SPUR 2019 Notes and Exercises

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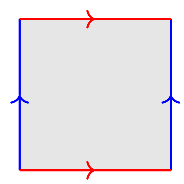
Collected here are a series of exercises and notes written for the Summer Program for Undergraduate Research at Cornell University. These notes are far from complete, and I strongly encourage any readers to refer to the references listed herein as well. Please do not hesitate to contact me with feedback or corrections.

1 Simplicial Sets

In algebraic topology, we don't often think of spaces as "sets with a topology on them," as we do in point-set topology. Instead, our spaces are built as cell complexes or simplicial complexes. For example, we construct a circle S^1 by attaching a line (1-cell) to a point (0-cell) at both of its ends. We build a torus T^2 by attaching two 1-cells to a point, and then gluing in a 2-cell appropriately, as below.



In algebraic topology, we only care about these spaces up to homotopy. So we might as well think of spaces *combinatorially* – as a collection of cells in each dimension, together with data for how to attach these cells to each other.

There are many good introductions to simplicial sets [GZ67, Rie11, Fri08]. You should go read some of them! The book [GJ09] is a good reference, but not a good introduction. In these notes we will mainly focus on examples. The goal is to try to get you used to working with simplicial sets and thinking simplicially.

We will begin with an abstract definition of simplicial sets, and then re-interpret this as something more concrete.

1.1 Abstract simplicial sets

Definition 1.1. The simplicial category Δ is the category whose objects are finite, totally ordered sets $[n] = \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$, and whose morphisms are order-preserving functions.

Definition 1.2. A **simplicial set** is a contravariant functor $X\colon \Delta^{op} \to \mathbf{Set}$. A **morphism of simplicial sets** is a natural transformation $f\colon X \to Y$. The category of simplicial sets is denoted **sSet**.

Example 1.3. For any $n \in \mathbb{N}$, there is a simplicial set $\Delta^n := \mathrm{Hom}_{\Delta}(-, [n])$. This is called the **standard n-simplex**.

Remark 1.4. We may replace the category **Set** by another category **C** in Definition 1.2 to get the category **sC** of simplicial objects in **C**. We will do this in the next lecture with the category of abelian groups, in which case the category **sAb** has a nice algebraic description.

Let's unwind this definition a little bit. Let $X\colon \Delta^{op} \to \mathbf{Set}$ be a simplicial set. For each $n \in \mathbb{N}$, we have a set $X([n])$, which we will write X_n. In addition to this sequence of sets $X_0, X_1, \ldots$, we must also specify how X behaves on morphisms. This is made easier by the following theorem.

Definition 1.5. There are two special classes of morphisms in Δ, the **cofaces** $d^i\colon [n-1] \to [n]$ and **degeneracies** $s^j\colon [n+1] \to [n]$, defined as follows:

$$d^i(k) = \begin{cases} k & (k < i) \\ k+1 & (k \geq i) \end{cases} \qquad s^j(k) = \begin{cases} k & (k \leq i) \\ k-1 & (k > i) \end{cases}$$

These morphisms satisfy the following **cosimplicial identities**:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && (i < j) \\ s^j s^i &= s^i s^{j+1} && (i \leq j) \\ s^j d^i &= \begin{cases} \mathrm{id} & (i = j \text{ or } i = j+1) \\ d^i s^{j-1} & (i < j) \\ d^{i-1} s^j & (i > j+1) \end{cases} \end{aligned}$$

Theorem 1.6. *Every morphism $f \in \mathrm{Hom}_{\Delta}([n], [m])$ may be written as a composite of the coface and codegeneracy maps. More precisely, any such $f\colon [n] \to [m]$ may be written in a unique form*

$$f = d^{i_1} d^{i_2} \cdots d^{i_r} s^{j_1} \cdots s^{j_s}$$

with $m = n - s + r$ and $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$

Proof. Exercise (1.4.6). □

1.2 Combinatorial simplicial sets

The above theorem gives generators and relations for the morphisms in Δ. It also allows us to rewrite Definition 1.2 to obtain a combinatorial description of simplicial sets.

Definition 1.7. A **simplicial set** X is a sequence of sets $X_0, X_1, X_2, \ldots$ together with

- **face maps** $d_i\colon X_n \to X_{n-1}$ for $i = 0, 1, 2, \ldots, n$

- **degeneracies** $s_j \colon X_n \to X_{n+1}$ for $j = 0, 1, 2, \ldots, n$

The face maps and degeneracies must satisfy the **simplicial identities**:

$$\begin{cases} d_i d_j = d_{j-1} d_i & (i < j) \\ d_i s_j = s_{j-1} d_i & (i < j) \\ d_j s_j = d_{j+1} s_j = \mathrm{id} & \\ d_i s_j = s_j d_{i-1} & (i > j+1) \\ s_i s_j = s_{j+1} s_i & (i \leq j) \end{cases}$$

An element $x \in X_n$ is referred to as an n**-simplex** of X. It is called **degenerate** if it is in the image of $s_j \colon X_{n-1} \to X_n$ for some $j \in \{0, \ldots, n-1\}$.

If we want to emphasize that an object is a simplicial set, we often write $X_\bullet$, where $\bullet$ is a stand-in for a natural number.

Example 1.8. The standard n-simplex Δ^n has k-simplicies

$$\Delta^n_k = \mathrm{Hom}_{\mathbb{\Delta}}([k], [n]),$$

and the face and degeneracy maps are given by pre-composition in $\mathbb{\Delta}$ by d^i and s^j, respectively.

$$\begin{aligned} d_i \colon \Delta^n_k &\longrightarrow \Delta^n_{k-1} & \qquad s_j \colon \Delta^n_k &\longrightarrow \Delta^n_{k+1} \\ f &\longmapsto f \circ d^i & f &\longmapsto f \circ s^j \end{aligned}$$

Example 1.9. For any set A, we may define a simplicial set $A_\bullet$ with n-simplicies $A_n = A$ for all $n \in \mathbb{N}$. All face and degeneracy morphisms are the identity function id_A. This is called a **discrete simplicial set**. The functor $A \mapsto A_\bullet$ defines a fully faithful embedding **Set** $\hookrightarrow$ **sSet**. In this sense, every set is also a simplicial set.

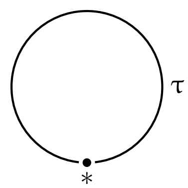
Example 1.10. The simplicial circle $S^1_\bullet$ is the simplicial set with n-simplicies $S^1_n = \{0, 1, \ldots, n\} = [n]$. The degeneracy map $s_j \colon [n] \to [n+1]$ is the unique monotone injection skipping $j+1$ in its image, and the face map $d_i \colon [n] \to [n-1]$ is given by

$$d_i(k) = \begin{cases} k & (k < i) \\ k-1 & (k > i) \\ i & (k = i < n) \\ 0 & (k = i = n) \end{cases}$$

It may seem that this has nothing to do with the topological circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, but in fact these are closely related. Almost all simplicies in the simplicial set $S^1_\bullet$ are degenerate, with the exception of $0 \in S^1_0$ and $1 \in S^1_1$.

$$\begin{aligned} S^1_0 &= \{0\} \\ S^1_1 &= \{0 = s_0(0), 1\} \\ S^1_2 &= \{0 = s_0(0), 1 = s_1(1), 2 = s_0(1)\} \\ S^1_3 &= \{0 = s_0(0), 1 = s_2(1), 2 = s_2(2), 3 = s_1(2)\} \\ &\vdots \end{aligned}$$

This is the smallest simplicial model of the topological circle S^1, built from the standard cell decomposition consisting of a single 0-cell $*$ and a single 1-cell τ:



The 0-cell corresponds to $0 \in S^1_0$ and the 1-cell corresponds to $1 \in S^1_1$.

1.3 Geometric realization

At the beginning of this lecture, we mentioned that simplicial sets would be a combinatorial model of topological spaces. To use them in this way, we need a way to convert between simplicial sets and topological spaces. This comes in the form of an adjunction.

Definition 1.11. Define the **geometric n-simplex** Δ^n_{geom} by

$$\Delta^n_{\mathrm{geom}} = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \sum_{i=0}^{n} t_i \leq 1 \right\}$$

The i-th face of the geometric n-simplex is the face opposite the i-th vertex.

Definition 1.12. The **geometric realization** of a simplicial set X is the topological space

$$|X| := \left(\coprod_{n \geq 0} X_n \times \Delta^n_{\mathrm{geom}} \right) / \sim$$

where $\sim$ is the equivalence relation given by $(d_i(x), t) \sim (x, d^i(t))$ and $(s_j(x), s) \sim (x, s^j(s))$, $x \in X_n$, $t \in \Delta^{n-1}$, $s \in \Delta^{n+1}$.

This may seem kind of abstract at first, but it is quite a reasonable definition. For each n-simplex $x \in X_n$, we get a copy of the topological n-simplex Δ^n_{geom}, and the relation $\sim$ describes how to glue these simplicies together using the face maps. Notice that by the equivalence relation, elements of $\{s_j(x)\} \times \Delta^{n+1}_{\mathrm{geom}}$ are identified with $\{x\} \times \Delta^n_{\mathrm{geom}}$, so degenerate simplicies of X make no contribution to the geometric realization. This is in fact why they are called degenerate.

Example 1.13. Δ^n_{geom} is the geometric realization of Δ^n.

Example 1.14. If A is any set, let $A_\bullet$ be the simplicial set from Example 1.9. The geometric realization of this discrete simplicial set is the set A with the discrete topology.

Example 1.15. The geometric realization of the simplicial set $S^1_\bullet$ from Example 1.10 is the topological circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

There is also a way to take any topological space and turn it into a simplicial set.

Definition 1.16. The **singular simplicial set** of a topological space Y is the simplicial set $\text{Sing}(Y)$ with n-simplicies

$$\text{Sing}(Y)_n = \text{Hom}_{\textbf{Top}}(\Delta^n_{\text{geom}}, Y)$$

with face and degeneracy maps defined appropriately.

Theorem 1.17. *There is an adjunction* $|-|\colon \textbf{sSet} \leftrightarrows \textbf{Top}\colon \text{Sing}$ *with* Sing *right adjoint to* $|-|$.

Remark 1.18. This adjunction is actually something called a **Quillen equivalence** of model categories. In short, this means that the categories **Top** and **sSet** have the same homotopy theory. So, for the purposes of homotopy theory, we can (and often do) work with simplicial sets instead of topological spaces.

1.4 Exercises

(1.4.1) Let X be a simplicial set. Show that there is a bijection $\text{Hom}_{\mathbb{\Delta}^{\text{op}}}(\Delta^n, X) \cong X_n$.

(1.4.2) Show that Δ^n has a unique nondegenerate n-simplex. Describe all nondegenerate simplicies of Δ^n. Verify that Δ^n_{geom} is the geometric realization of Δ^n.

(1.4.3) Define a simplicial set T such that $|T|$ is homeomorphic to the 2-torus. Can you do so with only four nondegenerate simplicies?

(1.4.4) What is the product of two simplicial sets? The coproduct? What is a quotient of simplicial sets?

(1.4.5) Describe a simplicial set $S^n_\bullet$ such that $|S^n_\bullet|$ is the topological n-sphere.

(1.4.6) Prove Theorem 1.6. Start by computing a few examples. Then prove that any morphism $f \in \text{Hom}_{\mathbb{\Delta}}([n], [m])$ is a composite of cofaces and codegeneracies. By imposing requirements on the order that these cofaces and codegeneracies must occur and using the cosimplicial identities, obtain the unique decomposition. See [GZ67, Lemma 2.2] or [Mac78, §7.5] if you're stuck.

2 Dold–Kan Correspondence

Simplicial sets are a good combinatorial model of spaces, but they also have many other uses – a subcategory of simplicial sets models infinity categories, for example. We may also define simplicial objects in other categories, which can describe algebraic objects, or topological spaces with additional structure.

In this lecture, we will investigate the relationship between simplicial abelian groups and chain complexes. This relation manifests itself in the form of an equivalence of categories known as the Dold–Kan correspondence. Most of the material in this section comes from [Meh17]. Other references are [Mat11], [Dwy01], [GJ09, Section III.2], and [Wei94, Chapter 8].

2.1 Simplicial objects

Previously, we defined in Definition 1.2 a simplicial set as a contravariant functor $X\colon \Delta^{op} \to$ **Set**. But perhaps the sets in the image have additional structure, or they might not be sets at all. In this case, we get simplicial objects in a different category.

Definition 2.1. Let **C** be a category. A **simplicial object in C** is a contravariant functor $\Delta^{op} \to \mathbf{C}$. The category of simplicial objects in **C** is denoted **sC**.

As in Definition 1.7, we can rewrite this as a sequence of objects together with faces and degeneracies satisfying certain relations. These faces and degeneracies must be morphisms in the category **C**. For us, the most important simplicial objects will be simplicial abelian groups.

Definition 2.2. A **simplicial abelian group** A is a sequence $A_0, A_1, A_2, \ldots$ of abelian groups with face and degeneracy maps as in Definition 1.7. The face and degeneracy maps must be homomorphisms of abelian groups. The category of simplicial abelian groups is **sAb**.

Example 2.3. Given any simplicial set X, we may define a simplicial abelian group $\mathbb{Z}X$ with n-simplicies $\mathbb{Z}X_n$ given by the free abelian group on the set X_n. The face and degeneracy homomorphisms are induced by the face and degeneracy maps of X.

2.2 Chain complexes and simplicial abelian groups

Let R be a ring. Let $\mathbf{Ch}(R)$ be the category of chain complexes of R-modules, and let $\mathbf{Ch}^+(R)$ be the category of **connective** chain complexes, i.e., chain complexes concentrated in nonnegative degrees.

Note that a simplicial abelian group looks a lot like a chain complex, but instead of a differential, there are many maps $d_i\colon A_i \to A_{i-1}$. However, this data is enough to build a chain complex.

Definition 2.4. Define a functor $C\colon \mathbf{sAb} \to \mathbf{Ch}^+(\mathbb{Z})$ that takes a simplicial abelian group A to the connective chain complex

$$\cdots \to A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

with differential

$$\partial_n = \sum_{i=0}^{n} d_i.$$

One can check that this is genuinely a differential ($\partial^2 = 0$) using the simplicial relations. See exercise (2.4.1).

Definition 2.5. For any simplicial abelian group A, let $(DA)_n$ be the subgroup of $(CA)_n$ generated by the image of the degenerate simplicies. Precisely,

$$(DA)_n = \left\{ \sum_i a_i \in A_n \;\middle|\; a_i = s_j(a_i') \text{ for some } a_i' \in A_{n-1} \right\}.$$

Define the chain complex DA with n-th group $(DA)_n$ and differential inherited from CA.

Definition 2.6. The **normalized chain complex** associated to a simplicial group A is the chain complex

$$NA = CA/DA.$$

This defines a functor $N\colon \mathbf{sAb} \to \mathbf{Ch}^+(\mathbb{Z})$.

Example 2.7. Let Y be any topological space. We may form the **singular chain complex** of Y by $N\mathbb{Z}\,\mathrm{Sing}(Y)$, where $\mathrm{Sing}(Y)$ is the singular simplicial set of Y (Definition 1.16). The homology of this chain complex is the homology $H_*(Y;\mathbb{Z})$ of the space Y.

Recall that two chain complexes A and B are **quasi-isomorphic** if there is a homomorphism $f\colon A \to B$ inducing isomorphisms on homology $f_*\colon H_*(A) \cong H_*(B)$. This next proposition shows that the chain complex depends only on the nondegenerate simplicies of A.

Proposition 2.8. *The chain complexes CA and NA are quasi-isomorphic for any simplicial abelian group A.*

Proof. To prove this, we will prove that DA is an acyclic chain complex (i.e. quasi-isomorphic to zero). This suffices because we have a short exact sequence of chain complexes

$$0 \to DA \to CA \to NA \to 0$$

which induces a long exact sequence in homology. If $H_*(DA) = 0$, then this long exact sequence yields isomorphisms $H_*(CA) \cong H_*(NA)$, as desired.

It remains to show that DA is acyclic. For this, see [Lur17, Proposition 1.2.3.17] or [Wei94, Theorem 8.3.8]. □

We have described a functor $N\colon \mathbf{sAb} \to \mathbf{Ch}^+(\mathbb{Z})$, but to fully realize the equivalence of categories between simplicial abelian groups and connective chain complexes, we must supply an inverse functor. This is somewhat more difficult.

Definition 2.9. Let Q be a chain complex. Define a simplicial abelian group ΓQ as follows. The n-simplicies of ΓQ are

$$(\Gamma Q)_n = \bigoplus_{\substack{p \leq n \\ [n] \twoheadrightarrow [p] \in \Delta}} Q_p.$$

For any map $\theta\colon [m] \to [n]$ in Δ, the corresponding homomorphism $\theta^*\colon (\Gamma Q)_n \to (\Gamma Q)_m$ is given by the following procedure.

It suffices to define

$$\theta^*\colon \bigoplus_{\substack{p \leq n \\ [n] \twoheadrightarrow [p] \in \Delta}} Q_p \to \bigoplus_{\substack{q \leq m \\ [m] \twoheadrightarrow [q] \in \Delta}} Q_q$$

on each summand $(\Gamma Q)_n$. For the summand Q_ℓ corresponding to some surjection $\sigma\colon [n] \twoheadrightarrow [\ell]$, we rewrite the composite $\sigma \circ \theta$ as an injection following a surjection:

$$\begin{array}{ccc} [m] & \xrightarrow{\theta} & [n] \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ [r] & \xhookrightarrow{\lambda} & [\ell] \end{array}$$

Then τ is a surjection, so defines a summand $Q_r \hookrightarrow (\Gamma Q)_m$. We obtain a map $\lambda^*\colon Q_\ell \to Q_r$ as follows:

$$\lambda^*\colon Q_\ell \to Q_r = \begin{cases} \mathrm{id}_{Q_\ell} & (r = \ell) \\ \partial_Q & (r = \ell - 1) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

This is summarized in the following diagram.

$$\begin{array}{ccc} Q_\ell & \xrightarrow{\lambda^*} & Q_r \\ \downarrow & & \downarrow \\ (\Gamma Q)_n & \overset{\theta^*}{\dashrightarrow} & (\Gamma Q)_m \end{array}$$

[Mat11, Section 2.2] gives a decent description of this operation. In essence, the three steps for determining where the factor $Q_\ell \hookrightarrow (\Gamma Q)_n$ corresponding to $\sigma\colon [n] \to [\ell]$ goes are:

(1) Factor the composite $[m] \xrightarrow{\theta} [n] \xrightarrow{\sigma} [\ell]$ as an injection following a surjection $[m] \xrightarrow{\tau} [r] \xrightarrow{\lambda} [\ell]$.

(2) Q_ℓ is sent to the factor $Q_r \hookrightarrow (\Gamma Q)_m$ corresponding to $\tau\colon [m] \twoheadrightarrow [r]$.

(3) The map $\lambda^*\colon Q_\ell \to Q_r$ is determined by the differential of Q if $r = \ell - 1$, or the identity of Q_ℓ if $r = \ell$, or zero otherwise.

A picture of ΓQ is as follows:

$$Q_0 \leftleftarrows Q_0 \oplus Q_1 \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; Q_0 \oplus Q_1 \oplus Q_1 \oplus Q_2 \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; Q_0 \oplus (Q_1)^{\oplus 3} \oplus (Q_2)^{\oplus 3} \oplus Q_3 \dashleftarrow$$

Theorem 2.10 (Dold–Kan)**.** *The functors* $N\colon \mathbf{sAb} \to \mathbf{Ch}^+(\mathbb{Z})$ *and* $\Gamma\colon\ \mathbf{Ch}^+(\mathbb{Z}) \to \mathbf{sAb}$ *form an equivalence of categories.*

Sketch. The full proof of this theorem is not too hard, but it is tedious. You may find complete proofs in [Wei94, Section 8.4] or [Mat11]. We will satisfy ourselves by describing the natural isomorphisms $\Psi\colon \Gamma N \to \mathrm{id}_{\mathbf{sAb}}$ and $\Phi\colon \mathrm{id}_{\mathbf{Ch}^+(\mathbb{Z})} \to N\Gamma$.

For any simplicial abelian group A, to describe $\Psi\colon \Gamma N \to \mathrm{id}_{\mathbf{sAb}}$, we must describe isomorphisms:

$$\bigoplus_{\substack{k\leq n\\ [n]\twoheadrightarrow[k]}} (NA)_k \to A_n$$

This is given on the summand indexed by $\sigma\colon [n] \twoheadrightarrow [k]$ by $NA_k \to A_k \xrightarrow{\sigma^*} A_n$, with σ^* defined as in (1).

On the other hand, for any chain complex Q, we may define $\Phi\colon Q \to N\Gamma Q$ as follows. First, define $Q \to C\Gamma Q$ as inclusion of the summand correspsonding to id_{Q_n}.

$$Q_n \hookrightarrow C\Gamma Q_n = \bigoplus_{\substack{k\leq n\\ [n]\twoheadrightarrow[k]}} Q_k.$$

Then compose this with the quotient map $C\Gamma Q \to N\Gamma Q$. ☐

Because this is an equivalence of categories, it sends weak equivalences of simplicial abelian groups to quasi-isomorphisms of chain complexes. In other words, we have

$$\pi_n|A| \cong H_n(NA)$$

for any simplicial abelian group A.

2.3 Monoidal Dold–Kan

While the Dold–Kan correspondence is useful, it isn't perfect. The two categories **sAb** and $\mathbf{Ch}^+(\mathbb{Z})$ have different tensor products on them that are not compatible with this equivalence of categories.

Definition 2.11. The tensor product of two chain complexes P and Q is

$$(P \otimes Q)_n = \bigoplus_{p+q=n} P_p \otimes Q_q.$$

Definition 2.12. The tensor product of two simplicial abelian groups A and B is

$$(A \otimes B)_n = A_n \otimes B_n.$$

Example 2.13. If we let $\mathbb{Z}[1]$ be the chain complex which is $\mathbb{Z}$ concentrated in degree 1, then

$$\mathbb{Z}[1] \otimes_{\mathbf{Ch}^+(\mathbb{Z})} \mathbb{Z}[1] = \mathbb{Z}[2]$$

as chain complexes. On the other hand,

$$\Gamma(\mathbb{Z}[2]) \neq \Gamma(\mathbb{Z}[1]) \otimes_{\mathbf{sAb}} \Gamma(\mathbb{Z}[1]),$$

which can be seen, for example, by checking their values in degree one.

Therefore, Γ cannot be a monoidal functor (it doesn't respect tensor products). A similar example works to show that N isn't monoidal either.

However, there is a remedy to this situation. If we relax the condition that N and Γ be an equivalence of categories, and ask instead that they only respect the homotopy theories of these categories, then it is a theorem of Schwede–Shipley that there is a *monoidal* Quillen equivalence between the categories of simplicial abelian groups and connective chain complexes.

2.4 Exercises

(2.4.1) Use the simplicial relations to show that $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$ satisfies $\partial_n \circ \partial_{n+1} = 0$.

(2.4.2) Describe $\mathbb{Z}\Delta_n$ as a chain complex.

(2.4.3) Describe $\mathbb{Z}S^n$ as a chain complex, using the simplicial model for S^n from Exercise (1.4.5).

(2.4.4) Let A be a simplicial abelian group. Prove that

$$(NA)_n \cong \bigcap_{i=0}^{n-1} \ker(d_i \colon A_n \to A_{n-1}).$$

Prove that the associated chain complex CA splits as $CA = NA \oplus DA$.

(2.4.5) For any chain complex Q, verify that ΓQ is a simplicial abelian group.

(2.4.6) Show that N and Γ are adjoint functors. That is, prove that there is a natural bijection

$$\mathrm{Hom}_{\mathbf{sAb}}(\Gamma Q, A) \cong \mathrm{Hom}_{\mathbf{Ch}^+(\mathbb{Z})}(Q, NA).$$

(2.4.7) Verify that $\Psi \colon \Gamma N \to \mathrm{id}_{\mathbf{sAb}}$ and $\Phi \colon \mathrm{id}_{\mathbf{Ch}^+(\mathbb{Z})} \to N\Gamma$ are inverse natural isomorphisms.

(2.4.8) This section has been written for simplicial sets and homology of chain complexes. What should we change if we instead want to think about cohomology of cochain complexes?

3 Classifying Spaces

The classifying space of a group G is a topological space BG such that homotopy classes of maps $X \to BG$ correspond bijectively to isomorphism classes of principal G-bundles on X. In this sense, the space BG is "classifying." More generally, we can form the classifying space of any category $\mathbf{C}$, which has a similar property which we won't state here.

A description of the spaces EG and BG as simplicial sets can be found in [GJ09, Chapter V]. For the classical story on vector bundles and classifying spaces, see [Hat17, Section 1.2] or [Mit01].

3.1 Nerves

Definition 3.1. Let $\mathbf{C}$ be a small category. The **nerve of C** is the simplicial set $N_\bullet\mathbf{C}$ with n-simplicies consisting of n-tuples of composable morphisms in $\mathbf{C}$.

$$N_n\mathbf{C} = \left\{ c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \right\}$$

The face map d_i is given by deleting the i-th object and composing adjacent morphisms where necessary:

$$d_i\left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) = \begin{cases} \left(c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_n\right) & (i = 0) \\ \left(c_0 \xrightarrow{f_1} \cdots c_{i-1} \xrightarrow{f_i \circ f_{i-1}} c_{i+1} \xrightarrow{f_{i+1}} \cdots \to c_{n-1}\right) & (0 < i < n) \\ \left(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} c_{n-1}\right) & (i = n) \end{cases}$$

The degeneracies are given by inserting an identity:

$$s_j\left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) = \left(c_0 \to c_1 \to \cdots \to c_j \xrightarrow{\mathrm{id}_{c_j}} c_j \to \cdots \to c_n\right)$$

With this definition, notice that $N_0\mathbf{C}$ is the set of objects of $\mathbf{C}$, and $N_1\mathbf{C}$ is the set of morphisms of $\mathbf{C}$.

We may describe the nerve somewhat more abstractly with the following observation. Recall that any ordered set A can be considered as a category, with a morphism $a \to b$ if $a \leq b$. If we consider $[n] = \{0 < 1 < \ldots < n\}$ as a category

$$0 \to 1 \to \cdots \to n,$$

then we have

$$N_k\mathbf{C} = \mathrm{Fun}([k], \mathbf{C}).$$

Example 3.2. Consider the ordered set $[n]$ as a category. We have

$$N_k[n] = \mathrm{Fun}([k], [n]) = \mathrm{Hom}_{\mathbb{\Delta}}([k], [n]) = \Delta^n_k.$$

Therefore, $N_\bullet[n] \cong \Delta^n$ as simplicial sets.

Example 3.3. Let G be a discrete group. Consider the category $\underline{G}$ with a single object $*$ and a morphism $g\colon * \to *$ for each $g \in G$. Composition is given by multiplication in G. In this case, $N_\bullet \underline{G}$ is the simplicial set with $N_n \underline{G} = G^n$ with face and degeneracy maps given by

$$s_j(g_1, \dots, g_n) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n)$$
$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & (i = 0) \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & (0 < i < n) \\ (g_1, \dots, g_{n-1}) & (i = n) \end{cases}$$

Definition 3.4. Let G be a group. The **classifying space** BG of G is the geometric realization of the nerve of G considered as a one-object category

$$BG := |N_\bullet \underline{G}|.$$

We may also define the classifying space for any category **C** in exactly the same way, but we will focus on classifying spaces for groups from here on.

Example 3.5. Let G be a group. Consider the category $\widetilde{G}$ with object set G and a unique morphism $g \to h$ for each pair $(g, h) \in G \times G$. An n-simplex in the nerve of this category is a diagram of the form

$$\left(g_0 \xrightarrow{!} g_1 \xrightarrow{!} \cdots \xrightarrow{!} g_n \right),$$

where ! denotes the unique morphism $g_i \to g_{i+1}$. The data of these morphisms adds nothing to this construction, so we may as well forget them: $N_\bullet \widetilde{G}$ is the simplicial set with n-simplicies $N_n \widetilde{G} = G^{n+1}$ and face and degeneracy maps

$$s_j(g_0, \dots, g_n) = (g_0, \dots, g_j, g_j, \dots, g_n)$$
$$d_i(g_0, \dots, g_n) = (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

The geometric realization of this simplicial set is denoted $EG := |N_\bullet \widetilde{G}|$.

There is a functor $\widetilde{G} \to \underline{G}$ given by sending the unique morphism $g \to h$ in $\widetilde{G}$ to the morphism gh^{-1} in $\underline{G}$. On nerves, this yields a morphism of simplicial sets

$$\begin{aligned} N_n \widetilde{G} &\longrightarrow N_n \underline{G} \\ (g_0, \dots, g_n) &\longmapsto (g_0 g_1^{-1}, \dots, g_{n-1} g_n^{-1}) \end{aligned}$$

Applying the geometric realization, we obtain a map of spaces $EG \to BG$.

Remark 3.6. The construction given above doesn't take into account any topology on the group G, so should only be applied to discrete groups. However, there is a version of the classifying space construction that makes sense for any topological group in a way that takes into account its topology.

3.2 Identifying classifying spaces

The construction of the classifying space given above is often a bit difficult to work with in practice. Instead, we will try to recognize spaces we already know as the classifying spaces of groups.

Definition 3.7. Let G be a group. A topological space X is an **Eilenberg–MacLane space** of type $K(G, n)$ if $\pi_n X = G$ and $\pi_i X = 0$ for $i \neq n$.

Proposition 3.8. *If* G *is a discrete group, then* BG *is a* $K(G, 1)$*-space.*

In fact, BG is the standard construction of a space of type $K(G, 1)$ for discrete groups.

Theorem 3.9. *Any two spaces of type* $K(G, n)$ *are homotopy equivalent.*

Proof Sketch. This theorem follows from Whitehead's theorem and CW-approximation. See [Hat02, §4.2] for details. $\square$

Knowing this theorem, we abuse notation and write $K(G, n)$ for any space with this homotopy type. The upshot of this theorem is that if X is any space of type $K(G, 1)$, then $X \simeq BG$. This allows us to identify spaces we already know as classifying spaces.

Example 3.10. S^1 is a classifying space for $\mathbb{Z}$, because $\pi_1 S^1 = \mathbb{Z}$ but $\pi_k S^1 = 0$ for $k \neq 1$.

Example 3.11. $\mathbb{RP}^\infty$ is a classifying space for $\mathbb{Z}/2$. More generally, $S^\infty/(\mathbb{Z}/n)$ is a classifying space for $\mathbb{Z}/n$.

Example 3.12. $\mathbb{CP}^\infty$ is a space of type $K(\mathbb{Z}, 2)$. In fact, $\mathbb{CP}^\infty \simeq BS^1$, but that doesn't make it a space of type $K(S^1, 1)$ because S^1 isn't a discrete group.

Example 3.13. The infinite-dimensional real Grassmannian $\mathrm{Gr}_n(\mathbb{R}^\infty) \simeq BO(n)$ is a classifying space for the orthogonal group $O(n)$. Similarly, $\mathrm{Gr}_n(\mathbb{C}^\infty) \simeq BU(n)$ is a classifying space for the unitary group $U(n)$.

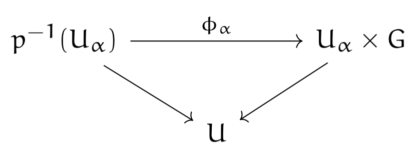
3.3 Classifying spaces and principal G-bundles

Definition 3.14. Let G be a group, and let P be a topological space with a G-action. A **principal** G**-bundle** $p\colon P \to B$ is a continuous G-equivariant map of topological spaces, such that G acts trivially on B, and satisfies the **local triviality** condition:

- there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of B and G-equivariant homeomorphisms

  $$\phi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times G$$

  where $U_\alpha \times G$ is given the right G-action $(u, g)h = (u, gh)$. These homeomorphisms must make the following diagram commute:

Note that by this condition, G acts freely on P and $p\colon P \to B$ factors through a homeomorphism $P/G \to B$. We say that a principal G-bundle is a locally trivial free G-space P with orbit space B.

Example 3.15. The **trivial** G**-bundle** over a base space B is the projection $B \times G \to B$, with G acting only on the second component of the total space.

Example 3.16. The covering map $\mathbb{R} \to S^1$ is an example of a principal $\mathbb{Z}$-bundle, with $\mathbb{Z}$ acting on the additive group of $\mathbb{R}$ and trivially on S^1.

Example 3.17. There is a particularly important vector bundle for any group G with base space given by the classifying space BG and total space EG. Note that the n-simplicies of $N_\bullet \widetilde{G}$ have a free G-action given by left-multiplication

$$g \cdot (g_0, \dots, g_n) = (gg_0, \dots, gg_n).$$

From this action on the n-simplicies, the geometric realization EG inherits a G-action. In fact, EG has a map $EG \to BG$ such that $EG/G \simeq BG$. Hence, $EG \to BG$ is a principal bundle.

Definition 3.18. The **universal** G**-bundle** is the G-bundle $EG \to BG$ described above.

Much as we had a way to recognize spaces we already know as classifying spaces, we also have a way to recognize the total space of the universal bundle given its classifying space.

Proposition 3.19. *The space* EG *is the universal cover of the classifying space* BG*.*

Finally, we arrive at the reason that the spaces BG are "classifying." Given any map $f\colon X \to BG$ of spaces, we can form a G-bundle by pullback as in the following diagram:

$$\begin{array}{ccc} f^*(EG) & \dashrightarrow & EG \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

This construction depends only on the homotopy class of the map f. In fact, every principal G-bundle can be constructed in this manner.

Theorem 3.20. *Principal* G*-bundles with base space* X *correspond bijectively to homotopy classes of maps* $f\colon X \to BG$ *via pullback of the universal bundle* $EG \to BG$*.*

3.4 Exercises

(3.4.1) Show that $N\mathbf{C} \simeq N\mathbf{C}^{op}$.

(3.4.2) Show that if $\mathbf{C}$ has an initial or terminal object, then $B\mathbf{C}$ is contractible (i.e. homotopy equivalent to a point) as a topological space.

(3.4.3) Let P be a poset considered as a category with a morphism $A \to B$ if $A \leq B$. Prove that if every chain in P has an upper bound, then BP is contractible.

(3.4.4) Let G be a discrete group. Using Dold–Kan, we can construct chain complexes from $N_\bullet \underline{G}$ and $N_\bullet \widetilde{G}$. Describe the homology groups of these chain complexes in terms of G. How are they related to each other? How do the homology and cohomology of BG relate to the group homology and cohomology of G?

(3.4.5) What are the properties of the functor $B\colon$ **Group** $\to$ **Top**? Does it preserve products? Coproducts?

(3.4.6) Describe a classifying space for the free abelian group of rank n.

(3.4.7) Describe a classifying space for the free group of rank n.

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