Lesson 12 - Review for Exam 1

Rebus 1

1. What is the definition of the *ideal associated with a variety*?

Answer: $\mathbf{I}(V) = \{f \in k[x_1, x_2, \ldots, x_n] : f(a_1, a_2, \ldots, a_n) = 0 \text{ for all } (a_1, a_2, \ldots, a_n) \in V\}$

(Recall that if U is contained in V, then $\mathbf{I}(V)$ is contained in $\mathbf{I}(U)$. The **I** map is inclusion reversing.)

2. State Mason's Theorem.

Let K be a field and A, B, C nonzero elements of $K[T]$ with $A + B = C$ and $\gcd(A, B, C) = 1$. If $\deg A \geq \deg \operatorname{rad} ABC$, then $A' = B' = C' = 0$.

And the corollary...If K has characteristic 0 and A, B, C are nonzero polynomials in $K[T]$ with $A + B = C$ with ABC nonconstant and $\gcd(A, B, C) = 1$, then

$$\max\{\deg A, \deg B, \deg C\} \leq \deg \operatorname{rad} ABC - 1.$$

3. What is the definition of the *ideal of leading terms* of an ideal?
The ideal of leading terms of an ideal I is the ideal generated by the set $\mathrm{LT}(I) = \{\mathrm{LT}(f) : f \in I\}$. That is, it's the ideal $\langle \mathrm{LT}(I) \rangle$.

Rebus 2

1. **True or False:** If $f \in I = \langle g_1, g_2, \dots, g_s \rangle$ and we apply the division algorithm to divide $f \in k[x_1, x_2, \dots, x_n]$ by the set $F = (g_1, g_2, \dots, g_s)$, then the remainder will be zero.

False. We aren't ensured a zero remainder unless the basis is a Groebner basis.

2. **True or False:** The infinite intersection of affine varieties is an affine variety.

True. Since the infinite intersection of ideals is an ideal, and every ideal is finitely generated (by the Hilbert basis theorem) we can express that intersection as an ideal of the form $I = \langle f_1, f_2, \dots, f_s \rangle$. Then the intersection of affine varieties is given by $V(f_1, f_2, \dots, f_s)$.

3. **True or False:** The function **I** defined in class provided a bijection between the set of varieties in k^n and the set of ideals in $k[x_1, x_2, \dots, x_n]$.

False. It was not surjective. The ideal $\langle x_1^2, x_3 \rangle$, for example, would not be in the image of **I**...because $\mathbf{I}\big(\mathbf{V}(\langle x_1^2, x_3 \rangle)\big) = \mathbf{I}\big(H_{x_1} \cap H_{x_3}\big) = \langle x_1, x_3 \rangle$.

4. **True or False:** Let $I = \langle f_1, f_2, \dots, f_s \rangle \subseteq k[x, y, z]$ be an ideal and assume there exists a polynomial $g \in I$ such that $\mathrm{LT}(f_i) \nmid \mathrm{LT}(g)$ for all $1 \le i \le s$. Then $\{f_1, f_2, \dots, f_s\}$ is not a Groebner basis for I.

True. By the HW a Groebner basis can also be defined as a basis that has the property that any element in the ideal has a leading term divisible by a leading term of (at least) one of the elements in the basis.


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Rebus 3

1. Find the equation of the affine variety determined by the parameterization
$$x = \frac{t}{1+t}, y = 1 - \frac{1}{t^2}, \qquad t \neq 1, t \neq 0.$$

Solution: Write first equation as $x + tx = t$ and solve for t to get $t = \frac{x}{1-x}$. This tx-equation describes the hyperbola whose asymptotes are the lines $t = -1$ and $x = 1$. Substituting this into the equation for y yields $y = 1 - \left(\frac{1-x}{x}\right)^2$ or $x^2y - x^2 + (1-x)^2 = 0$. So the variety is $\mathbf{V}(x^2y + 1 - 2x)$. Now we found that varying t will yield any value for x except for $x = 1$. Thus any point on the curve $x^2y + 1 - 2x = 0$ or except for $y = 1$, since y is uniquely determined by the value of x. Thus the parameterization covers all points of the variety except for (1, 1).

2. Let k be a field and $I = \langle f_1, f_2, \dots, f_s \rangle \subseteq k[x]$ an ideal. Explain (in words, not pseudocode) how one can determine whether or not a given polynomial $f \in k[x]$ is in I.

Answer: Since $k[x]$ is a PID, every ideal is generated by a single polynomial. In particular $I = \langle f_1, f_2, \dots, f_s \rangle = \langle \mathrm{GCD}(f_1, f_2, \dots, f_s) \rangle$. This GCD can be computed using the Euclidean Algorithm over and over again. To be more explicit,
$D = \mathrm{GCD}(f_1, f_2, \dots, f_s) = \mathrm{GCD}\big(f_1, \mathrm{GCD}(f_2, \dots, f_s)\big) = \mathrm{GCD}\big(f_1, \mathrm{GCD}(f_2, \mathrm{GCD}(f_3, \dots, f_s)\big).$
Now f will be in I if D divides f. So we can determine ideal membership by dividing f by D. The polynomial f is in I iff division by D yields a remainder is zero.

3. Let k be an algebraically closed field. Characterize those $f \in k[x]$ for which $\mathbf{V}(f) = \emptyset$.

Answer: Every nonconstant polynomial f will have a root in k unless $\deg f = 0$. So the polynomials $f \in k[x]$ satisfying $\mathbf{V}(f) = \emptyset$ are precisely those with $\deg f = 0$ (i.e., the constant functions).

4. A basis $\langle x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)} \rangle$ for a monomial ideal I is *minimal* if no $x^{\alpha(i)}$ divides another $x^{\alpha(j)}$ for $i \neq j$. Show that every monomial ideal has a unique minimal basis.

Proof: First apply Dickson's Lemma to choose a finite basis $\{x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)}\}$ for I and delete any monomial in this basis that is a multiple of another monomial in the basis. The result is still a basis and it is minimal. Now consider two minimal bases for the monomial idea I:
$\{x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)}\}$ and $\{x^{\beta(1)}, x^{\beta(2)}, \dots, x^{\beta(t)}\}$.
Since $x^{\alpha(1)}$ is a monomial in I it follows that it must be divisible by (at least) one monomial from the basis $\{x^{\beta(1)}, x^{\beta(2)}, \dots, x^{\beta(t)}\}$, say $x^{\beta(1)}$. For the same reason, $x^{\beta(1)}$ is divisible by one of the $x^{\alpha(i)}$'s, but because $\{x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)}\}$ is minimal there is only one candidate: $x^{\beta(1)}$ is divisible by $x^{\alpha(1)}$. It follows that $x^{\alpha(1)} = x^{\beta(1)}$. Taking up $x^{\alpha(2)}$ we can argue similarly that it is equal to precisely one of the $x^{\beta(i)}$'s, say $x^{\beta(2)}$, and continuing in this manner, it follows that these two minimal bases for I are identical.

5. Describe the variety $\mathbf{V}(x^4 - zx, x^3 - xy)$.

Solution: $\mathbf{V}(x^4 - zx, x^3 - xy) = \mathbf{V}(x(x^3 - z), x(x^2 - y)) = \mathbf{V}(x^3 - z, x^2 - y) \cup \mathbf{V}(x)$. So the variety consists of all points on the yz-plane together with the twisted cubic.

6. Consider the equation $X^3 + Y^3 = Z^2$ for polynomials $X, Y, Z \in \mathbb{C}[T]$. Use Mason's Theorem to derive bounds for deg X, deg Y, deg Z of a solution in nonconstant coprime polynomials of this equation.

Solution: Let $A = X^3$, $B = Y^3$, and $C = Z^2$. Then Mason's theorem says that the maximum of the degrees of A, B, C are bounded above by deg rad $ABC - 1$.

deg A = 3deg$X \le$ deg rad$(X^3\, Y^3\, Z^2)$ -1 $\le$ deg XYZ -1 = deg X + deg Y + deg Z -1
(1) So 2deg$X \le$ deg Y + deg Z -1

Similarly, deg B = 3deg$Y \le$ deg rad$(X^3\, Y^3\, Z^2)$ -1 $\le$ deg XYZ -1= deg X + deg Y + deg Z -1, so
(2) 2deg$Y \le$ deg X + deg Z -1

and 2deg$Z \le$ deg rad$(X^3\, Y^3\, Z^2)$ -1 $\le$ deg XYZ -1 = deg X + deg Y + deg Z -1, so
(3) deg$Z \le$ deg X + deg Y -1

Add and cancel the three inequalities...
2deg$X$+2deg$Y$ + deg$Z \le$ deg Y + deg Z -1 + deg X + deg Z -1 + deg X + deg Y -1 implies **deg $Z \ge 3$**

Applying third bound to previous two yields: 2deg$X \le$ 2deg Y + deg X -2 or **deg$X \le$ 2deg Y -2** and similarly, 2deg$Y \le$ 2deg X + deg Y -2, so **deg$Y \le$ 2deg X -2**. So **deg$X \le$ 2(2deg X -2) -2** or

deg$X \le$ 4deg X - 6 . Hence **$2 \le$ deg X and by applying a symmetric argument, $2 \le$ deg Y.**

7. Divide $f(x, y, z) = 4x^2yz^2 - xy^3z - x^2y^2z - 1$ by the set $F = (x^2y - 1, yz + 1)$ using the grevlex monomial order.

Answer: Dividing f by F should yield:

$$\begin{aligned} f(x,y,z) &= 4x^2yz^2 - xy^3z - x^2y^2z - 1 \\ &= (-yz + 4z^2)(x^2y - 1) + (-xy^2 - 1)(yz + 1) + xy^2 + 4z^2 \end{aligned}$$

8. What is $\mathbf{I}(\mathbf{V}(x^m, y^n))$ for m and n positive integers? Prove your claim.

Claim: $\mathbf{I}(\mathbf{V}(x^m, y^n)) = \langle x, y \rangle$.

Proof. ($\supseteq$) The ideal $\mathbf{I}(\mathbf{V}(x^m, y^n))$ consists of all those polynomials which vanish at (0, 0) in k^2. Thus, $x, y \in \mathbf{I}(\mathbf{V}(x^m, y^n))$, and therefore $\mathbf{I}(\mathbf{V}(x^m, y^n)) \supseteq \langle x, y \rangle$.

($\subseteq$) For the reverse inclusion, observe that any polynomial $f \in \mathbf{I}(\mathbf{V}(x^m, y^n))$ vanishes at (0, 0) and so has a zero constant term. But then $f \in \langle x, y \rangle$.

RECALL: $\mathbf{I}(V) = \{f \in k[x_1, x_2, \dots, x_n]: f(a_1, a_2, \dots, a_n) = 0$ for all $(a_1, a_2, \dots, a_n) \in V\}$
so $\mathbf{I}(\mathbf{V}(x^m, y^n))$ consists of the set of polynomials having zero set equal to $\mathbf{V}(x^m, y^n)$. Clearly, $x^m = 0$ iff $x = 0$, and similarly, $y^n = 0$ iff $y = 0$.