *There is a global homotopy cofiber sequence*

$$\Sigma^\infty_+ B_{gl}\mathcal{F}_n \to \mathrm{Sym}^{n-1} \to \mathrm{Sym}^n \to \Sigma^\infty B_{gl}\mathcal{F}^\diamond,$$

where the diamond denotes unreduced suspension.

Proof sketch. $\mathrm{Sym}^{n-1} \hookrightarrow \mathrm{Sym}^n$ is an h-cofibration of orthogonal spectra, so $\mathrm{Sym}^n / \mathrm{Sym}^{n-1}$ globally models the mapping cone. On the other hand, we find the

$$\begin{aligned}
\mathrm{Sym}^n(V)/\mathrm{Sym}^{n-1}(V) &= (S^V)^n/\Sigma_n \big/ (S^V)^{n-1}/\Sigma_{n-1} \\
&= (S^V)^{\wedge n}/\Sigma_n \\
&= S^{V \otimes \rho_n}/\Sigma_n \\
&= S^{V \oplus (V \otimes \overline{\rho}_n)}/\Sigma_n \\
&= S^V \wedge S^{V \otimes \overline{\rho}_n}/\Sigma_n \\
&= S^V \wedge (S(V \otimes \overline{\rho}_n, n)/\Sigma_n)^\diamond .
\end{aligned}$$

∎

The right-hand spectrum is globally connective, so automatically the map $\mathrm{Sym}^{n-1} \to \mathrm{Sym}^n$ is surjective on π_0. The key calculation is to figure out where w_n goes under the map $\Sigma^\infty_+ B_{gl}\mathcal{F}_n \to \mathrm{Sym}^{n-1}$, which is

$$w_n \mapsto \mathrm{im}(t_n) \subseteq \pi_0^{\Sigma_n}(\mathrm{Sym}^{n-1}).$$

Then Theorem 2.35 follows from an inductive argument.

3 Assembly maps and trace methods

3.1 Introduction

Here we will state and sketch the proof of a theorem that illustrates the kinds of methods and ideas this lecture course will think about.

Theorem 3.1. *For every discrete group* G*, the following map is injective:*

$$\operatorname*{colim}_{H \leq G \text{ finite}} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} C \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C}$$

This is an example of an **assembly map**. The proof of this result uses the **trace map**, in particular the **Hattori–Stallings rank**

$$\mathrm{tr}\colon K_0(\mathbb{C}G) \to \mathbb{C}[G/\operatorname{conj}],$$

where G/conj is the set of conjugacy classes of G. This is induced by the map

$$\begin{aligned} M_n(\mathbb{C}G) &\to \mathbb{C}[G/\operatorname{conj}] \\ A &\mapsto \mathrm{tr}(A) \end{aligned}$$

Any finitely-generated $\mathbb{C}G$-module M is isomorphic to $(\mathbb{C}G)^n P$ for some projection matrix $P \in M_n(\mathbb{C}G)$. Then $\mathrm{tr}(M) = \mathrm{tr}(P)$. Hence, we arrive at a map

$$K_0(\mathbb{C}G) \to M_n(\mathbb{C}G)$$

given by $[M] \mapsto \mathrm{tr}(P)$.

Lemma 3.2. *For* $H \leq G$ *finite, the following diagram commutes:*

$$\begin{tikzcd} K_0(\mathbb{C}H) \arrow[r, "\mathrm{tr}"] \arrow[d, "\cong"] & \mathbb{C}[H/\operatorname{conj}] \arrow[d, "\phi"] \\ R_{\mathbb{C}}(H) \arrow[r, "\chi"] & \mathbb{C}[H/\operatorname{conj}] \end{tikzcd}$$

where $\phi([g]) = \#(Z_H\langle g\rangle)[g]$*, where* $Z_H\langle g\rangle$ *is the size of the centralizer in* G *of* $\langle g\rangle$.

The proof of this lemma is a simple exercise in writing down the definitions and using the definition of the trace.

The colimit in Theorem 3.1 is taken over the **orbit category**, $\mathcal{O}(G, \mathrm{Fin})$, which is the full subcategory of G-sets and G-maps spanned by G/H with $H \leq G$ finite.

Proof of Theorem 3.1. To prove the theorem, consider the diagram

$$\begin{tikzcd} \operatorname{colim}_{G/H\in\mathcal{O}(G,\mathrm{Fin})} K_0(\mathbb{C}H)\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r, "\text{assembly}"] \arrow[d, "\mathrm{tr}"] & K_0(\mathbb{C}G)\otimes_{\mathbb{Z}}\mathbb{C} \arrow[d, "\mathrm{tr}"] \\ \operatorname{colim}_{G/H\in\mathcal{O}(G,\mathrm{Fin})} \mathbb{C}[H/\operatorname{conj}] \arrow[r] & \mathbb{C}[G/\operatorname{conj}] \end{tikzcd}$$

To show that the assembly map is injective, it is enough to show that the counterclockwise composite is injective. So it suffices to show that the map along the bottom is injective. The functor $\mathbb{C}[-]$ is a left adjoint, and so it commutes with colimits, Hence,

$$\operatorname*{colim}_{G/H\in\mathcal{O}(G,\mathrm{Fin})} \mathbb{C}[H/\operatorname{conj}] \cong \mathbb{C}\left[\operatorname*{colim}_{G/H\in\mathcal{O}(G,\mathrm{Fin})} H/\operatorname{conj}\right],$$

and the map

$$\operatorname*{colim}_{G/H\in\mathcal{O}(G,\mathrm{Fin})} H/\operatorname{conj} \to G/\operatorname{conj}$$

is injective even before applying $\mathbb{C}$. □

In these lectures, we will prove generalizations of these kinds of results. The goal is to generalize from K_0 to higher K-theory, and from complex group rings to integral group rings.

Here's an outline of what we will accomplish in these three talks.

(1) Assembly maps and topological Hochschild homology

(2) The Farrell–Jones conjecture

(3) Bökstedt–Hsiang–Madsen's Theorem and its generalization by Lück–Reich–Rognes–Varisco

(4) Proofs of these results and related results

3.2 Topological Hochschild homology

Earlier, we introduced the trace map $\mathrm{tr}\colon K_0(\mathbb{C}G) \to \mathbb{C}[G/\operatorname{conj}]$. In fact, $\mathbb{C}[G/\operatorname{conj}]$ is the zeroth Hochschild homology group $HH_0(\mathbb{C}G)$.

Definition 3.3. The **cyclic nerve** or **cyclic bar construction** of a ring A is the simplicial object

$$N^{\mathrm{cyc}}_{\otimes}(A)_n = A^{\otimes n+1}$$

with face and degeneracy maps

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0, \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n, \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n, \end{cases}$$

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

Definition 3.4. The **Hochschild homology** of a ring A is the geometric realization of the cyclic nerve of A.

$$\mathrm{HH}(A) = |N_\otimes^{\mathrm{cyc}}(A)|$$

The n**-th Hochschild homology group** of A is

$$\mathrm{HH}_n(A) = \pi_n \, \mathrm{HH}(A).$$

Example 3.5. If G is a discrete group, and A is a ring, then

$$\mathrm{HH}_0(A[G]) = \operatorname{coker}(d_1 - d_0) = {}^{AG}\!/_{[AG, AG]} = A\left[{}^{G}\!/_{\mathrm{conj}}\right].$$

Remark 3.6. There is an obvious extra structure in the construction of $N_\otimes^{\mathrm{cyc}}(A)$; in each degree, $N_\otimes^{\mathrm{cyc}}(A)_n$ has an action of the cyclic group C_{n+1} of order $n+1$. This gives $N_\otimes^{\mathrm{cyc}}(A)$ the structure of a **cyclic object**, not just a simplicial object. Hence, S^1 acts on $|N_\otimes^{\mathrm{cyc}}(A)|$. We won't construct this action explicitly, but only use the fact that it exists; a nice conceptual explanation is in Nikolaus–Scholze Appendix T.

This construction $N_\otimes^{\mathrm{cyc}}(A)$ works more generally for monoids A in any symmetric monoidal category $(\mathbf{C}, \otimes, I)$.

Example 3.7.

- $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ $\mathrm{HH}(A) = |N_{\otimes_{\mathbb{Z}}}^{\mathrm{cyc}}(A)|$
- $(\mathbf{Set}, \times, \mathrm{pt})$. For any monoid M, the composite map

$$S^1 \times B^{\mathrm{cyc}}(M) \xrightarrow{\text{action}} B^{\mathrm{cyc}}(M) = |N_\times^{\mathrm{cyc}}(M)| \longrightarrow |N(M)| = BM$$

  gives a map

$$B^{\mathrm{cyc}}(M) \to \mathrm{Map}(S^1, BM) \tag{3.1}$$

Theorem 3.8. *If M is a group, then (3.1) is a homotopy equivalence.*

Definition 3.9. Consider $(\mathbf{Sp}^{\mathrm{O}}, \wedge, \mathbb{S})$. If A is any orthogonal spectrum, then we define

$$\mathrm{THH}(A) := |N^{\mathrm{cyc}}_{\wedge}(A)|$$

if A is sufficiently cofibrant.

Remark 3.10. This is not how Bökstedt defined topological Hochschild homology originally. When he first worked with it, no suitable monoidal model categories of spectra existed, so he worked with spectra on a point-set level.

The equivalence of what we define as THH above to Bökstedt's original definition is due to many people, among them Angelveit–Blumberg–Gerhardt–Hill–Lawson, Doto–Malkiewich–Sagave–Wu.

Fact 3.11. *If A is a ring spectrum and G is a group, then define $AG = A \wedge G_+$. Then*

$$\mathrm{THH}(AG) \cong \mathrm{THH}(A) \wedge B^{\mathrm{cyc}} G_+.$$

Fact 3.12. *If A is a discrete ring, then the natural map*

$$\mathrm{THH}(A) = \mathrm{THH}(HA) \to \mathrm{HH}(A)$$

is a rational isomorphism.

Theorem 3.13 (Bökstedt)**.**

$$\mathrm{THH}_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \cong \mathrm{HH}_0(\mathbb{Z}) & (n = 0), \\ \mathbb{Z}/k\mathbb{Z} & (n = 2k-1), \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 3.14. *The trace map $K_0(\mathbb{C}G) \to \mathrm{HH}_0(\mathbb{C}G)$ generalizes to a trace map $\mathrm{tr}\colon K(A) \to \mathrm{THH}(A)$.*

3.3 Assembly maps

Let G be a discrete group, and let A be a ring or a ring spectrum.

Let $\mathcal{F}$ be a family of subgroups of G. Denote the collection of finite subgroups of G by $\mathcal{F}\mathrm{in}$, the collection of cyclic subgroups of G by $\mathcal{C}\mathrm{yc}$, the collection of finite cyclic subgroups by $\mathcal{FC}\mathrm{yc}$, and the $\mathcal{VC}\mathrm{yc}$ the collection of **virtually finite subgroups:** subgroups containing a subgroup of finite index. $\mathcal{A}\ell\ell$ is the collection of all subgroups.

$$\begin{tikzcd}
 & & \mathcal{F}\mathrm{in} \arrow[dr, hook] & & \\
\mathcal{F} = \{1\} \arrow[r, hook] & \mathcal{FC}\mathrm{yc} \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{VC}\mathrm{yc} \arrow[r, hook] & \mathcal{A}\ell\ell \\
 & & \mathcal{C}\mathrm{yc} \arrow[ur, hook] & &
\end{tikzcd}$$

Definition 3.15. The **orbit category** $\mathcal{O}(G, \mathcal{F})$ is the full subcategory of G-sets and G-maps spanned by G/H with $H \in \mathcal{F}$.

Example 3.16. $\mathcal{O}(G, \mathcal{A}\ell\ell)$ has a terminal object $G/G = \mathrm{pt}$. $\mathcal{O}(G, \{1\})$ is isomorphic to G as a one-object category.

let T be a functor $\mathbf{Sp} \to \mathbf{Sp}$ (e.g. $T = K, \mathrm{THH}, \mathrm{TC}, \ldots$) and let A be a ring or a ring spectrum. Assume that the functor $T(A(-))\colon \mathbf{Group} \to \mathbf{Sp}$ extends to a functor $T(A(-))\colon \mathbf{Groupoid} \to \mathbf{Sp}$ and this extension preserves equivalences.

Example 3.17. We may extend $\mathrm{THH}(A(-))$ to a functor from groupoids to spectra by modifying the construction of the cyclic nerve. If $\mathbf{C}$ is a groupoid or any category, then

$$(N_\times^{\mathrm{cyc}}\mathbf{C})_n = \coprod_{(x_0, x_1, \ldots, x_n) \in \mathrm{Ob}(\mathbf{C})^{\times(n+1)}} \mathbf{C}(x_0, x_n) \times \mathbf{C}(x_1, x_0) \times \cdots \times \mathbf{C}(x_n, x_{n-1}).$$

Definition 3.18. If G is a group and S is a G-set, then the **action groupoid** $G \int S$ has objects the elements of S and morphisms

$$G \textstyle\int S(x, y) = \{g \in G \mid gx = y\}.$$

Now consider

$$\mathcal{O}(G, \mathcal{F}) \hookrightarrow G\text{-}\mathbf{Sets} \xrightarrow{G \int -} \mathbf{Groupoid} \xrightarrow{T(A(-))} \mathbf{Sp}$$

Example 3.19.

$$G/H \mapsto G \textstyle\int G/H \simeq H \mapsto T(A[G \int G/H]) \simeq T(AH)$$

Definition 3.20. $f\colon X \to Y$ is **split injective** if there is some $g\colon Y \to Z$ such that $g \circ f$ is a weak equivalence.

Definition 3.21. The **assembly map** for $T(AG)$ with respect to $\mathcal{F}$ is the map

$$\alpha_{\mathcal{F}}\colon \operatorname{hocolim}_{G/_H \in \mathcal{O}(G, \mathcal{F})} T(A[G \textstyle\int G/H]) \to T(AG).$$

Theorem 3.22 (Lück–Reich–Rognes–Varisco)**.** *For all rings or ring spectra A, for all groups G, and for all families $\mathcal{F}$ of subgroups of G, the assembly map for* $\mathrm{THH}(AG)$ *with respect to $\mathcal{F}$ is split injective. Moreover, if $\mathcal{F} \supseteq \mathcal{C}\mathrm{yc}$, then it is a weak equivalence.*

Proof. Consider

$$\operatorname{hocolim}_{\mathcal{O}(G, \mathcal{F})} \mathrm{THH}(A[G \textstyle\int -]) \xrightarrow{\alpha_{\mathcal{F}}} \mathrm{THH}(AG) \cong \mathrm{THH}(A) \wedge B^{\mathrm{cyc}}G_+.$$

If we consider the map $B^{cyc}(G) \to G/\operatorname{conj}$ given by taking the conjugacy class of the product of group elements, then we have a cyclic map. Denote by $B^{cyc}_{[g]}G$ the preimage of a class $[g] \in G/\operatorname{conj}$ in $B^{cyc}G$. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}(A[G\int -]) \arrow[r, "\alpha_{\mathcal{F}}"] \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] & \mathrm{THH}(AG) \cong \mathrm{THH}(A) \wedge B^{cyc}G_+ \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] & \mathrm{THH}(A) \wedge \bigvee_{[g]\in G/\operatorname{conj}} B^{cyc}_{[g]}G_+ \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] \\
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}_{\mathcal{F}}(A[G\int -]) \arrow[r, "\alpha_{\mathcal{F}}"] & \mathrm{THH}_{\mathcal{F}}(AG) \arrow[r, "\mathrm{def}"] & \mathrm{THH}(A) \wedge \bigvee_{[g]\in G/\operatorname{conj}, \langle g\rangle \in \mathcal{F}} B^{cyc}_{[g]}G_+
\end{tikzcd}$$

Claim that the map $\mathrm{pr}_{\mathcal{F}}\colon \mathrm{THH}(AG) \to \mathrm{THH}_{\mathcal{F}}(A)$ is a weak equivalence, and an isomorphism if $\mathcal{F} \supseteq \mathcal{C}yc$. This follows from a computation that for all G-sets S,

$$B^{cyc}_{[g]}GS \simeq (S^{\langle g\rangle})_{hZ_G\langle g\rangle}.$$

There is a formal correspondence

$$\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} B^{cyc}_{[g]}(G\int -) \simeq (E(G,\mathcal{F})^{\langle g\rangle})_{hZ_G\langle g\rangle}.$$

Here $E(G,\mathcal{F})$ is the G-cell complex such that for all $H \leq G$, $E(G,\mathcal{F})^H$ is empty if $H \notin \mathcal{F}$ or contractible if $H \in \mathcal{F}$. □

3.4 THE FARRELL–JONES CONJECTURE

Example 3.23. If $\mathcal{F} = \{1\}$, then $\mathcal{O}(G,\{1\}) \cong G$, where we consider G as a category with one object. Then

$$\operatorname{hocolim}_{\mathcal{O}(G,\{1\})} T(A[G\int -]) = T(A[G\int {}^{G}/_{1}])_{hG} \simeq BG_+ \wedge T(A).$$

So in this case, the assembly map looks like $\alpha_{\{}1\}\colon BG_+ \wedge T(A) \to T(A[G])$. We call these **classical assembly maps**.

Let $K(-)$ denote the non-connective algebraic K-theory spectrum.

Definition 3.24. A ring A is **regular** if it is Noetherian and each module has a projective resolution of finite length.

Example 3.25. Any field or PID is a regular ring, and so is $\mathbb{Z}$.

Fact 3.26. *If* A *is a regular ring, then* $K_n(A) = 0$ *for all* $n < 0$.

Conjecture 3.27 (Farrell–Jones, special case)**.** *If* G *is torsion-free and* A *is regular, then*

$$\alpha_{\{1\}}\colon BG_+ \wedge K(A) \to K(AG)$$

is a weak equivalence.

If we know this conjecture, then we could compute the K-groups of a group ring AG given the K-groups of A:

$$K_n(AG) = \pi_n K(AG) \cong \pi_n(BG_+ \wedge K(A)) = H_n(G; K(A))$$

The right-hand-side can be computed by means of an Atiyah-Hirzebruch spectral sequence:

$$E^2_{s,t} = H_s(G; K_t(A)) \implies H_n(G; K(A)). \tag{3.2}$$

If A is regular, this is a first-quadrant spectral sequence.

Example 3.28. If $A = \mathbb{Z}$, then there is a homomorphism

$$\begin{array}{rcl} H_0(G; K_1(\mathbb{Z})) \oplus H_1(G; K_0(\mathbb{Z})) & \xrightarrow{\pi_1(\alpha_{\{1\}})} & K_1(\mathbb{Z}G) \\ (\pm 1, [g]) & \longmapsto & [(\pm g)] \end{array}$$

Hence, $\operatorname{coker}(\pi_1(\alpha_{\{1\}})) \cong \mathrm{Wh}(G)$, the Whitehead group of G. So the Farrell–Jones conjecture implies that $\mathrm{Wh}(G) = 0$ if G is torsion free.

Remark 3.29 (Warning!)**.** Neither the assumption that G is torsionfree nor that A is regular can be dropped.

If $G = C_n$, then $\mathrm{Wh}(C_n) = 0$ only if $n \in \{1, 2, 3, 4, 6\}$. So if G has torsion, then $\alpha_{\{1\}}$ may not be surjective even for $A = \mathbb{Z}$.

If $G = \mathbb{Z}$, then $BG = S^1$, and $AG = A[x^{\pm 1}]$. In this case,

$$\pi_n(\alpha_{\{1\}}) \colon \pi_n(S^1_+ \wedge K(A)) \to K_n(A[x^{\pm 1}])$$

Here, $\pi_n(S^1_+ \wedge K(A)) \cong K_n(A) \oplus K_{n-1}(A)$, because $S^1_+ \cong S^1 \wedge S^0$. So

$$\operatorname{coker}(\pi_n(\alpha_{\{1\}})) \cong NK_n(A) \oplus NK_n(A)$$

where $NK_n(A) = \ker(K_n(A[x]) \to K_n(A))$. When A is regular, this kernel is zero by the fundamental theorem of algebraic K-theory. So if A is not regular, then $\alpha_{\{1\}}$ may not be surjective even for $G = \mathbb{Z}$.

It may however be the case that the map $\pi_k(\alpha_{\{1\}})$ is an isomorphism for some but not all integers k.

Conjecture 3.30 (Farrell-Jones, general case)**.** *For all discrete groups G and all rings A,*

$$\alpha_{\mathcal{VC}\mathrm{yc}} \colon \operatorname{hocolim}_{G/_H \in \mathcal{O}(G, \mathcal{VC}\mathrm{yc})} K(A[G \textstyle\int G/H]) \to K(AG)$$

is a weak equivalence.

This next proposition shows that this general case subsumes the special case from before.

Proposition 3.31. *If* G *is torsionfree and* A *is regular, then*

$$\operatorname{hocolim}_{\mathcal{O}(G,\{1\})} K(A[G\textstyle\int -]) \to \operatorname{hocolim}_{\mathcal{O}(G,\mathcal{VC}yc)} K(A[G\textstyle\int -])$$

is a weak equivalence.

Proof. To prove this, we will use the **transitivity principle:** given $\mathcal{F} \subseteq \mathcal{H}$,

$$\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} T(A[G\textstyle\int -]) \to \operatorname{hocolim}_{\mathcal{O}(G,\mathcal{H})} T(A[G\textstyle\int -])$$

is a weak equivalence if for all $H \in \mathcal{H}$,

$$\operatorname{hocolim}_{\mathcal{O}(H,\mathcal{F}|_H)} T(A[G\textstyle\int -]) \to T(AH)$$

is a weak equivalence. Apply this to $\{1\} \subseteq \mathcal{VC}yc$. Since G is torsion free we know that $\mathcal{VC}yc = \mathcal{C}yc$. The proposition then follows from the general Farrell–Jones conjecture. $\square$

Theorem 3.32 (Lück–Steimle)**.** *If* A *is a regular ring, then for all groups* G*, we may reduce from virtual cyclic subgroups of* G *to finite cyclic subgroups; there are isomorphisms*

$$\pi_n\left(\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{FC}yc)} K(A[G\textstyle\int -])\right) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n\left(\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{VC}yc)} K(A[G\textstyle\int -])\right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Theorem 3.33 (Lück's Chern characters)**.** *There is an isomorphism*

$$\pi_n\left(\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{FC}yc)} K(A[G\textstyle\int -])\right) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{[C]\in \mathcal{FC}yc/_{\mathrm{conj}}} \bigoplus_{s+t=n} H_s(Z_G(C);\mathbb{Q}) \otimes_{\mathbb{Q}[W'_G C]} \left(\widehat{K}_t(AC) \otimes_{\mathbb{Z}} \mathbb{Q}\right),$$

where

- $Z_G C$ *is the centralizer of* C *in* G*,*
- $N_G C$ *is the normalizer of* C *in* G*,*
- $W'_G C = {}^{N_G C}/_{Z_G C}$
- $\widehat{K}(AC) = \operatorname{coker}\left(\bigoplus_{D \lneq C} K_t(AD) \to K_t(AC)\right)$

Inside this object, there is a summand corresponding to the trivial subgroup

$$\bigoplus_{s+t=n} H_s(G;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(A) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H_n(G;K(A)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The following diagram commutes:

$$
\begin{tikzcd}
\pi_n\left(\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{FC}yc)} K(A[G\textstyle\int -])\right) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\cong"] & \pi_n\left(\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{VC}yc)} K(A[G\textstyle\int -])\right) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\alpha_{\mathcal{VC}yc}"] & K_n(AG) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\displaystyle\bigoplus_{[C]\in \mathcal{FC}yc/_{\mathrm{conj}}} \bigoplus_{s+t=n} H_s(Z_G(C);\mathbb{Q}) \otimes_{\mathbb{Q}[W'_G C]} \left(\widehat{K}_t(AC) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \arrow[u, "\cong"] & & \\
\displaystyle\bigoplus_{s+t=n} H_s(G;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(A) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H_n(G;K(A)) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[u, hook] \arrow[r, "\cong"] & H_n(G;K(A)) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[uur, "\alpha_{\{1\}}"'] &
\end{tikzcd}
$$

The isomorphism across the bottom is the degeneration of the Atiyah–Hirzebruch spectral sequence as in (3.2), and the two maps $\alpha_{\mathcal{VC}yc}$ and $\alpha_{\{1\}}$ represent the general and special cases of the Farell–Jones conjecture, respectively.

Theorem 3.34 (Bökstedt–Hsiang–Madsen, Lück–Reich–Rognes–Varisco)**.** *The assembly map for connective algebraic* K*-theory* $K^{\geq 0}(-)$

$$\operatorname{hocolim}_{G/_H \mathcal{O}(G,\mathcal{F})} K^{\geq 0}(\mathbb{Z}[G\textstyle\int {}^G/_H]) \to K^{\geq 0}(\mathbb{Z}G)$$

is rationally injective if $\mathcal{F} \subseteq \mathcal{FC}\mathrm{yc}$ *and for all* $C \in \mathcal{F}$*,*

(a) for all $s \geq 0$*,* $H_s(Z_G C;\mathbb{Z})$ *is finitely generated, and*

(b) for all $t \geq 0$*,*

$$K_t(\mathbb{Z}[\zeta_c]) \otimes_{\mathbb{Z}} \mathbb{Q} \to \prod_{p \textit{ prime}} K_t(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c];\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.3}$$

is injective,

where $c = \#C$*,* ζ_c *is a primitive* c*-th root of unity, and* $K_t(-;\mathbb{Z}_p) = \pi_t(K(-)^{\wedge}_p)$.

The theorem is due to Bökstedt–Hsiang–Madsen for $\mathcal{F} = \{1\}$, and Lück–Reich–Rognes–Varisco for arbitrary $\mathcal{F}$.

Remark 3.35. The map (3.4) is injective in the following cases:

- if $c = 1$ for all t, (so condition (b) is moot in the Bökstedt–Hsiang–Madsen version)
- if $t \in \{0,1\}$ for any c,
- for fixed c, and all but infinitely many t,
- if the Leopoldt–Schneider conjecture is true for $\mathbb{Q}(\zeta_c)$.

So we adopt the motto that condition (b) is conjecturally *always* true.

If we assume only condition (a), then we have a similar statement about injectivity of assembly maps for the Whitehead groups.

Corollary 3.36. *The assembly map*

$$\operatorname*{colim}_{G/C \in \mathcal{O}(G,\mathcal{F})} \mathrm{Wh}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective if $\mathcal{F} \subseteq \mathcal{FC}\mathrm{yc}$ *and for all* $C \in \mathcal{F}$, $H_s(\mathbb{Z}_G C; \mathbb{Z})$ *is finitely generated for all* $s \geq 0$.

Remark 3.37. The Farrell–Jones conjecture is still open, but it is known to be true for many classes of groups, including but not limited to:

- fundamental groups of closed Riemannian manifold with negative sectional curvature (Farell–Jones),
- hyperbolic groups and CAT(0) groups (non-positive curvature conditions on abstract groups) (Bartels–Lück–Reich),
- arithmetic groups (Bartels–Lück–Reich–Rüping),
- mapping class groups (Bartels–Bestivina),
- virtually solvable groups (Wegner).

Notice that the first four types of groups are groups with some geometric condition, yet in Theorem 3.34 here we had a homological finiteness condition. However, Theorem 3.34 is much weaker than the Farell–Jones conjecture.

It is known that the class of groups for which the conjecture is true is closed under colimits, free products, amalgamated free products, subgroups, and some other constructions.

3.4.1 Geometric applications

Definition 3.38. If M is a closed topological manifold, we say that M is **topologically rigid** if for all closed manifolds N, if N is homotopy equivalent to M, then N is homeomorphic to M.

Definition 3.39. A topological manifold M is **aspherical** if its universal cover is contractible.

Conjecture 3.40 (Poincaré)**.** S^n *is topologically rigid.*

Conjecture 3.41 (Borel)**.** *Aspherical manifolds are topologically rigid.*

Remark 3.42. If the Farell–Jones conjecture is known for both $K(\mathbb{Z}G)$ and $L(\mathbb{Z}G)$ (algebraic L-theory) then the Borel conjecture is also true in dimensions at least 5.

Remark 3.43. If $\mathrm{Wh}(1) = 0$, then the Poincaré conjecture holds for $n \geq 4$.

3.5 Proof of $\mathbb{Q}$-injectivity of assembly maps for $K(\mathbb{Z}[G])$

Bökstedt–Hsiang–Madsen invented TC and the cyclotomic trace to compute the $\mathbb{Q}$-injectivity of assembly maps. We will present a simplified version of the argument for rational homotopy groups, and at the same time, present a generalization due to Lück–Rognes–Reich–Varisco.

The Farell–Jones conjecture states that there is an equivalence,

$$\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} K(\mathbb{Z}-) \to K(\mathbb{Z}G).$$

If $\mathcal{F} = \mathcal{VC}\mathrm{yc}$, this is the general case of the Farell–Jones conjecture. If $\mathcal{F} = \{1\}$, then this reduces to the case Bökstedt–Hsaing–Madsen considered.

Rationally, we can instead consider

$$\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} K(\mathbb{S}-) \to \mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} K(\mathbb{Z}-) \to K(\mathbb{Z}G).$$

The first map is always $\mathbb{Q}$-injective for $\mathcal{F} \subseteq \mathcal{FC}\mathrm{yc}$, and a $\mathbb{Q}$-isomorphism if $\mathcal{F} = \mathcal{FC}\mathrm{yc}$. The second is the Farell–Jones conjecture.

The strategy of the proof is contained in the following diagram:

$$\begin{tikzcd}
\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} K(\mathbb{Z}-) \arrow[r] & \mathrm{hocolim}_{\mathcal{O}(G,\mathcal{VC}\mathrm{yc})} K(\mathbb{Z}-) \arrow[r] & K(\mathbb{Z}G) \\
\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} K(\mathbb{S}-) \arrow[u, "\simeq_{\mathbb{Q}}"] \arrow[d] \arrow[dd, bend right=60, "\beta"'] & & K(\mathbb{S}G) \arrow[u, "\simeq_{\mathbb{Q}}"'] \arrow[d, "\mathrm{tr}"] \\
\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} \prod_{p \text{ prime}} \mathrm{TC}(\mathbb{S}-;p) \arrow[d] & & \prod_{p \text{ prime}} \mathrm{TC}(\mathbb{S}G;p) \arrow[d] \\
\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}(\mathbb{S}-) \times \prod_{p \text{ prime}} C(\mathbb{S}-;p) \arrow[rr, "\alpha"] & & \mathrm{THH}(\mathbb{S}G) \times \prod_{p \text{ prime}} C(\mathbb{S}G;p)
\end{tikzcd}$$

To show rational injectivity across the top, we will show rational injectivity across the bottom and left sides.

Recall our assumptions on G and $\mathcal{F} \subseteq \mathcal{FC}\mathrm{yc}$:

(a) For all $C \in \mathcal{F}$ and all $s \geq 0$, $H_s(Z_G C; \mathbb{Z})$ is finitely generated.

(b) for all $t \geq 0$,

$$K_t(\mathbb{Z}[\zeta_c]) \otimes_{\mathbb{Z}} \mathbb{Q} \to \prod_{p \text{ prime}} K_t(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.4}$$

is injective, where $c = \#C$, ζ_c is a primitive c-th root of unity, and $K_t(-;\mathbb{Z}_p) = \pi_t(K(-)^{\wedge}_p)$.

The assumption (b) is conjecturally always true, and definitely true if $\mathcal{F} = \{1\}$.

There are two theorems that we will use to prove the injectivity of α and β.

Theorem 3.44 (Splitting Theorem). *Assume (a). Then α is rationally injective.*

Theorem 3.45 (Detection Theorem). *Assume (b). Then $\beta^{\geq 0}$ is rationally injective, where $\beta^{\geq 0}$ is the connective cover of β.*

Before we prove these theorems, a warning is in order.

Remark 3.46 (Warning). Without assumption (a), then $\alpha\beta^{\geq 0}$ may not be rationally injective. For a counterexample, choose $G = \mathbb{Q}$ and $\mathcal{F} = \{1\}$. Then $H_1(\mathbb{Q};\mathbb{Z}) = \mathbb{Q}$, is not finitely generated, and $\alpha\beta^{\geq 0}$ is therefore not rationally injective (even though the Farrell–Jones conjecture holds for $\mathbb{Q}$).

3.5.1 The Detection Theorem

To prove the detection theorem, apply Lück's Chern character to both source and target of $\beta^{\geq 0}$. For all $C \in \mathcal{F}$, consider the diagram.

$$\begin{tikzcd}
K(\mathbb{S}C) \arrow[r, "\simeq_{\mathbb{Q}}"] \arrow[d, "\psi"] & K(\mathbb{Z}C) \arrow[r, "\phi"] & \prod_p K(\mathbb{Z}_p C)^{\wedge}_p \arrow[d, "\mathrm{tr}"] \\
\prod_p TC(\mathbb{S}C; p) \arrow[rr] & & \prod_p TC(\mathbb{Z}_p C)^{\wedge}_p
\end{tikzcd}$$

We want to know that the map ψ on the left is rationally injective; it will suffice to show that ϕ and tr are rationally injective.

Under assumption (b), $\phi^{\geq 0}$ is rationally injective.

Hesselholt–Madsen show that the trace map

$$\prod_p K(\mathbb{Z}_p C)^{\wedge}_p \xrightarrow{\mathrm{tr}} \prod_p TC(\mathbb{Z}_p C)^{\wedge}_p$$

is an equivalence on the connective covers, so ψ is rationally injective.

This is one half of the Detection theorem; we now need to know that the map

$$\prod_{p \text{ prime}} TC(\mathbb{S}-; p) \to THH(\mathbb{S}-) \times \prod_{p \text{ prime}} C(\mathbb{S}-; p)$$

is rationally injective on connective covers.

3.5.2 The Splitting Theorem

Theorem 3.47. *For all rings or ring spectra A and all groups G and all collections $\mathcal{F}$ of subgroups of G, there is a commutative square*

$$\begin{tikzcd}
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}(A-) \arrow[r] \arrow[d, "\simeq"] & \mathrm{THH}(AG) \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] \\
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}_{\mathcal{F}}(A-) \arrow[r, "\simeq"] & \mathrm{THH}_{\mathcal{F}}(AG)
\end{tikzcd}$$

such that $\mathrm{pr}_{\mathcal{F}}$ is an isomorphism if $\mathcal{F} \supseteq \mathcal{C}yc$.

We will use the BHM description of TC here: we have two maps $R, F\colon \mathrm{THH}(A)^{C_p} \to \mathrm{THH}(A)$ called the restriction and Frobenius maps. Then we can schematically illustrate TC in the following diagram:

$$\begin{tikzcd}
 & & \mathrm{TC}(A;p) \arrow[r, equal] \arrow[d] & \operatorname{hoeq}\left(\mathrm{TF}(A;p), \overset{R}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{TF}(A;p) \right) \\
 & C(A;p) \arrow[r] \arrow[d, dashed] & \mathrm{TF}(A;p) \arrow[r, "R"] \arrow[d, dashed] & \mathrm{TF}(A;p) = \operatorname{holim}_{\mathbb{N}} \arrow[d, dashed] \\
\mathrm{THH}(A)_{hC_{p^2}} \arrow[r, "\simeq"] & \operatorname{hofib}(R) \arrow[r] \arrow[d] & \mathrm{THH}(A)^{C_{p^2}} \arrow[r] \arrow[d] & \mathrm{THH}(A)^{C_p} \arrow[d, "F"] \\
\mathrm{THH}(A)_{hC_p} \arrow[r, "\simeq"] & \operatorname{hofib}(R) \arrow[r] & \mathrm{THH}(A)^{C_p} \arrow[r, "R"] & \mathrm{THH}(A)
\end{tikzcd}$$

The identification $\mathrm{THH}(A)_{hC_{p^2}} \simeq \operatorname{hofib}(R)$ only works when A is connective; from now on, assume that A is connective.

Now consider the diagram from **??**. We may amend the theorem by taking homotopy orbits of the C_{p^n} action, so we have commutative squares

$$\begin{tikzcd}
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}(A-)_{hC_{p^n}} \arrow[r] \arrow[d, "\simeq"] & \mathrm{THH}(AG)_{hC_{p^n}} \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] \\
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{THH}_{\mathcal{F}}(A-)_{hC_{p^n}} \arrow[r, "\simeq"] & \mathrm{THH}_{\mathcal{F}}(AG)_{hC_{p^n}}
\end{tikzcd}$$

Now take the homotopy limit over $n \in \mathbb{N}$, and assume that A is connective. In this case, we have a diagram

$$\begin{tikzcd}
\operatorname{holim}_{\mathbb{N}} \operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \operatorname{hofib} R \arrow[r] & \operatorname{holim}_{\mathbb{N}} \operatorname{hofib} R \arrow[d, equal] \\
\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \operatorname{holim}_{\mathbb{N}} \operatorname{hofib} R \arrow[u, "\gamma"] \arrow[r, "\alpha_{\mathcal{F}}^{C}"] & C(AG;p)
\end{tikzcd}$$

The map across the top is split injective.

Theorem 3.48 (Lück–Reich–Varisco)**.** *Assume (a) and that the number of conjugacy classes of subgroups in $\mathcal{F}$ is finite. Then γ is rationally injective.*

Corollary 3.49. *Assume (a). Then $\alpha^{\mathrm{C}}_{\mathcal{F}}$ is rationally injective.*

This proves the Splitting Theorem.

So now it remains to show how to travel from TC to C. Since we're working over the sphere spectrum, the map $R\colon \mathrm{THH}^{C_p}(\mathbb{S}) \to \mathrm{THH}(\mathbb{S})$ splits; in fact we may add splittings to a previous diagram

$$\begin{tikzcd}
 & & \mathrm{TC}(A;p) \arrow[dl] \arrow[d] \arrow[r, equal] & \mathrm{hoeq}\left(\mathrm{TF}(A;p), \overset{R}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{TF}(A;p)\right) \\
 & \mathrm{C}(A;p) \arrow[r, shift left] \arrow[d, dashed] & \mathrm{TF}(A;p) \arrow[r, "R", shift left] \arrow[l, bend left] \arrow[d, dashed] & \mathrm{TF}(A;p) = \mathrm{holim}_{\mathbb{N}} \arrow[l, bend left] \arrow[d, dashed] \\
\mathrm{THH}(A)_{hC_{p^2}} \arrow[r, "\simeq"] & \mathrm{hofib}(R) \arrow[r, shift left] \arrow[d] & \mathrm{THH}(A)^{C_{p^2}} \arrow[r, shift left] \arrow[l, bend left] \arrow[d] & \mathrm{THH}(A)^{C_p} \arrow[l, bend left] \arrow[d, "F"] \\
\mathrm{THH}(A)_{hC_p} \arrow[r, "\simeq"] & \mathrm{hofib}(R) \arrow[r, shift left] & \mathrm{THH}(A)^{C_p} \arrow[r, "R", shift left] \arrow[l, bend left] & \mathrm{THH}(A) \arrow[l, bend left]
\end{tikzcd}$$

Remark 3.50. R restricts to a map $\mathrm{THH}^{C_p}_{\mathcal{F}} \to \mathrm{THH}_{\mathcal{F}}$ if and only if for all $g \in G$, $\langle g \rangle \in \mathcal{F} \iff \langle pg \rangle \in \mathcal{F}$. If $\mathcal{F}$ satisfies this condition, then we say that $\mathcal{F}$ is p**-radicable.**

Example 3.51. If $\mathcal{F} = \mathcal{F}\mathcal{C}\mathrm{yc}$ or $\mathcal{F} = \mathcal{C}\mathrm{yc}$ are p-radicalizable, but $\mathcal{F} = \{1\}$ is not if G has p-torsion.

3.6 The Farrell–Jones conjecture for TC

Define

$$\mathrm{TC}(A;p) = \mathrm{hoeq}\left(\mathrm{THH}(A;p)^{C_{p^n}}, \overset{R}{\underset{F}{\rightrightarrows}} \mathrm{THH}(A;p)^{C_{p^{n-1}}} \right)$$

Note that, because homotopy equalizers and homotopy limits commute,

$$\mathrm{TC}(A;p) = \mathrm{TC}^n(A;p).$$

Theorem 3.52 (Lück–Reich–Rognes–Varisco)**.** *For all connective ring spectra A and for all groups G and primes p,*

$$\left\{\mathrm{hocolim}_{\mathcal{O}(G,\mathcal{C}\mathrm{yc})} \mathrm{TC}^n(A-;p)\right\}_n \xrightarrow{\{\alpha^{\mathrm{TC}^n}_{\mathcal{C}\mathrm{yc}}\}_n} \{\mathrm{TC}^n(AG;p)\}_n$$

is a levelwise weak equivalence of pro-spectra.

We think this is the correct version of the Farell–Jones conjecture for TC; it is really a statement about pro-systems of spectra rather than a statement for just TC itself.

Consider the assembly map for TC for a fixed prime p:

$$\operatorname{hocolim}_{\mathcal{O}(G,\mathcal{F})} \mathrm{TC}(A-;p) \xrightarrow{\alpha^{\mathrm{TC}}_{\mathcal{F}}} \mathrm{TC}(AG;p).$$

Theorem 3.53 (Lück–Reich–Rognes–Varisco)**.** *For all connective A and for all p,*

(1) Assume (a) and that the number of conjugacy classes of subgroups in $\mathcal{FC}\mathrm{yc}$ is finite. Then $\alpha^{\mathrm{TC}}_{\mathcal{FC}\mathrm{yc}}$ is rationally injective.

(2) $\alpha^{\mathrm{TC}}_{\mathcal{FC}\mathrm{yc}}$ is a weak equivalence if G is finite.

(3) $(\alpha^{\mathrm{TC}}_{\mathcal{VC}\mathrm{yc}})_*$ is injective on homotopy groups if G is hyperbolic or virtually finitely-generated abelian.

Remark 3.54. In general, $(\alpha^{\mathrm{TC}}_{\mathcal{VC}\mathrm{yc}})_*$ is not surjective on homotopy groups; there are explicit counterexamples. If G is torsion-free hyperbolic or $\mathbb{Z}^n$ for $n > 1$, and $\pi_0 A = \mathbb{Z}_{(p)}$, then $(\alpha^{\mathrm{TC}}_{\mathcal{VC}\mathrm{yc}})_*$ is not surjective on π_{-1}.

Question 3.55. Can we remove the assumption that the number of conjugacy classes in $\mathcal{FC}\mathrm{yc}$ is finite? We have such a result for K-theory, but we don't for TC. This seems to be a weird case when we know more for K-theory than we do for TC.