THE SERRE SPECTRAL SEQUENCE NAME: ____________________

January 29, 2018

For a homotopy fibration sequence $F \to E \to B$, there is a long exact sequence of homotopy groups, but there is no associated long exact sequence of homology groups. Nevertheless, we may relate the homology of the total space E to the homology of the fiber and base using the **Serre spectral sequence**. By the end of these exercises, you should be able to apply this spectral sequence to compute both homology and cohomology for simple examples.

SPECTRAL SEQUENCES

Definition 1. A (homologically graded, first quadrant) **spectral sequence** of R-modules is a sequence of bigraded R-modules (called **pages**)

$$E^r = \bigoplus_{p,q=0}^{\infty} E^r_{p,q}$$

for $r = 0, 1, 2, \ldots$, together with linear differentials

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $E^{r+1}_{p,q} = H_*(E^r_{p,q})$, that is,

$$E^{r+1}_{p,q} = \ker\left(d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}\right) \Big/ \operatorname{im}\left(d^r_{p+r,q-r+1} \colon E^r_{p+r,q-r+1} \to E^r_{p,q}\right)$$

By convention, $E^r_{p,q} = 0$ for either $p < 0$ or $q < 0$. Notice that when $p < r$ and $q < r - 1$, then $E^r_{p,q}$ is the target of a differential with domain 0, and the source of a differential with codomain 0. Hence,

$$E^{r+1}_{p,q} = \ker\left(d^r \colon E^r_{p,q} \to 0\right) \Big/ \operatorname{im}\left(d^r \colon 0 \to E^r_{p,q}\right) = E^r_{p,q}$$

For fixed p and q and $r > \max(p, q+1)$, we have $E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \cdots$ We denote this common module by $E^\infty_{p,q}$.

Definition 2. The E^∞**-page** of a spectral sequence is the direct sum of all the groups $E^\infty_{p,q}$, $E^\infty = \bigoplus_{p,q} E^\infty_{p,q}$.

Let $M = \bigoplus_{i=0}^\infty M_i$ be a graded R-module with a bounded-above filtration by submodules

$$0 \lneq \ldots \lneq M_{(i)} \lneq M_{(i+1)} \lneq \ldots \lneq M_{(n)} = M.$$

Definition 3. A spectral sequence **converges** to M if

$$E^\infty_{p,q} \cong M_{(p+1)} \cap M_{p+q} \Big/ M_{(p)} \cap M_{p+q}$$

Definition 4. A spectral sequence **collapses** on the E_N-page if $d^n = 0$ for all $n \geq N$.

If the spectral sequence collapses on the N-th page, then $E^N \cong E^{N+1} \cong E^{N+2} \cong \cdots \cong E^\infty$.

Theorem 5 (The Serre Spectral Sequence for homology)**.** *Let $f \colon X \to B$ be a Serre fibration with path-connected base B and homotopy fiber F, and let A be an abelian group. If $\pi_1(B)$ acts trivially on $H_*(F; A)$, then there is a spectral sequence with*

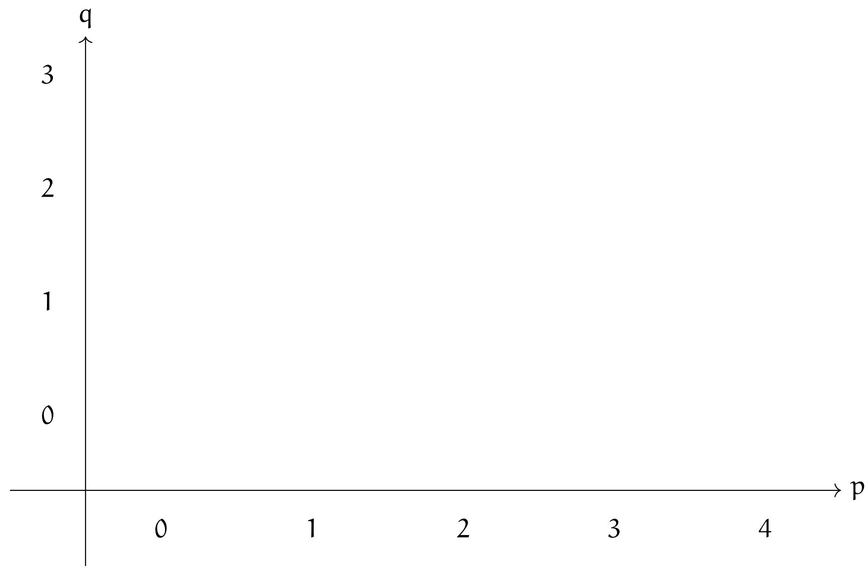
$$E^2_{p,q} \cong H_p(B; H_q(F; A))$$

and converging to $H_{p+q}(X)$.

Exercises

(1) In this exercise, we will compute the homology of $\mathbb{CP}^\infty$ with the fibration $S^1 \to P(\mathbb{CP}^\infty) \twoheadrightarrow \mathbb{CP}^\infty$, where $P(\mathbb{CP}^\infty)$ is the space of paths in $\mathbb{CP}^\infty$ from a fixed basepoint.

(a) Write down the E^2 page in the table below. Draw any differentials that are not obviously zero.



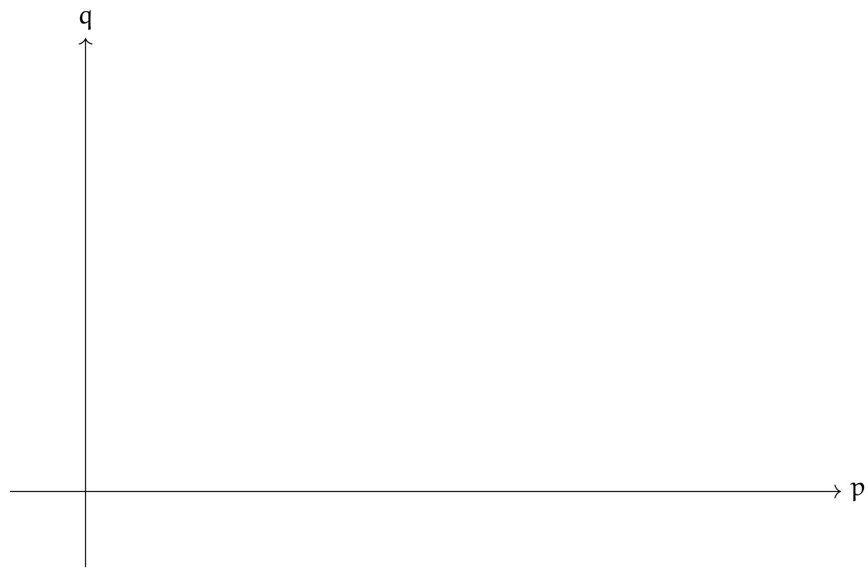
(b) The total space of this fibration is contractible. What does that mean the E^∞ page must look like?

(c) Combining the information from the previous parts, what implication does this have for the E^3-page? (Hint: look at the differentials.)

(d) Conclude that every nonzero differential on the E^2-page must be an isomorphism. Use this to write down the homology groups of $\mathbb{CP}^\infty$.

(2) Consider the pathspace/loopspace fibration $\Omega S^n \to P(S^n) \to S^n$ for S^n, $n \geq 2$. We will use this to compute the homology of ΩS^n.

(a) Write down the nonzero terms of the E_2 page on the diagram below.



(b) What does the E_∞ page look like?

(c) This spectral sequence collapses on a finite page. When? Use this to find $H_i(\Omega S^n)$ for all i.

Spectral Sequences of Algebras

Definition 6. A (cohomologically graded, first quadrant) **spectral sequence of R-algebras** consists of the following data:

(a) A sequence of bigraded differential R-algebras (called **pages**)

$$E_r = \bigoplus_{p,q=0}^{\infty} E_r^{p,q}$$

for $r = 0, 1, 2, \ldots$, each with a product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+q,s+t}$.

(b) Linear differentials $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying the **Leibniz rule:** for $a \in E_r^{s,t}$,

$$d_r(ab) = d_r(a)b + (-1)^{s+t} a d_r(b).$$

such that $E_{r+1}^{p,q} = H^*(E_r^{p,q})$, i.e.,

$$E_{r+1}^{p,q} = {\ker\left(d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}\right)} \Big/ {\operatorname{im}\left(d_r^{p-r,q+r-1} \colon E_r^{p-r,q+r-1} \to E_r^{p,q}\right)}$$

and the product on E_{r+1} is induced by the product of E_r on cohomology.

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded R-algebra with a bounded-below filtration by submodules

$$A = A^{(0)} \supsetneq \ldots \supsetneq A^{(i)} \supsetneq A^{(i+1)} \supsetneq \ldots \supsetneq 0.$$

Write $A_p^{(i)}$ for $A^{(i)} \cap A_p$. Assume moreover that this filtration is **stable**, i.e. $A^{(s)} \cdot A^{(t)} \subseteq A^{(s+t)}$. The associated graded algebra is

$$\operatorname{gr} A = \bigoplus_i A^{(i)} / A^{(i+1)},$$

but this is actually a *bigraded* algebra: there is a grading both from the grading on A and from the filtration.

Definition 7. A spectral sequence of algebras to **converges** to A as a graded algebra if there is an isomorphism of bigraded algebras $E_\infty \cong \operatorname{gr} A$. In particular, for each p, q, we have isomorphisms of R-modules

$$E_{p,q}^{\infty} \cong A_{p+q}^{(p)} / A_{p+q}^{(p+1)}$$

There is another Serre spectral sequence with homology replaced by cohomology.

Theorem 8 (The Serre Spectral Sequence for cohomology)**.** *Let $f \colon X \to B$ be a Serre fibration with path-connected base B and homotopy fiber F, and let A be an abelian group. If $\pi_1(B)$ acts trivially on $H^*(F; A)$, then there is a spectral sequence with*

$$E_2^{p,q} \cong H^p(B; H^q(F; A))$$

and converging to $H^{p+q}(X)$.

Fact 9. *For the Serre spectral sequence, the product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ on the E_2 page of the spectral sequence is $(-1)^{qs}$ times the standard cup product*

$$H^p(B; H^q(F)) \times H^s(B; H^t(F)) \longrightarrow H^{p+s}(B; H^{q+t}(F))$$

sending a pair of cocycles (α, β) to $\alpha \smile \beta$, where coefficients are multiplied via $H^q(F) \times H^t(F) \xrightarrow{\smile} H^{q+t}(F)$.

Fact 10. *For the Serre spectral sequence, the product is also* ***graded commutative****: for $a \in E_r^{p,q}$ and $b \in E_r^{s,t}$,*

$$ab = (-1)^{(p+q)(s+t)} ba.$$

EXERCISES

(3) In this exercise, we will compute the ring structure on $H^*(\mathbb{CP}^\infty)$ using the same fibration as before: $S^1 \to P(\mathbb{CP}^\infty) \twoheadrightarrow \mathbb{CP}^\infty$.

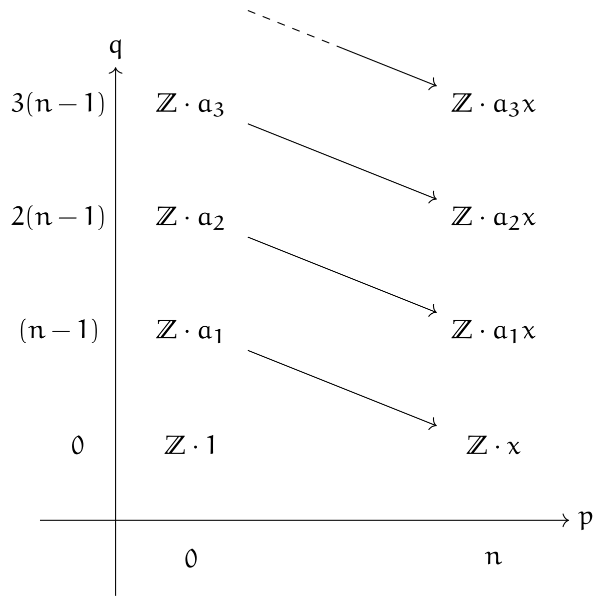
(a) Here is part of the E_2 page for the *cohomology* Serre spectral sequence. The letters indicate choices of generators for each copy of $\mathbb{Z}$. Draw the nonzero differentials. What is $d_2(y_n)$?

q \ p	0	1	2	3	4
1	$\mathbb{Z} \cdot y$	0	$\mathbb{Z} \cdot y_2$	0	$\mathbb{Z} \cdot y_4$
0	$\mathbb{Z} \cdot 1$	0	$\mathbb{Z} \cdot x_2$	0	$\mathbb{Z} \cdot x_4$

(b) What can you say about the relationship between $y \cdot x_n$, and y_n, using the fact that the product on E_2 comes from the cup product in cohomology?

(c) Use the fact that the nonzero differentials are isomorphisms to prove that $x_2 x_{2n} = x_{2n+2}$. What is $H^*(\mathbb{CP}^\infty)$ as a ring?

(4) In this exercise, we will compute the ring structure on $H^*(\Omega S^n)$ for $n \geq 2$ using the same fibration as before: $\Omega S^n \to P(S^n) \to S^n$. The n-th page of the spectral sequence is reproduced below.



(a) First assume that n is odd. Prove by induction that $a_1^k = k! a_k$. Write down generators and relations for $H^*(\Omega S^n)$. This is called a **divided power algebra**.

(b) If n is even, show inductively that $a_1 a_{2k} = a_{2k+1}$ and $a_1 a_{2k+1} = 0$. Also show that $a_2^k = k! a_{2k}$. Write down generators and relations for $H^*(\Omega S^n)$ in this case. Can you recognize $H^*(\Omega S^n)$ as the tensor product of two other algebras?

(5) Recall that $\Omega K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n-1)$ and $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$. Compute $H^6(K(\mathbb{Z}, 3))$ using the pathspace fibration $K(\mathbb{Z}, 2) \to P \to K(\mathbb{Z}, 3)$.

(6) Prove the following:

Proposition 11. $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]$ *when* n *is even and* $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[x] = \mathbb{Q}[x]/\langle x^2 \rangle$ *when* n *is odd, where* $x \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$.