Friday Fun: Exploring the Ideal Membership Problem

Given an ideal $I = \langle f_1, f_2, \dots, f_s \rangle \subseteq k[x_1, x_2, \dots, x_n]$ and a polynomial $f \in k[x_1, x_2, \dots, x_n]$, we would like to decide whether $f \in I$. The idea is simple; $f \in I$ if and only if it can be written as $f = \sum_{i=1}^{s} h_i f_i$ for some $h_i \in k[x_1, x_2, \dots, x_n]$. Of course, when there is only one variable, then $k[x_1]$ is a PID, and the ideal membership problem is easy; $I = \langle f_1, f_2, \dots, f_s \rangle = \langle \mathrm{GCD}(f_1, f_2, \dots, f_s) \rangle$, and $f \in I$ iff $\mathrm{GCD}(f_1, f_2, \dots, f_s) \mid f$. Today we examine what happens in the multivariate case. Feel free to make use of the chalkboards while working through these exercises.

Exercise 1
a) Using lex order, divide the polynomial $f(x,y,z) = x^3 - x^2y - x^2z + x$ by (f_1, f_2), where $f_1(x,y,z) = x^2y - z$ and $f_2(x,y,z) = xy - 1$. Record your quotients and remainder below.

$$f(x,y,z) = \underline{\hspace{3cm}} \cdot (x^2y - z) + \underline{\hspace{3cm}} \cdot (xy - 1) + \underline{\hspace{5cm}}$$

b) Now switch the order; that is, divide the polynomial $f(x,y,z) = x^3 - x^2y - x^2z + x$ by (f_2, f_1). Record the result below.

$$f(x,y,z) = \underline{\hspace{3cm}} \cdot (x^2y - z) + \underline{\hspace{3cm}} \cdot (xy - 1) + \underline{\hspace{5cm}}$$

Exercise 2 Let r_1 be the remainder you obtained in part a) above and r_2 the remainder from part b). Define r to be the difference of the two: $r = r_1 - r_2$. Is $r \in \langle f_1, f_2 \rangle$? If yes, then find $h_1, h_2 \in k[x,y,z]$ such that $r = h_1 f_1 + h_2 f_2$. If no, then explain why $r \notin \langle f_1, f_2 \rangle$.

Exercise 3 – True or False
If f divided by $F = (f_1, f_2, \dots, f_s)$ produces a remainder r_1 and f divided by $G = (f_{i_1}, f_{i_2}, \dots, f_{i_s})$ produces a remainder r_2, where $(i_1, i_2, \dots, i_s)$ is a permutation of $(1, 2, \dots, s)$, then $r_1 - r_2 \in \langle f_1, f_2, \dots, f_s \rangle$. Prove your claim.

Exercise 4 (Exercises 4-7 refer to the example introduced in Exercise 1, so $f_1 = x^2y - z$, $f_2 = xy - 1$.)
a) Compute the remainder of $r = x - z$ on division by (f_1, f_2). Why could you have predicted your answer before doing the division?
b) What does your answer from part a) tell you about the Ideal Membership Problem?

Exercise 5 Show that $g(x,y,z) = yz - 1 \in \langle f_1, f_2 \rangle$. (Observe: $yz - 1 = (xy+1)(xy-1) - y(x^2y - z)$.) What is the remainder of g on division by (f_1, f_2)?

Exercise 6 Prove that $\langle x - z, yz - 1 \rangle = \langle f_1, f_2 \rangle$.

Exercise 7 Can any element $h_1 \cdot (x - z) + h_2 \cdot (yz - 1) \in \langle x - z, yz - 1 \rangle$ have a nonzero remainder when it is divided by $\langle x - z, yz - 1 \rangle$? Prove your claim.

One Final Question: What is the point of these exercises?

Quick Recap of the Multivariate Division Algorithm:

Theorem (The Division Algorithm in $k[x_1, x_2, \dots, x_n]$) Fix a monomial order > on $\mathbb{Z}_{\geq 0}^n$ and let $F = (g_1, g_2, \cdots, g_s)$ be an ordered s-tuple of polynomials in $k[x_1, x_2, \dots, x_n]$. Then for any $f \in k[x_1, x_2, \dots, x_n]$ there exist $a_1, a_2, \dots, a_s, r \in k[x_1, x_2, \dots, x_n]$ such that

1. $f = a_1 g_1 + a_2 g_2 + \cdots + a_s g_s + r$ and r is reduced with respect to G.
2. $\max\{\mathrm{LT}(a_1)\mathrm{LT}(g_1), \dots, \mathrm{LT}(a_s)\mathrm{LT}(g_s), \mathrm{LT}(r)\} = \mathrm{LT}(f)$

(Condition 2 stated above is stronger than and implies the textbook's second condition "if $a_i g_i \neq 0$, then $\mathrm{multideg}(f) \geq \mathrm{multideg}(a_i g_i)$".)

INITIAL VALUES: $a_1 := 0, a_2 := 0, \dots, a_s := 0, r := 0, h := f$

ALGORITHM: While $h \neq 0$ do

-if $\exists i$ such that $\mathrm{LT}(g_i) | \mathrm{LT}(h)$ then choose the least such i and

$$a_i := a_i + \frac{\mathrm{LT}(h)}{\mathrm{LT}(g_i)}$$

$$h := h - \frac{\mathrm{LT}(h)}{\mathrm{LT}(g_i)} g_i$$

-else $r := r + \mathrm{LT}(h)$

$h := h - \mathrm{LT}(h)$

Example Divide $x^2y^2z + 2x^2z + x^2 - xy^2$ by the set $F = (xy^2 + 2, x^2z - 2)$. Use the lex order.

Solution:

$a_1 :$

$a_2 :$

$$\begin{array}{l|l} xy^2 + 2 & \overline{x^2y^2z + 2x^2z + x^2 - xy^2} \\ x^2z - 2 & \end{array}$$