HOMOTOPY THEORY READING GROUP: RECAP ON WEAK 2-LIMITS IN THE HOMOTOPY 2-CATEGORY

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The 2-categorical definition of adjunctions between ∞ -categories does not capture the expected universal properties of cones nor do we see how an adjunction $f \dashv u$ induces an isomorphism in "hom-spaces" $\operatorname{Hom}_A(fb, a) \simeq \operatorname{Hom}_B(b, ga)$ for generalized elements. In order to define a notion of ∞ -category of cones and these hom-spaces, we need an appropriate notion of comma ∞ -category which is in a sense weaker than the standard definition of a comma object in a strict 2-category. This weaker notion of universal property will be expressed via a *smothering functor*.

1. Smothering Functors

Definition 1. A functor $f : A \to B$ between 1-categories is **smothering** if it is surjective on objects, full, and conservative (Namely, it reflects invertibility of morphisms but is neither injective on objects nor faithful).

Example. If \mathcal{Q} is a quasi-category and \mathcal{J} is a 1-category, then one might be interested in understanding the relationship between the homotopy category of $\mathcal{Q}^{\mathcal{J}}$ and the category of \mathcal{J} -diagrams in the homotopy category of \mathcal{Q} . There is a canonical functor $\mathbf{h}(\mathcal{Q}^{\mathcal{J}}) \to (\mathbf{h}\mathcal{Q})^{\mathcal{J}}$ defined by applying $\mathbf{h} : \mathcal{Q}\mathbf{Cat} \to \mathbf{Cat}$ to the evaluation functor $\mathcal{Q}^{\mathcal{J}} \times \mathcal{J} \to \mathcal{Q}$ and taking its adjoint mate. One can show that if \mathcal{J} is free on a directed graph, then this canonical functor is smothering (Lemma 3.1.4).

Lemma 2. Each fibre of a smothering functor is a non-empty connected groupoid.

Lemma 3. For any pullback diagram of quasi-categories in which p is an isofibration

$$\begin{array}{ccc} A \times_B E \longrightarrow E \\ & & & \downarrow^p \\ A \longrightarrow B \end{array}$$

the canonical functor $\mathbf{h}(A \times_B E) \to \mathbf{h}A \times_{\mathbf{h}B} \mathbf{h}E$ is smothering.

Lemma 4. For any tower of isofibrations between quasi-categories

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathbf{h}(\lim_{n} E_n) \to \lim_{n} \mathbf{h} E_n$ is smothering.

Lemma 5. For any cospan between quasi-categories $C \xrightarrow{g} A \xleftarrow{f} B$ consider the quasi-category defined by the pullback

$$\begin{array}{c|c} Hom_A(f,g) \longrightarrow A^2 \\ \downarrow^{(p_1,p_0)} & & \downarrow^{(cod,dom)} \\ C \times B \xrightarrow[g \times f]{} A \times A \\ 1 \end{array}$$

The canonical functor $\mathbf{h}Hom_A(f,g) \to Hom_{\mathbf{h}A}(\mathbf{h}f,\mathbf{h}g)$ is smothering.

Now, we see how these smothering functors express a "weak" notion of universal properties in the homotopy 2-category of any ∞ -cosmos.

2. ∞ -categories of arrows

Definition 6. Let A be an ∞ -category. The ∞ -category of arrows in A is the simplicial cotensor A^2 together with the canonical endpoint-evaluation isofibration

$$A^{\mathbf{2}}: A^{\Delta[1]} \stackrel{(p_0, p_1)}{\twoheadrightarrow} A^{\partial \Delta[1]} \cong A \times A$$

induced by $\partial \Delta[1] \to \Delta[1]$.

Lemma 7. For any ∞ -category A, the ∞ -category of arrows comes equipped with a canonical 2-cell

$$A^{2} \underbrace{\stackrel{p_{0}}{\overbrace{\downarrow}\kappa}}_{p_{1}} A$$

that we refer to as the generic arrow with codomain A.

The idea of this lemma is to use the universal property of the cotensor A^2 given by

$$\operatorname{Fun}(X, A^2) \cong \operatorname{Fun}(X, A)^2$$

and consider $X = A^2$. We get κ by taking the image of the identity under this isomorphism which is a 1-simplex in Fun (A^2, A) representing a 2-cell in the homotopy 2-category.

Proposition 8. The generic arrow with codomain A has a weak universal property in the homotopy 2-category given by three operations (see Proposition 3.2.5): 1-cell induction, 2-cell induction, and 2-cell conservativity.

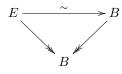
Remark. This weak universal property comes from the fact that the natural map

 $\mathbf{h}\mathrm{Fun}\left(X,A^{2}\right) \to \mathbf{h}\mathrm{Fun}\left(X,A\right)^{2}$

of homotopy categories is a smothering functor. Surjectivity of objects expresses the notion of 1-cell induction, fullness expresses 2-cell induction, and conservativity expresses 2-cell conservativity.

This universal property is also enjoyed by objects $(e_0, e_1) : E \to A \times A$ that are equivalent to the arrow ∞ -category $(p_0, p_1) : A^2 \to A \times A$ in the following sense.

Definition 9. A fibered equivalence over an ∞ -category B in an ∞ -cosmos \mathcal{K} is an equivalence



in the sliced ∞ -cosmos $\mathcal{K}_{/B}$.

Proposition 10. For any isofibration $(e_0, e_1) : E \twoheadrightarrow A \times A$ with a fibered equivalence $e : E \xrightarrow{\sim} A^2$, the corresponding 2-cell $\epsilon : e_0 \Rightarrow e_1$ satisfies the defined weak universal properties. Furthermore, arrow ∞ -categories are unique up to fibered equivalence.