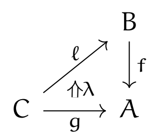
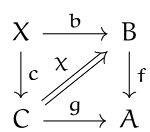
Limits, Colimits, and Preservation

2.3: Limits and Colimits

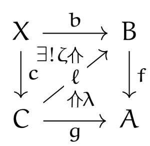
Definition 1 ([RV18, Definition 2.3.4]). Give a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in any 2-category, an **absolute left lifting** of g through f is given by a 1 cell $\ell \colon C \to B$ and a 2-cell $\lambda \colon g \implies f\ell$, as below



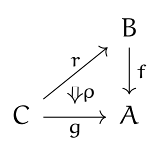
such that any two cell



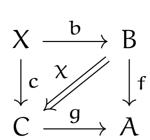
factors uniquely through λ as



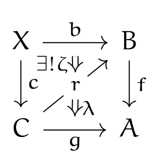
Definition 2 ([RV18, Definition 2.3.4]). Give a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in any 2-category, an **absolute right lifting** of g through f is defined dually: it is given by a 1-cell $r \colon C \to B$ and a 2-ell $\rho \colon fr \implies g$



such that any two cell



factors uniquely through λ as



Definition 3 ([RV18, Definition 2.3.7]). A **colimit** of a family of diagrams $d\colon D \to A^J$ indexed by a simplicial set J in an ∞-category A is given by an absolute left lifting diagram

$$\begin{array}{ccc} & & A \\ & \overset{\text{colim}}{\nearrow} \ \Uparrow\eta & \downarrow \Delta \\ D & \xrightarrow[d]{} & A^J \end{array}$$

comprised of a **colimit functor** $\text{colim}\colon D \to A$ and a **colimit cocone** $\eta\colon d \implies \Delta \circ \text{colim}$.

Definition 4 ([RV18, Definition 2.3.7]). A **limit** of a family of diagrams $d\colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute right lifting diagram

$$\begin{array}{ccc} & & A \\ & \overset{\lim}{\nearrow} \ \Downarrow\varepsilon & \downarrow \Delta \\ D & \xrightarrow[d]{} & A^J \end{array}$$

comprised of a **limit functor** $\lim\colon D \to A$ and a **limit cone** $\varepsilon\colon \Delta \circ \lim \implies D$.

2.4: Preservation of Limits

Theorem 5 ([RV18, Theorem 2.4.2]). *Right adjoints preserve limits and left adjoints preserve colimits.*

Corollary 6 ([RV18, Corollary 2.4.4]). *Equivalences preserve limits and colimits.*

Discussion

During the meeting, we discussed exercises 2.3.i, 2.3.ii, 2.4.i, and 2.4.iv. These exercises solved by diagram chases, essentially, hints are below:

- 2.3.i Draw what you want to prove and what you know, and then compare the two. They're basically the same!
- 2.3.ii For uniqueness, paste ε on to the right-hand-side of the square for a 2-cell χ as in [RV18, Definition 2.3.4] to show that any 2-cell ζ (again as in [RV18, Definition 2.3.4]) must have the correct form. This also serves as a definition of ζ.

  For the converse, draw out the triangle identities that you want to show as pasting diagrams in the homotopical 2-category of the ∞-cosmos. One of the two triangle identities comes from the existence part of the universal property of the absolute left lifting (f, η), while the other comes from the uniqueness part (use $\chi = \eta$).
- 2.4.i Use uniqueness of left adjoints and the fact that the square below

  $$\begin{array}{ccc} B^J & \xleftarrow[u]{} & A^J \\ {\scriptstyle\Delta}\uparrow & & {\scriptstyle\Delta}\uparrow \\ B & \xleftarrow[u]{} & A \end{array}$$

  commutes because cotensoring with simplicial sets in an ∞-cosmos $\mathcal{K}$ is natural in each variable separately.
- 2.4.iv Use the dual version of [RV18, Lemma 2.4.1] involving left lifting diagrams, and [RV18, Exercise 2.4.ii]. The absolute left lift σ (the upper one in the diagram) will be a horizontal composite of the unit for one of the adjunctions with one of the left adjoints.

References

[RV18] Emily Riehl and Dominic Verity. Elements of ∞-category theory. Available online at http://www.math.jhu.edu/~eriehl/elements.pdf, September 2018.