Lesson 18 – Elimination with Groebner Bases

We have seen that Groebner bases can be used to answer the "ideal membership problem", and last lesson we saw several other applications. Today we will examine how Groebner bases can be used to solve systems of polynomial equations (i.e., to determine affine varieties). A classical way to approach this problem is to eliminate variables sequentially until one obtains a polynomial in only one variable, which may be solved explicitly or numerically, and then proceed by back substitution (or *extension*).

I. Motivating Example

Consider the polynomials:

$$\begin{aligned} f_1 &= x - y - z \\ f_2 &= x + y - z^2 \\ f_3 &= x^2 + y^2 - 1 \end{aligned}$$

and let us find the set $\mathbf{V}(f_1, f_2, f_3) \subseteq \mathbb{C}[x, y, z]$ of common zeros.

Exercise 1

a) In "GaussianElimination-like" way, use the x in f_1 to eliminate terms in the latter two polynomials, arriving at the new system:

$$\begin{aligned} g_1 &= x - y - z \\ g_2 &= \quad 2y - z^2 + z \\ g_3 &= \quad 2y^2 + 2yz + z^2 - 1, \end{aligned}$$

which satisfies $\mathbf{V}(f_1, f_2, f_3) = \mathbf{V}(g_1, g_2, g_3)$.

b) Next we choose the term $2y$ in polynomial g_2 as the term most suitable for eliminating terms containing y in the other polynomials. Do this to produce the system:

$$\begin{aligned} h_1 &= 2x \qquad - z^2 - z \\ h_2 &= \quad 2y \ - z^2 + z \\ h_3 &= \qquad\quad z^4 + z^2 - 2, \end{aligned}$$

which satisfies $\mathbf{V}(f_1, f_2, f_3) = \mathbf{V}(g_1, g_2, g_3) = \mathbf{V}(h_1, h_2, h_3)$.

In the previous example, we replaced a system of polynomial equations with a new system having the same zero set as the original system. The new system is nice because its upper triangular form implies that it can be solved more easily. Let's see what a reduced Groebner basis does for us. Maple's **Basis** procedure with the lex ordering ($x > y > z$), yields:

```
>   with(Groebner):
>   G:={x^2+y+z-1,x+y^2+z-1,x+y+z^2-1};
>   Basis(G,plex(x,y,z));
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$$[-2 + z^2 + z^4,\ 2y - z^2 + z,\ - z - z^2 + 2x]$$

The Upshot: Groebner Bases achieve elimination!

II. Elimination Ideals

Referring again to the reduced Groebner basis computed above:

$$h_1 = 2x - z^2 - z$$
$$h_2 = 2y - z^2 + z$$
$$h_3 = z^4 + z^2 - 2,$$

observe that $h_3 = z^4 + z^2 - 2 \in I \cap \mathbb{C}[z]$, where $I = \mathbf{V}(f_1, f_2, f_3) = \mathbf{V}(h_1, h_2, h_3)$.
The ideal $I \cap \mathbb{C}[z]$ is said to be the *second elimination ideal* of I because the two variables x and y have been eliminated. More generally, we have:

Definition Given an ideal $I = \langle f_1, \dots, f_s \rangle \subseteq k[x_1, \dots, x_n]$, **the *j*th elimination ideal** is

$$I_j = I \cap k[x_{j+1}, \dots, x_n] \subseteq k[x_{j+1}, \dots, x_n].$$

It is an easy exercise to check that the I_j are indeed ideals.

Remarks:

1. We define the 0^{th} elimination ideal to be I, itself; i.e., $I_0 = I$.
2. With elimination we use the lex ordering, and different orderings on the variables will lead to different elimination ideals (where different variables are eliminated). One chooses the ordering on the variables purposefully, to eliminate the variable of choice. In the above definition, it is typically the case that $x_1 > \cdots > x_n$. (See the Elimination Theorem below.)

Exercise 2 Referring to the example on page 1, identify each of the elimination ideals: I_1, I_2, where $x > y > z$.

So here's the important theorem that says that Groebner bases achieve elimination:

Theorem (Elimination Theorem) If G is a Groebner basis for $I \subseteq k[x_1, \dots, x_n]$ with respect to the lex order $x_1 > x_2 > \cdots > x_n$, then for $0 \leq j < n$, $G_j = G \cap k[x_{j+1}, \dots, x_n]$ is a Groebner basis for the elimination ideal I_j.

Proof.

III. Extension

While the Groebner basis technique gives us bases for the elimination ideals, the extension process has its own problems. It is not clear in advance whether a solution $(a_{j+1}, \ldots, a_n) \in \mathbf{V}(I_j)$ extends to a solution $(a_j, a_{j+1}, \ldots, a_n) \in \mathbf{V}(I_{j-1})$. Let us consider some examples.

Exercise 3 Consider the equations

$$x^2 = y$$
$$x^2 = z$$

in $\mathbb{R}[x, y, z]$.

a) What is the first elimination ideal, I_1? Find a partial solution $(a_2, a_3) \in \mathbf{V}(I_1)$.

b) Does your partial solution from part a) extend to a full solution? That is, does there exist $a_1 \in \mathbb{R}$ an such that $(a_1, a_2, a_3) \in \mathbf{V}(I)$?

Exercise 4 Find all solutions to the equations

$$xy = 1$$
$$xz = 1$$

in $\mathbb{C}[x, y, z]$.

So when can we extend a partial solution to get a full solution???

The Extension Theorem tells us when we can extend a partial solution in $(a_2, \dots, a_n) \in \mathbf{V}(I_1)$ to a solution $(a_1, a_2, \dots, a_n) \in \mathbf{V}(I)$.

Theorem (Extension Theorem) Suppose $I = \langle f_1, \dots, f_s \rangle \in \mathbb{C}[x_1, \dots, x_n]$ and I_1 is the first elimination ideal of I. For each let $1 \leq i \leq s$ write

$$f_i = g_i(x_2, \dots, x_n)x_1^{N_i} + terms\ in\ which\ x_1\ has\ degree <\ N_i$$

where $N_i > 0$ and $g_i \in \mathbb{C}[x_2, \dots, x_n]$ is non-zero. If $(a_2, \dots, a_n) \in \mathbf{V}(I_1)$ is a partial solution such that $(a_2, \dots, a_n) \notin \mathbf{V}(g_1, \dots, g_r)$, then there exists $a_1 \in \mathbb{C}$ so that $(a_1, a_2, \dots, a_n) \in \mathbf{V}(I)$.

Actually, the Extension Theorem can be stated more generally for the j^{th} elimination ideal, but the additional indices required make the statement of the theorem more complicated than it's worth. Here's the more general statement, but I recommend that you work with the above statement instead.

Theorem (Extension Theorem for extending a partial soln in $\mathbf{V}(I_j)$ to a soln in $\mathbf{V}(I_{j-1})$.)
Suppose $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{C}[x_1, \dots, x_n]$ and $I_{j-1} = \langle f_1, \dots, f_r \rangle \subseteq \mathbb{C}[x_j, \dots, x_n]$. For each $1 \leq i \leq r$ let

$$f_i = g_i\left(x_{j+1}, \dots, x_n\right)x_j^{N_i} + terms\ in\ which\ x_j\ has\ degree <\ N_i$$

where $N_i > 0$ and $g_i \in \mathbb{C}\left[x_{j+1}, \dots, x_n\right]$ is non-zero. If $\left(a_{j+1}, \dots, a_n\right) \in \mathbf{V}(I_j)$ is a partial solution such that $\left(a_{j+1}, \dots, a_n\right) \notin \mathbf{V}(g_1, \dots, g_r)$, then there exists $a_j \in \mathbb{C}$ so that $\left(a_j, a_{j+1}, \dots, a_n\right) \in \mathbf{V}(I_{j-1})$.

Exercise 5 Consider the equations

$$x^2 + y^2 - z^2 = 1$$
$$2xyz = 1$$

Using Maple, we find that $G = \{4y^4z^2 - 4y^2z^2 - 4y^2z^4 + 1, x + 2y^3z - 2yz^3 - 2yz\}$ is a Groebner basis of $I = \langle x^2 + y^2 - z^2 - 1, 2xyz - 1 \rangle$ using the lex order with $x > y > z$.

a) What is the first elimination ideal I_1?

b) Given a partial solution (b, c) in $\mathbf{V}(I_1)$, can it be extended to a solution $(a, b, c) \in \mathbf{V}(I)$?

Exercise 6 What does the Extension Theorem say about the examples presented in Exercise 4?