Homotopy theory reading group
Recap of sections 1.1 and 1.2

MARU SARAZOLA

1.1 Quasi-categories

Definition 1 (quasi-categories)**.** A *quasi-category* is a simplicial set X such that every inner horn has a filler.

Two examples to keep in mind:

Example 2. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a quasicategory.

In this case, all fillers will be unique. In fact, the converse is also true: any quasi-category with unique fillers comes from the nerve of a category.

Example 3. If X is a topological space, recall that its singular complex is

$$\operatorname{Sing}_n X = \mathbf{Top}(\Delta_n, X)$$

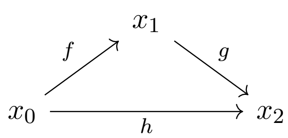
where Δ_n denotes the geometric n-simplex.

In this case, we don't necessarily have unique fillers, but we find fillers for *all* horns, not just inner ones. Such a simplicial set is called *Kan complex*, and they play an important role in studying the homotopy theory of **sSet**.

Quasi-categories already come with a natural notion of homotopy, which can be used to define the following.

Definition 4 (homotopy category)**.** If X is a quasi-category, its *homotopy category* hX has

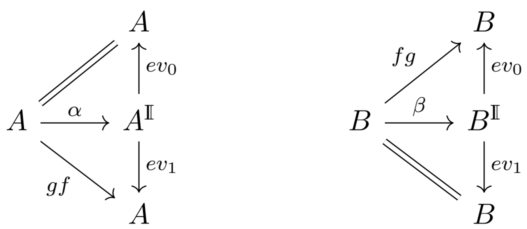
- as objects, the set X_0,
- as morphisms, the set of homotopy classes of 1-simplices in X_1,
- a composition relation $h = g \circ f$ if and only if, for any choices of 1-simplices representing these maps, there exists a 2-simplex



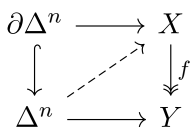
The following are some important classes of maps.

Definition 5 (isofibrations)**.** A simplicial map $f : X \to Y$ is an *isofibration* if it lifts against the inner horn inclusions, and against the inclusion of either vertex into the free standing isomorphism $\mathbb{I}$

Definition 6 (equivalences between quasi-categories)**.** A map $f : A \to B$ between quasi-categories is an *equivalence* if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval; that is, if there exist maps $g : B \to A, \alpha$ and β such that



Definition 7 (trivial fibrations)**.** A simplicial map $f : X \to Y$ is a *trivial fibration* if it lifts against all boundary inclusions



Remark 8. If this last nomenclature reminds you of model categories, that is exactly right: there exists a model structure on simplicial sets (the Joyal model structure) whose fibrant objects are the quasi-categories. The fibrations, weak equivalences, and trivial fibrations between fibrant objects are precisely the classes of isofibrations, equivalences, and trivial fibrations, respectively. If you don't know what any of this means, that's totally fine; you can simply remember the following.

Proposition 9. *As suggested by the notation,*

Trivial fibration = isofibration + equivalence

1.2 ∞-cosmoi

In the book, they don't define exactly what an ∞-category should be; rather, they axiomatize the "universe" in which ∞-categories live, and give an idea of how they interact with each other via some special classes of maps, and as usual in category theory, these probings should give us some idea of what these things look like.

Definition 10 (∞-cosmoi)**.** An ∞-*cosmos* $\mathcal{K}$ is a category enriched over quasi-categories, meaning that it has

- objects A, B, that we call ∞-categories, and
- its morphisms define the vertices of functor-spaces $\mathrm{Fun}(A, B)$, which are quasi-categories,

that is also equipped with a specified class of maps that we call isofibrations.

From these classes, we define a map $f : A \to B$ to be an equivalence if and only the induced map $f_* : \mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathcal{K}$, and we define f to be a trivial fibration just when f is both an isofibration and an equivalence.

Aside from this, there are two axioms:

(i) (completeness) "$\mathcal{K}$ has a bunch of (enriched) limits".

(ii) (isofibrations) "isofibrations behave like fibrations in a model category, and everything is fibrant".

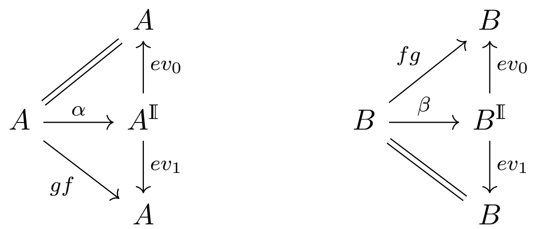
Example 11. Some examples of ∞-cosmoi are:

1. **qCat**,
2. **Cat**, through the nerve functor,
3. (the category of) complete Segal spaces,
4. (the category of) Segal categories,
5. (the category of) 1-complical sets, or "naturally marked quasi-categories", if you know what those are (I don't),
6. the subcategory of cofibrant-fibrant objects in a model category enriched (as a model category) over the Joyal model structure on simplicial sets.

As a consequence of the axioms in definition 10, the class of trivial fibrations enjoys the same stability properties as the class of fibrations.

Another thing that works just like in quasi-categories is that we can characterize equivalences as "homotopy equivalences".

Lemma 12 (equivalences are homotopy equivalences)**.** *A map $f : A \to B$ in an ∞-cosmos $\mathcal{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence", that is, if there exist maps $g : B \to A, \alpha$ and β such that*



Questions

1. Why is it okay that simplicial categories aren't an example of ∞-cosmos?
2. I'm not convinced that they put so much emphasis on quasi-categories when defining ∞-cosmoi.