Lesson 9 – Monomial Ideals and Dickson's Lemma

Definition An ideal $I \subseteq k[x_1, x_2, \dots, x_n]$ is a **monomial ideal** if it is generated by a set of monomials $\{x^\alpha : \alpha \in A\}$, where $A \subseteq \mathbb{Z}^n_{\geq 0}$. In this case, we write $I = \langle x^\alpha : \alpha \in A \rangle$.

Properties of Monomial Ideals

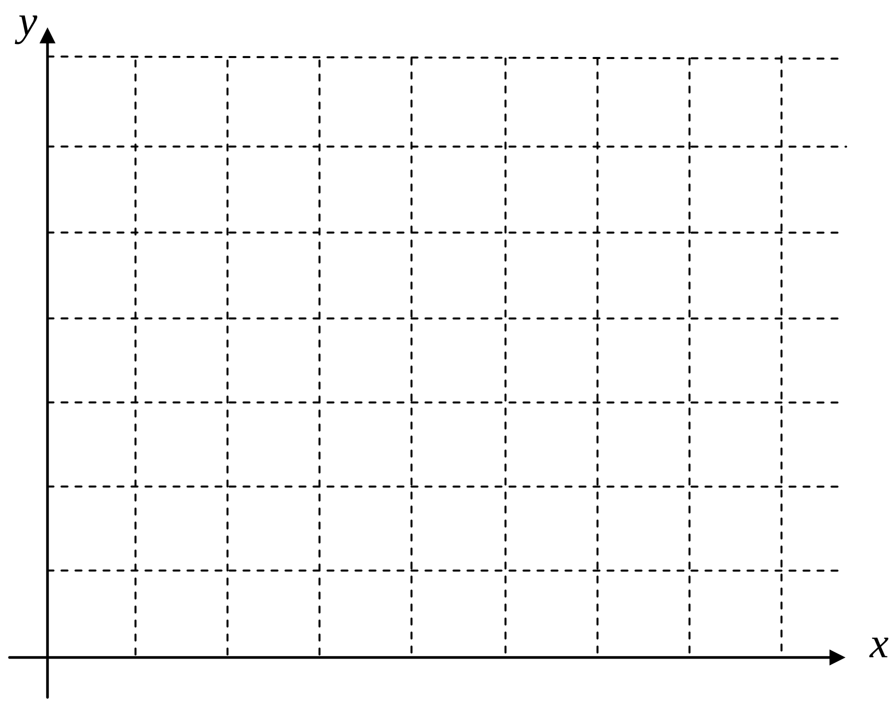
1. If $I = \langle x^\alpha : \alpha \in A \rangle$, then $x^\beta \in I \Rightarrow x^\beta = x^\gamma \cdot x^\alpha$ for some $\alpha \in A$.

2. A polynomial f is in a monomial ideal $I = \langle x^\alpha : \alpha \in A \rangle$ iff every term of f is in I. Or equivalently,

$$I = \langle x^\alpha : \alpha \in A \rangle = \{f \in k[x_1, x_2, \dots, x_n] \text{: every term of } f \text{ is divisible by some } x^\alpha, \alpha \in A\}$$

Exercise 1 (Visualizing Monomial Ideals) Consider the ideal $I = \langle x^6, x^2y^3, xy^5 \rangle \subseteq k[x, y]$.

a) In the plane below, plot the set of all possible exponent vectors (m, n) appearing as exponents of monomials $x^m y^n$ in I.



b) (And just for fun...) If we apply the Division Algorithm to an element $f \in k[x, y]$, using the generators of I as divisors, what terms can appear in the remainder?

Definition. If $I \subseteq k[x_1, x_2, \dots, x_n]$ is a monomial ideal, we define the **signature** of I to be the subset

$$\text{signature}(I) = \{\beta \in \mathbb{Z}^n_{\geq 0} : x^\beta \in I\}.$$

Exercise 2 What is the signature of $I = \langle x^6, x^2y^3, xy^5 \rangle \subseteq k[x, y]$?

Exercise 3 More generally, what is the signature of $I = \langle x^\alpha : \alpha \in A \rangle \subseteq k[x_1, \dots, x_n]$?

The main result of Section 2.4 is Dickson's Lemma...

Dickson's Lemma A monomial ideal has a finite basis. In particular, if $I = \langle x^\alpha : \alpha \in A \rangle$, there is a finite subset $\{\alpha(1), \alpha(2), \dots \alpha(s)\} \subseteq A$ for which

$$\langle x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)} \rangle = \langle x^\alpha : \alpha \in A \rangle$$

Exercise 4 Since monomial ideals are completely determined by their signatures, we should be able to restate Dickson's Lemma using the language of signatures - instead of monomials. Do this!

Proof of Dickson's Lemma.

We will proceed by induction on the number of variables, *n*. The base case amounts to the following lemma.

Lemma (Dickson's Lemma when *n* = 1) If $I = \langle x^\alpha : \alpha \in A \rangle$ is a monomial ideal in $k[x_1]$, then I is finitely generated.

Proof of Dickson's Lemma continued...

Next consider $n > 1$. For notational purposes, write the variables as $x_1, x_2, \dots, x_{n-1}, y$ and the exponents as (α, j), where $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$ and $j \in \mathbb{Z}$. Then the monomials in $k[x_1, x_2, \dots, x_{n-1}, y]$ can be written in the form $x^\alpha y^j$.

Suppose $I \subseteq k[x_1, x_2, \dots, x_{n-1}, y]$ is a monomial ideal; so $I = \langle x^\alpha y^j : \alpha \in \mathbb{Z}_{\geq 0}^{n-1}, j \in \mathbb{N} \rangle$

Define $J = \langle x^\alpha \in k[x_1, x_2, \dots, x_{n-1}] : x^\alpha y^m \in$ I for some $m \in \mathbb{N}$
Then J is a monomial ideal in $k[x_1, x_2, \dots, x_{n-1}]$ (but we won't prove this here), so by the inductive hypothesis,

$$J = \langle x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)} \rangle$$

for some choice of $\alpha(i) \in \mathbb{Z}_{\geq 0}^{n-1}$, where for each i there is an $x^{\alpha(i)} y^{m_i} \in I$.

Choose such m_i's and let

$$m = \max\{m_1, m_2, \dots, m_s\} .$$

Now for each q with $0 \leq q \leq m-1$ let J_q be the ideal in $k[x_1, x_2, \dots, x_{n-1}]$ generated by the x^β for which $x^\beta y^q \in I$.

J_q has a finite generating set by the inductive assumption, say $J_q = \langle x^{\alpha_q(1)}, x^{\alpha_q(2)}, \dots, x^{\alpha_q(s_q)} \rangle$

Claim: the monomials in the following $m+1$ sets generate I.

from J_0: $\{x^{\alpha_0(1)}, x^{\alpha_0(2)}, \dots, x^{\alpha_0(s_0)}\}$

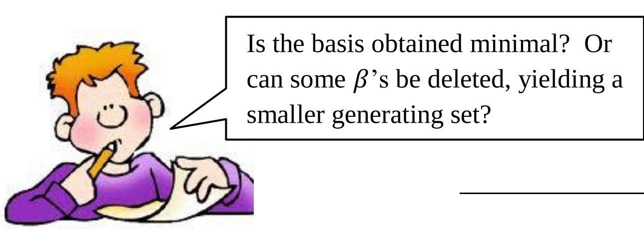
from $J_1 y$: $\{x^{\alpha_1(1)} y, x^{\alpha_1(2)} y, \dots, x^{\alpha_1(s_1)} y\}$

from $J_2 y^2$: $\{x^{\alpha_2(1)} y^2, x^{\alpha_2(2)} y^2, \dots, x^{\alpha_2(s_2)} y^2\}$

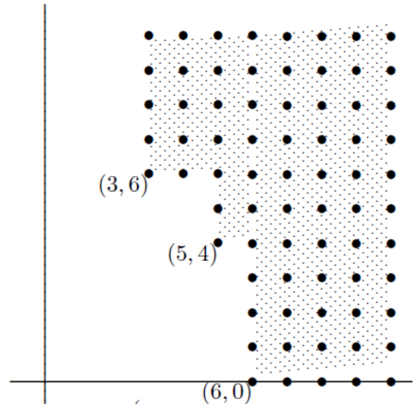
$\vdots$

from $J_{m-1} y^{m-1}$: $\{x^{\alpha_{m-1}(1)} y^{m-1}, \dots, x^{\alpha_{m-1}(s)} y^{m-1}\}$

from $J y^m$: $\{x^{\alpha(1)} y^m, x^{\alpha(2)} y^m, \dots, x^{\alpha(s)} y^m\}$



Example Let $I \subseteq k[x, y]$ be the monomial ideal spanned over k by the monomials x^β corresponding to β in the shaded region below:



a) What is the ideal J in this example?

b) Find m. (Is the value unique?)

c) Find J_q for all $0 \leq q \leq m-1$.

d) Find an ideal basis for I.

Claim: Let L be the ideal generated by the monomials in list below. Then $L = I$

$$\begin{aligned}
&\text{from } J_0: && \{x^{\alpha_0(1)}, x^{\alpha_0(2)}, \dots, x^{\alpha_0(s_0)}\} \\
&\text{from } J_1 y: && \{x^{\alpha_1(1)}y, x^{\alpha_1(2)}y, \dots, x^{\alpha_1(s_1)}y\} \\
&\text{from } J_2 y^2: && \{x^{\alpha_2(1)}y^2, x^{\alpha_2(2)}y^2, \dots, x^{\alpha_2(s_2)}y^2\} \qquad (*) \\
& && \vdots \\
&\text{from } J_{m-1} y^{m-1}: && \{x^{\alpha_{m-1}(1)}y^{m-1}, \dots, x^{\alpha_{m-1}(s_{m-1})}y^{m-1}\} \\
&\text{from } J y^m: && \{x^{\alpha(1)}y^m, x^{\alpha(2)}y^m, \dots, x^{\alpha(s)}y^m\}
\end{aligned}$$

Proof of Claim:

So why are we so fixated on monomial ideals?

(1) It is often easy to “see” what is true for monomial ideals
(2) What is true for monomial ideals is often true for general ideals (Hilbert Basis Theorem)
(3) There is frequently an easy reduction of the general to the monomial case.