# Lesson 5 – From Varieties to Ideals

**I. Quick Review of Ideals** (As always $k$ is a field.)

**Definition 1** A subset $I \subseteq k[x_1, x_2, \dots, x_n]$, is an **ideal** if it satisfies:

i) $0 \in I$
ii) If $f, g \in I$, then $f + g \in I$.
iii) If $f \in I$ and $h \in k[x_1, x_2, \dots, x_n]$, then $hf \in I$

**Definition 2** Let $f_1, f_2, \dots, f_s \in k[x_1, x_2, \dots, x_n]$. Then we set

$$\langle f_1, f_2, \dots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, h_2, \dots, h_s \in k[x_1, x_2, \dots, x_n] \right\}.$$

It is easy to see that $\langle f_1, f_2, \dots, f_s \rangle$ is an ideal of $k[x_1, x_2, \dots, x_n]$[1]; it is called the **ideal generated by** $f_1, f_2, \dots, f_s$.

**Question:** Your textbook is titled "**Ideals**, Varieties, and Algorithms". Why are ideals so important to Algebraic Geometry?

**Proposition 3** Let $I \subseteq k[x_1, x_2, \dots, x_n]$ be an ideal, and let $f_1, f_2, \dots, f_s \in k[x_1, x_2, \dots, x_n]$. Prove that the following statements are equivalent:

i) $f_1, f_2, \dots, f_s \in I$.
ii) $\langle f_1, f_2, \dots, f_s \rangle \subseteq I$.

[1] This short proof is in the textbook.

Proposition 3 is helpful in proving that one ideal is contained in another...

**Exercise 1** Prove that $\langle x + xy, y + xy, x^2, y^2 \rangle = \langle x, y \rangle$ in $\mathbb{Q}[x, y]$

**Definition** Let $k$ be a field, and let $f_1, f_2, \dots, f_d \in k[x_1, x_2, \dots, x_n]$,. Then the **affine variety**, denoted by $\mathbf{V}(f_1, f_2, \dots, f_d)$, is defined by:

$$\mathbf{V}(f_1, f_2, \dots, f_d) = \{(a_1, a_2, \dots, a_n) \in k^n : f_i(a_1, a_2, \dots, a_n) = 0 \text{ for all } 1 \leq i \leq n\}.$$

Let's make sure we have a good understanding of the definition...

**Exercise 2 -** For each of the statements below, assume $k$ is a field and $f_1, f_2, \dots, f_d \in k[x_1, x_2, \dots, x_n]$

**True** **False** a) For any point $P = (a_1, a_2, \dots, a_n) \in \mathbf{V}(f_1, f_2, \dots, f_d)$ we have $f_i(P) = 0$.

**True** **False** b) For any point $P = (a_1, a_2, \dots, a_n) \in \mathbf{V}(f_1, f_2, \dots, f_d)$, and any polynomial $g \in k[x_1, x_2, \dots, x_n]$, $gf_i(P) = 0$

**True** **False** c) For any point $P = (a_1, a_2, \dots, a_n) \in \mathbf{V}(f_1, f_2, \dots, f_d)$, $(f_i + f_j)(P) = 0$ for all $1 \leq i, j \leq d$.

**True** **False** d) For any point $P = (a_1, a_2, \dots, a_n) \in \mathbf{V}(f_1, f_2, \dots, f_d)$ and $g_1, g_2, \dots, g_d \in k[x_1, x_2, \dots, x_n]$, if $f = g_1 f_1 + g_2 f_2, \dots, g_d f_d$, then $f(P) = 0$.

**True** **False** e) For any point $P = (a_1, a_2, \dots, a_n) \in \mathbf{V}(f_1, f_2, \dots, f_d)$, $f(P) = 0$ for all $f \in \langle f_1, f_2, \dots, f_d \rangle$

Hopefully the previous exercise has motivated the following definition of the vanishing *ideal* of $V$. (That is, hopefully, you are convinced that the set $\mathbf{I}(V)$ *is indeed an ideal*. Check this, if you like.)

**Definition** Let $V \subseteq k^n$ be an affine variety. Then define the **ideal of** $\boldsymbol{V}$:

$$\mathbf{I}(V) = \{f \in k[x_1, x_2, \dots, x_n] : f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in V\}$$

**Exercise 3** – (Below we examine $\mathbf{I}(V)$ for various affine varieties $V \subseteq k^n$; $k$ is an infinite field.)

1) What is the vanishing ideal of the empty set? That is, what is $\mathbf{I}(\phi)$?

2) What is $\mathbf{I}(k^n)$?

3) What is $\mathbf{I}(\mathbf{V}(x_1))$?

As the above examples might indicate, the function $\mathbf{I}$ : {varieties in $k^n$} → {ideals in $k[x_1, x_2, \dots, x_n]$} is inclusion-reversing. (Note that item ii) below ensures that $\mathbf{I}$ is indeed a function, and it's injective.)

**Proposition 2** Let $V$ and $W$ be affine varieties in $k^n$. Then

i) $V \subseteq W$ iff $\mathbf{I}(V) \supseteq \mathbf{I}(W)$.
ii) $V = W$ iff $\mathbf{I}(V) = \mathbf{I}(W)$.
**(True or False:** $V \subsetneq W$ iff $\mathbf{I}(V) \supsetneq \mathbf{I}(W)$)

**Upshot:** Given the function **I**: {varieties in $k^n$} → {ideals in $k[x_1, x_2, \dots, x_n]$} defined by:

$$\mathbf{I}(V) = \{f \in k[x_1, x_2, \dots, x_n] : f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in V\}$$

$\mathbf{I}(V) = \mathbf{I}(W) \Rightarrow V = W$, so **I** is an injection.

**A Natural Question to Ask:**

> Is the function **I** *surjective*? That is, given an ideal $I$ in $k[x_1, x_2, \dots, x_n]$, is $I$ the vanishing ideal for some affine variety *V*?

Later we will see that every ideal of $k[x_1, x_2, \dots, x_n]$ is finitely generated; this means there exist polynomials (i.e., generators) $f_1, f_2, \dots, f_d \in k[x_1, x_2, \dots, x_n]$ such that $I = \langle f_1, f_2, \dots, f_d \rangle$.

It might be natural then to guess that $\mathbf{V}(f_1, f_2, \dots, f_d)$ is the variety we seek. So we ask:

Given $f_1, f_2, \dots, f_d \in k[x_1, x_2, \dots, x_n]$, is $\mathbf{I}(\mathbf{V}(f_1, f_2, \dots, f_d)) = \langle f_1, f_2, \dots, f_d \rangle$? We will examine this question more carefully later; for now let's look at an example to convince you that there is much to think about...

**Exercise 4** Consider the ideals $I_1 = \langle x^2, xy \rangle$ and $I_2 = \langle x, xy \rangle$ in $k[x, y]$.

a) Describe the sets $\mathbf{V}(x^2, xy)$ and $\mathbf{V}(x, xy)$.

b) What are $\mathbf{I}(\mathbf{V}(x^2, xy))$ and $\mathbf{I}(\mathbf{V}(x, xy))$? Does the function **I** appear to be surjective?

c) Can you find other functions $f_1, f_2 \in k[x, y]$ such that $\mathbf{V}(f_1, f_2) = \mathbf{V}(x^2, xy)$ and hence $\mathbf{I}(\mathbf{V}(f_1, f_2)) = \mathbf{I}(\mathbf{V}(x^2, xy))$? Can you say anything about when an ideal $\langle f_1, f_2 \rangle \subseteq k[x, y]$ is in the image of **I**?