Here's a fun problem to finish off the week...

Recall from previous lessons that $G = \{y - x^2, z - x^3\}$ is a Groebner basis for the ideal defining the twisted cubic under the lex order with $z > y > x$. Defining $f(x, y, z) = y - x^2$ and $g(x, y, z) = z - x^3$, observe that $\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f) \cdot \mathrm{LM}(g)$. More generally, the following is true:

Claim: If $G = \{f, g\}$ satisfies $\mathrm{LCM}\big(\mathrm{LM}(f), \mathrm{LM}(g)\big) = \mathrm{LM}(f) \cdot \mathrm{LM}(g)$ under a given monomial ordering, then G is a Groebner basis under that ordering. (And actually, this result extends to bases containing more than two elements, though we'll focus on the case where there are only two here.)

Proof Assume $G = \{f, g\}$ satisfies $\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f) \cdot \mathrm{LM}(g)$ under a given monomial ordering. For the sake of simplicity we may assume that f and g have been multiplied by the appropriate constant so that they are monic; i.e., assume $\mathrm{LC}(f) = \mathrm{LC}(g) = 1$. Additionally, write

$$f = \mathrm{LM}(f) + p$$
$$g = \mathrm{LM}(g) + q,$$

so that $\mathrm{multideg}(p) < \mathrm{multideg}(f)$ and $\mathrm{multideg}(q) < \mathrm{multideg}(g)$.

Okay...now it's your turn. Compute the S-polynomial and finish off the proof.