1 Introduction

Let $k$ be a commutative ring. To any $k$-algebra $A$, there is a simplicial $A$-module $L_{A/k}$ such that $\pi_\ast L_{A/k}$ defines a homology theory of algebras, called Andrè–Quillen homology [Qui70]. This gives a very sensitive invariant of the geometry of the algebra. For example:

1. A morphism $A \to B$ is smooth if and only if $L_{B/A} \to \Omega_{B/A}$ is a weak equivalence and $\Omega_{B/A}$ is projective as a $B$-module [GS07, Remark 4.33].

2. A morphism $A \to B$ is étale if and only if $L_{B/A} \cong 0$ [GS07, Remark 4.33].

3. A morphism $A \to B$ of Noetherian rings is a locally complete intersection if and only if $\pi_n(M \otimes_B L_{B/A}) = 0$ for all $n \geq 2$ and all $B$-modules $M$ [Iye07, Theorem 8.4].

4. A morphism $A \to B$ of Noetherian rings is regular if and only if $\pi_n(M \otimes_B L_{B/A} = 0$ for all $n \geq 1$ and all $B$-modules $M$ [Iye07, Theorem 9.5].

Moreover, the cotangent complex provides the setting for obstructions of commutative $k$-algebra structures on $k$-modules. The Hochschild and cyclic homology of $A$ admit a filtrations by Andrè–Quillen homology [Mor19, Proposition 2.28] and in characteristic zero, this yields a spectral sequence

$$E^2_{i,j} = \pi_i(L_{A/k}^j) \Rightarrow HH_{p+q}(A).$$

In this talk, we will define the cotangent complex and Andrè–Quillen homology, state some of its properties, and describe how to perform some calculations.

Remark 1.1 (A note to the reader). Where things have already been done in the literature, I have tried to provide careful citations. When things need more description, however, I have tried to spell it out. Please let me know if there’s anything you’d like to see in more detail, or if I have made any errors!
2 Quillen homology of algebras

If we aim to develop a homology theory of algebras, we immediately run into a problem: the category $\text{Alg}_k$ is not an abelian category. Therefore, the usual notion of a resolution in homological algebra doesn’t make sense. Instead, another approach is needed. We take the perspective of Quillen homology, following [GS07, Section 4.4].

**Definition 2.1.** Let $\mathcal{C}$ be a category, and let $A \in \text{Ob}(\mathcal{C})$. We say that $A$ is an abelian group object if $\mathcal{C}(−, A): \mathcal{C}^{\text{op}} \to \text{Set}$ factors through abelian groups.

Equivalently, if $\mathcal{C}$ is nice enough, this states that there are morphisms $m: A \times A \to A$ and $u: * \to A$ and $i: A \to A$ that are associative, unital, and describe inverses.

Let $(\mathcal{C})_{ab}$ be the category of abelian group objects in $\mathcal{C}$. Suppose that the forgetful functor $U: (\mathcal{C})_{ab} \to \mathcal{C}$ has a left adjoint $F: \mathcal{C} \to (\mathcal{C})_{ab}$ (called abelianization) and that both $\mathcal{C}$, $(\mathcal{C})_{ab}$ are model categories making the adjunction $F \dashv U$ into a Quillen equivalence. Then we may define:

**Definition 2.2.** The Quillen homology of $X \in \text{Ob}(\mathcal{C})$ is $L_F(X)$, the total left derived functor of abelianization $F: \mathcal{C} \to (\mathcal{C})_{ab}$.

To compute the Quillen homology of an object $X$, we take a cofibrant replacement $QX$ of $X$: $L_F(X) \simeq F(QX)$.

**Example 2.3.** Consider the category $\text{sSet}$ of simplicial sets. Then $(\text{sSet})_{ab} \simeq \text{sAb}$. If $X$ is a simplicial set, then its abelianization is $\mathbb{Z}[X]$, the free simplicial abelian group whose $n$-simplicies are the free abelian group on the set $X_n$. Since all simplicial sets are cofibrant, the Quillen homology of $X$ is $\mathbb{Z}[X]$, with $\pi_\ast(\mathbb{Z}[X]) = H_\ast(X; \mathbb{Z})$.

So this is a reasonable framework for constructing homology theories in categories that are not abelian. Let’s apply this to commutative algebras.

Let $k$ be a commutative ring. Unfortunately, the only abelian group object in $\text{Alg}_k$ is the zero ring, because any abelian group object $A$ must admit a morphism from the terminal object $0$, and therefore $0 = 1$ in $A$.

The fix is instead to artificially introduce a new terminal object to the category $\text{Alg}_k$. Fix a $k$-algebra $A$, and consider the category $\text{Alg}_{k/A}$ of $k$-algebras over $A$. Then we may ask what are the abelian objects in this category. It turns out they are not all trivial. The next example gives a nice class of abelian objects.

**Example 2.4.** If $M$ is an $A$-module, define a new $k$-algebra $A \ltimes M$ on the set $A \otimes M$ with multiplication

$$(a_0, m_0) \cdot (a_1, m_1) = (a_0 a_1, a_0 m_1 + a_1 m_0).$$

To see that this is an abelian group object, note that there is a function $\phi$

$$\text{Alg}_{k/A}[B, A \ltimes M] \xrightarrow{\phi} \text{Der}_k(B, M)$$

$$\xrightarrow{f} \text{pr}_2 \circ f$$

(2.5)
where \( pr_2 : A \ltimes M \cong A \oplus M \to M \) is the projection homomorphism. In fact, this function \( \phi \) is an bijection, and therefore \( \text{Alg}_{k/A}(B, A \ltimes M) \) is an abelian group (in fact, a \( k \)-module), so \( A \ltimes M \) is an abelian group object in \( \text{Alg}_{k/A} \).

In fact, every abelian group object in \( \text{Alg}_{k/A} \) is of this form.

**Proposition 2.6.** There is an equivalence of categories \( A \ltimes (-) : \text{Mod}_A \to (\text{Alg}_{k/A})_{ab} \).

We may use this to determine the abelianization functor on \( \text{Alg}_{k/A} \), which is left adjoint to the forgetful functor \( U : (\text{Alg}_{k/A})_{ab} \to \text{Alg}_{k/A} \). By the previous proposition, the forgetful functor is equivalent to \( A \ltimes (-) : \text{Mod}_A \to (\text{Alg}_{k/A})_{ab} \), so it suffices to find a left adjoint to \( A \ltimes (-) \).

**Definition 2.7.** Let \( B \) be a \( k \)-algebra and let \( I = \ker(B \otimes_k B \to B) \) be the kernel of the multiplication homomorphism. The **module of Kähler differentials of \( B \) over \( k \)** is the \( B \)-module \( \Omega_{B/k} := I/I^2 \).

In fact, \( \Omega_{B/k} \) represents the functor \( \text{Mod}_B(\Omega_{B/k}, M) \cong \text{Der}_k(B, M) \).

This gives us a candidate for the abelianization functor on \( \text{Alg}_{A/k} \).

**Proposition 2.9.** The abelianization functor on \( \text{Alg}_{A/k} \) is given by \( B \mapsto A \otimes_B \Omega_{B/k} \).

**Proof.** Let \( B \) be a \( k \)-algebra over \( A \), and let \( M \) be an \( A \)-module. Via \( B \to A \), we may also consider \( M \) as a \( B \)-module. Then combining the isomorphisms (2.5) and (2.8), we have

\[
\text{Alg}_{k/A}(B, A \ltimes M) \cong \text{Der}_k(B, M) \cong \text{Mod}_B(\Omega_{B/k}, M) \cong \text{Mod}_A(A \otimes_B \Omega_{B/k}, M).
\]

To get the Quillen homology, we would take the total left derived functor. However, since \( \text{Alg}_{k/A} \) and \( \text{Mod}_A \) are not model categories, we can’t take derived functors. Instead, we will pass to simplicial \( k \)-algebras over \( A \) and simplicial \( A \)-modules. This is roughly analogous to doing homological algebra not with modules and algebras, but with chain complexes of modules and differential graded algebras.

Before moving on to simplicial \( k \)-algebras in the next section, here we record some properties of the functor \( \Omega(-)/k \).

**Proposition 2.10.** Let \( k \to A \to B \) be homomorphisms of commutative rings. Then there is an exact sequence

\[
\Omega_{A/k} \otimes_A B \to \Omega_{B/k} \to \Omega_{B/A} \to 0.
\]

This sequence is called the **Jacobi–Zariski sequence**.

**Proof.** Let \( N \) be a \( B \)-module. By the homomorphism \( A \to B \), \( N \) may also be considered an \( A \)-module, and by the composite \( k \to A \to B \), it also becomes a \( k \)-module. There is a left-exact sequence

\[
0 \to \text{Der}_A(B; N) \to \text{Der}_k(B; N) \to \text{Der}_k(A; N)
\]
where the first homomorphism is given by considering an $A$-linear derivation as a $k$-linear derivation via $k \to A$, and the second homomorphism is restriction of domain. By the universal property (2.8) of the Kähler differentials, the sequence above is isomorphic to

$$0 \to \text{Hom}_A(\Omega_{A/k}, N) \to \text{Hom}_B(\Omega_{B/k}, N) \to \text{Hom}_B(\Omega_{B/A}, N).$$

Note that the first term is isomorphic to $\text{Hom}_B(\Omega_{A/k} \otimes_A B, N)$ via the restriction/induction adjunction. Naturality of this sequence in $N$ yields the Jacobi–Zariski sequence. □

**Remark 2.11.** The cotangent complex may be considered the left-derived functor of this right-exact sequence.

**Example 2.12.** If $\varphi : k \to A$ is surjective, then $\Omega_{k/A} \cong 0$. To see this, note that a derivation $\delta \in \text{Der}_k(A, M)$ is equivalent to a $k$-linear homomorphism $A \to M$ that obeys the Leibniz rule and such that $\delta \circ \varphi = 0$. In case $\varphi$ is surjective, then every $k$-linear derivation from $A$ to $M$ is zero. Hence, the $\Omega_{k/A}$ represents the zero functor, and is zero itself.

**Example 2.13 ([Iye07, Exercise 2.3]).** $\Omega_{k[x_1, \ldots, x_n]/k} \cong \bigoplus_{i=1}^n k[x_1, \ldots, x_n] dx_i$

## 3 Simplicial $k$-algebras

Recall the Dold–Kan correspondence:

**Theorem 3.1** (Dold–Kan). There is an equivalence of categories $\text{sMod}_k \simeq \text{Ch}^+_{k}$ between connective chain complexes of $k$-modules and simplicial $k$-modules.

There is a standard model structure on $\text{Ch}^+_{k}$, called the projective model structure, with

- weak equivalences given by quasi-isomorphisms of chain complexes,
- fibrations given by homomorphisms which are surjective in positive degree,
- cofibrations given by injective homomorphisms with projective cokernel.

This translates to a model structure on $\text{sMod}_k$, where $f : X \to Y$ is

- a weak equivalence if $f_* : \pi_* X \to \pi_* Y$ is an isomorphism,
- a fibration if $f$ is a fibration of the underlying simplicial sets,
- a cofibration if it has the left-lifting property against all acyclic fibrations.

Equivalently, we may say that $f$ is a fibration if $X \to \pi_0 X \times_{\pi_0 Y} Y$ is surjective, or if the corresponding homomorphism of chain complexes is a fibration.

To determine a model structure on $\text{sAlg}_k$, we will use the model structure on $\text{sMod}_k$ and transfer it over the free/forgetful adjunction

$$\text{Sym}_k : \text{sMod}_k \xrightarrow{\text{L}} \text{sAlg}_k : \text{U},$$

where $\text{U}$ is the forgetful functor and $\text{Sym}_k$ is the symmetric $k$-algebra functor applied levelwise to simplicial modules.
**Theorem 3.2.** There is a simplicial model category structure on $\text{sAlg}_k$ where a morphism $f : A \to B$ is

(a) a weak equivalence if $\pi_* f : \pi_* A \to \pi_* B$ is an isomorphism,

(b) a fibration if $A \to \pi_0 A \times_{\pi_0 B} B$ is surjective,

(c) a cofibration if it has the left-lifting property with respect to acyclic fibrations.

A characterization of the cofibrations is provided by the following,

**Definition 3.3** ([Iye07, Definition 4.1]). Say that a $k$-algebra homomorphism $f : A \to B$ is free if there is a sequence $X = \{X_n\}_{n \geq 0}$ of sets such that $B_n \cong A_n[X_n]$ and $s_j(X_n) \subseteq X_{n+1}$, and $f$ is isomorphic the inclusion $A_n \hookrightarrow A_n[X_n]$.

Informally, $A \to B$ is free if $B$ is polynomial over $A$, compatibly with the degeneracies.

**Proposition 3.4.** A morphism in $\text{sAlg}_k$ is a cofibration if and only if it is a retract of a free morphism. A simplicial $k$-algebra $A$ is cofibrant if and only if there are projective $k$-modules $P_j$ and isomorphisms $A_n \cong \coprod_{\phi : [n] \to [j]} \phi^* \text{Sym}_k(P_j)$.

**Definition 3.5.** If $f : A \to B$ is a homomorphism of simplicial $k$-algebras, then a simplicial resolution of $B$ as an $A$-algebra is a factorization of $f$ as a cofibration followed by an acyclic fibration $A \hookrightarrow P \twoheadrightarrow B$.

Such a simplicial resolution always exists by the axioms of model categories, or alternatively, explicit general constructions can be found in [Lod13, 3.5.1] or [Wei94, paragraph preceding Definition 8.8.2]. Later we will see nicer constructions for specific examples.

**Example 3.6** ([Iye07, Construction 4.13]). Let’s compute the a simplicial resolution of $k$ as an $k[y]$-algebra, where $y$ acts by zero on $k$.

Consider the simplicial bar complex $B$ with $n$-simplicies $B_n = k[y] \otimes_k [y]^\otimes n \otimes_k k$ face maps

$$d_i(a \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda) = \begin{cases} \a a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \lambda & (i = 0) \\ a \otimes a_1 \otimes \cdots \otimes a_{i-1} a_{i+1} \otimes \cdots \otimes a_n \otimes \lambda & (0 < i < n) \\ a \otimes a_1 \otimes \cdots \otimes a_n \cdot \lambda & (i = n), \end{cases}$$

and degeneracies

$$s_j(a \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda) = \begin{cases} a \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda & (j = 0) \\ a \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i} \otimes \cdots \otimes a_n \otimes \lambda & (0 < j < n) \\ a \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \lambda & (j = n). \end{cases}$$

I’m pretty certain that this is wrong in the cited reference. At the least, there are confusing typos in the definitions of the faces and degeneracies there.
Evidently, $B_n$ is polynomial over $R[y]$; it is isomorphic to $R[y][x_1, \ldots, x_n]$ via
\[
    x_i \mapsto 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1
\]
with $y$ in the $i$-th spot. Therefore, the map $R[y] \to B$ is a cofibration. It is standard that $B \to R$ is an acyclic fibration, i.e. surjective quasi-isomorphism.

**Proposition 3.7** ([Iye07, Construction 4.16]). There is a weak equivalence of simplicial $R[y]$-modules $B \simeq K$ where $K$ is the Koszul complex
\[
    0 \to R[y] \xrightarrow{y} R[y] \to 0
\]
given by $\text{id}_{R[y]}$ in degree zero and $- \otimes y$ in degree one.

## 4 The cotangent complex

Now that we have model structures on simplicial $k$-algebras and simplicial $k$-modules, we can define the cotangent complex. The constructions in Section 2 extend levelwise to functors of simplicial objects
\[
    \Omega(-)/k \otimes (-) A : \text{sAlg}_{k/A} \rightleftarrows \text{sMod}_{A} : A \otimes (-)
\]
It is easy to see that the right adjoint $A \otimes (-)$ preserves weak equivalences and fibrations, and hence the adjunction is Quillen. Hence, the total left derived functor makes sense and we may take Quillen homology.

**Definition 4.1.** The **cotangent complex** of any $k$-algebra $A$ is
\[
    L_{A/k} := \Omega_{Q/k} \otimes_{Q} A
\]
where $Q$ is any cofibrant replacement for $A$ in $\text{sAlg}_{k}$.

**Definition 4.2.** The **André–Quillen homology** of $A$ is the homotopy of the cotangent complex:
\[
    D_n(A/k) := \pi_n L_{A/k}.
\]
There is a natural map $L_{A/k} \to \Omega_{A/k}$ coming from the Jacobi–Zariski sequence for $k \to Q \to A$; see Proposition 2.10.

**Example 4.3.** If $A$ is a cofibrant $k$-algebra, then we may take $A$ as its own cofibrant replacement and $L_{A/k} \simeq \Omega_{A/k} \otimes_{A} A \cong \Omega_{A/k}$. In particular,
\[
    L_{k[x_1, \ldots, x_n]/k} \simeq \Omega_{k[x_1, \ldots, x_n]/k} \cong \bigoplus_{i=1}^{n} k[x_1, \ldots, x_n] dx_i.
\]

**Example 4.4** ([Mor19, Example 2.26]). For any $k$-algebra $A$, there is an isomorphism $\pi_0 L_{A/k} \cong \Omega_{A/k}$. 

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Example 4.5 (Conormal sequence, [Mor19, Example 2.26] or [Iye07, Exercise 5.5]). If \( k \to A \) is surjective with cokernel \( J \), then \( \pi_0 L_{A/k} = 0 \) and \( \pi_1 L_{A/k} = J/J^2 \).

Example 4.6. Let’s compute the cotangent complex of \( k \) as an \( k[y] \)-algebra, where \( y \) acts by zero on \( k \). Recall that in Example 3.6, we described a simplicial \( k \)-algebra \( B \) such that \( k[y] \to B \to k \) is a factorization of \( k[y] \to k \) as a cofibration followed by an acyclic fibration. This simplicial \( k \)-algebra is isomorphic to a polynomial \( k \)-algebra

\[ B_n \cong k[y][x_1, \ldots, x_n] \]

with face maps determined by

\[
d_i(y) = y, \quad d_i(x_j) = \begin{cases} y & (i = 0, j = 1) \\ x_{j-1} & (i < j \text{ and } (i, j) \neq (0, 1)) \\ x_j & (i > j \text{ or } i = j \neq n) \\ 0 & (i = j = n) \end{cases}
\] (4.7)

From Example 2.13, we find that

\[ \Omega_{B_n/k[y]} \cong \bigoplus_{i=1}^n k[y][x_1, \ldots, x_n] dx_i. \]

To get the \( n \)-simplicies of the cotangent complex, we tensor this with \( k \) over \( B_n \). Hence,

\[ (L_{k/k[y]})_n = \Omega_{B_n/k[y]} \otimes_{B_n} k \cong \left( \bigoplus_{i=1}^n B_n dx_i \right) \otimes_{B_n} k \cong \bigoplus_{i=1}^n k dx_i \]

The degeneracy maps from (4.7) determine the face maps on the cotangent complex. In particular, we have \( k \)-linear face maps such that

\[
d_i(dx_j) = \begin{cases} 0 & (i = j = n \text{ or } i = 0, j = 1) \\ dx_{j-1} & (i < j \text{ and } (i, j) \neq (0, 1)) \\ dx_j & (i > j \text{ or } i = j \neq n). \end{cases}
\] (4.8)

We may use this to compute the homotopy groups of the cotangent complex by converting it into a chain complex using the Dold–Kan correspondence and then taking the homotopy. This chain complex has the \( n \)-simplicies of \( L_{k/k[y]} \) in degree \( n \) and the differential \( \delta \) of the corresponding chain complex is given by the alternating sum of face maps,

\[ \delta_n = \sum_{i=0}^n (-1)^i d_i. \]

Altogether, the complex looks as follows:

\[ 0 \xleftarrow{\delta_1} k dx_1 \xleftarrow{\delta_2} k dx_1 \oplus k dx_2 \xleftarrow{\delta_3} k dx_1 \oplus k dx_2 \oplus k dx_3 \xleftarrow{\delta_4} \cdots \]
From (4.8), we see that the first two differentials are zero, so we have \( \pi_0 L_{k/k[y]} \cong 0 \) and \( \pi_1 L_{k/k[y]} \cong k \). The third differential \( \partial_3 = d_0 - d_1 + d_2 - d_3 \) is surjective, sending \( dx_1 \mapsto -dx_1, \) \( dx_2 \mapsto 0, \) and \( dx_3 \mapsto dx_2, \) hence \( \pi_3 L_{k/k[y]} \cong 0 \). The fourth differential \( \partial_4 \) is described by \( dx_1 \mapsto 0, \) \( dx_2 \mapsto dx_2, \) \( dx_3 \mapsto dx_2, \) \( dx_4 \mapsto 0, \) and its image thus fills the kernel of \( \partial_3 \) to given \( \pi_3 L_{k/k[y]} \cong 0 \). In fact, this pattern continues and \( \pi_i L_{k/k[y]} \cong 0 \) when \( i \neq 1 \).

It can be proved this using Quillen’s fundamental spectral sequence \([Qui70, \text{Theorem 6.3}]\). See also \([Qui70, \text{Corollary 6.14}]\) or \([Iye07, \text{Proposition 5.11}]\).

**Remark 4.9.** The above calculation of the cotangent complex can be modified to calculate the cotangent complex of the quotient of any ring \( R \) by a regular sequence. See \([Iye07, \text{Construction 4.16 \& Exercise 4.17}]\).

Finally, we summarize some properties of the cotangent complex.

**Proposition 4.10.**

(a) The augmentation map \( L_{A/k} \to \Omega_{A/k} \) induces an isomorphism on \( \pi_0 \).

(b) If \( A \) is a cofibrant \( k \)-algebra, then \( L_{A/k} \to \Omega_{A/k} \) is a weak equivalence.

**Proposition 4.11** (Künneth Theorem, \([GS07, \text{Lemma 4.30}]\) or \([Mor19, \text{Proposition 2.27}]\)). If either \( A \) or \( B \) is a flat \( k \)-algebra, then there is an isomorphism
\[
L_{A \otimes_k B/k} \cong (A \otimes_k L_{B/k}) \oplus (L_{A/k} \otimes_k B).
\]

**Proposition 4.12** (Flat Base Change, \([GS07, \text{Theorem 4.31}]\) or \([Mor19, \text{Proposition 2.27}]\)). Let \( K \) and \( A \) be \( k \)-algebras, and suppose that \( k \to A \) is flat. Then \( K \otimes_k L_{A/k} \to L_{A \otimes_k K/k} \) is a weak equivalence.

**Theorem 4.13** (\([Mor19, \text{Proposition 2.27}]\) or \([GS07, \text{Proposition 4.32}]\)). Given a sequence of homomorphisms of commutative rings \( k \to A \to B \), there is a cofiber sequence of simplicial \( k \)-algebras
\[
L_{A/k} \otimes_A B \to L_{B/k} \to L_{B/A}.
\]

This last theorem yields a long exact sequence in André–Quillen homology.

**References**


