Riehl-Verity 2.1-2.2 Summary and Questions

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These chapters begin the generalization of traditional 1-category theory to arbitrary ∞-cosmoi, where the 2-category of categories, functors, and natural transformations are replaces by the homotopy 2-category of the cosmos.

1. 2.1 : Adjoints

Definition 1. An adjunction between ∞-categories consists of a pair of ∞-categories A, B, a pair of functors $u : A \to B$, $f : B \to A$, and a pair of natural transformations $\eta : 1_B \Rightarrow uf$ and $\epsilon : fu \Rightarrow 1_A$ which satisfy the triangle identities in the homotopy 2-category.

Note that this definition is precisely that of an adjunction in the homotopy 2-category, though there is some (purposeful!) vagueness in what is meant by natural transformation in this context (does it exist in the cosmos or its homotopy 2-category).

Question 2. *What does this look like explicitly in the functor spaces of a cosmos, and how does it simplify possible definitions of an adjunction in that context?*

Lemma 3. *Adjunctions in a 2-category are preserved by any 2-functor.*

Note that this follows directly from the definition of adjunction using unit and counit rather than the alternative definition by isomorphisms of Hom sets.

Example 4. Take any adjunction of 1-categories and apply the nerve 2-functor to get an adjunction of quasicategories.

Proposition 5. *Any adjunction in an ∞-cosmos $\mathcal{K}$ is preserved by the following 2-functors:*

- $Fun(X, -) : K \to QCat$ *for any ∞-category X in $\mathcal{K}$*
- $hFun(X, -) : K \to Cat$ *for any ∞-category X in $\mathcal{K}$*
- $(-)^U : \mathcal{K} \to \mathcal{K}$ *for any simplicial set U*
- $(-)^C : \mathcal{K} \to \mathcal{K}$ *for any ∞-category C in $\mathcal{K}$ if $\mathcal{K}$ is cartesian closed*

Proposition 6. *For adjunctions $f : B \rightleftharpoons A : u$ and $f' : C \rightleftharpoons B : u'$ the composites $ff' : C \rightleftharpoons A : u'u$ form an adjunction.*

Proposition 7. *Any two left adjoints of $u : A \to B$ are isomorphic, and any 1-cell isomorphic to a left adjoint of u is also a left adjoint of u.*

Proposition 8. *Any equivalence $f : A \rightleftharpoons B : g$ can be promoted to an adjoint equivalence by modifying just one of the 2-cells that define it.*

Proposition 9. *Adjunctions are preserved and reflected by equivalences.*

Lemma 10. *For any ∞-category A, the composition functor $A^{\nvDash} \times_A A^{\nvDash} \xrightarrow{\circ} A^{\nvDash}$ admits left and right adjoints which pair an arrow with an identity on the left or right respectively.*

Note that all of these results are purely 2-categorical; no additional structure of a cosmos goes into them (other than the cotensoring in the previous lemma).

2. Initial and Terminal Elements

We now begin with an example of defining (co)limits in this formalism. Instead of reusing the word "object" to no end, we call maps $a : 1 \to A$ elements of the ∞-category A instead.

Definition 11. An initial element in an ∞-category A is a left adjoint to the unique functor $! : A \to 1$, and a terminal element is a right adjoint to $!$.

Question 12. *Why does this make sense as a definition of initial and terminal objects?*

Lemma 13. *To define an initial element in A it suffices to specify an element $i : 1 \to A$ and a natural transformation $\epsilon : i! \Rightarrow 1_A$ such that the component $\epsilon i : i \Rightarrow i$ is the identity in hA.*

Lemma 14. *An element $i : 1 \to A$ is initial if and only if for all $f : X \to A$ there exists a unique 2-cell $i! \Rightarrow f$.*

This shows an initial element of A to be in fact representably initial, and in particular specializes to show it is also initial among elements in the homotopy category hA.

Lemma 15. *Equivalences preserve initial elements up to isomorphism.*

Proof. An equivalence can be promoted to an adjunction, and adjunctions compose, so if we have an adjunction $i : 1 \rightleftharpoons A :!$ and an equivalence $f : A \rightleftharpoons A' : g$, we get an adjunction $fi : 1 \rightleftharpoons A' :!g$. □