Lesson 20 – More Practice with Elimination, Extension, Implicitization and the Closure Theorem

Let's recall what the Closure Theorem says:

The Closure Theorem Let $V = \mathbf{V}(f_1, f_2, \dots, f_s) \subseteq \mathbb{C}^n$, and let I_k be the k^{th} elimination ideal of $I = \langle f_1, f_2, \dots, f_s \rangle$. Then
(i) $\mathbf{V}(I_k)$ is the smallest affine variety containing $\pi_k(V) \subseteq \mathbb{C}^{n-k}$.
(ii) When $V \neq \emptyset$, there is an affine variety $W \subsetneq \mathbf{V}(I_k)$ such that $\mathbf{V}(I_k) - W \subseteq \pi_k(V)$.

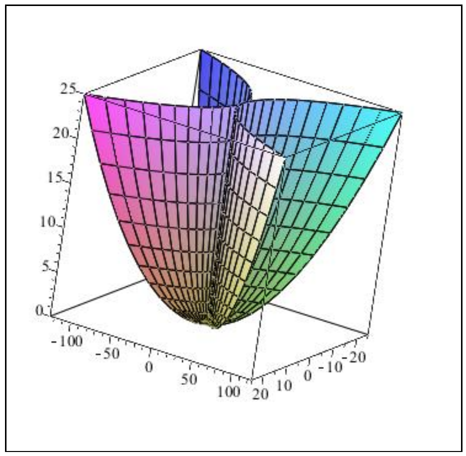
Example 1 For our first example, we return to the surface parameterization appearing in the last exercise of Wednesday's Maple lab: $x = u \cdot v,\ \ y = u \cdot v^2,\ \ z = u^2$.

The implicitization process involves solving the system

$$x - uv = 0$$
$$y - uv^2 = 0$$
$$z - u^2 = 0$$

via elimination of the variables *u* and *v*.

To this end, we use Maple to compute a Groebner basis...

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> with(Groebner) :
  Basis([x − u·v, y−u·v^2, z−u^2], plex(u, v, x, y, z));
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$$[-y^2 z + x^4,\ yzv - x^3,\ xv - y,\ -x^2 + zv^2,\ -x^2 + yu,\ -zv + xu,\ -x + uv,\ -z + u^2]$$

a) What is the equation of the smallest variety V' containing the surface *S* defined by the parameterization $F(u, v) = \{x - uv, y - uv^2, z - u^2\}$ (also denoted by $F(\mathbb{R}^2) = \pi_2(\mathbf{V}(x - uv, y - uv^2, z - u^2))$?

b) Over $\mathbb{C}$ what are the points in V' missing from the parameterization?

c) Applying part ii) of the Closure Theorem to this situation, can you identify the affine variety *W*?

Example 2 In this example, we examine how the Closure Theorem can fail when we work over $\mathbb{R}$ as opposed to $\mathbb{C}$ (also known as 👍).

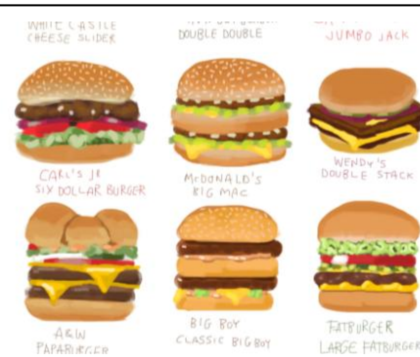
Consider the ideal: $I = \langle x^2 + y^2 + z^2 + 2,\ \ 3x^2 + 4y^2 + 4z^2 + 5 \rangle$

Let $V = \mathbf{V}(I)$, and let π_1 be the projection taking (x, y, z) to (y, z).

a) Working over $\mathbb{C}$, prove that $\mathbf{V}(I_1) = \pi_1(V)$. (It's not hard to see that $GB = \{y^2 + z^2 - 1, x^2 + 3\}$ is a Groebner basis for I under the lex ordering with $x > y > z$.)

b) Working over $\mathbb{R}$, what is $V = \mathbf{V}(I)$? What is $\mathbf{V}(I_1)$? What is the smallest variety containing $\pi_1(V)$?

MATHEMATICAL REBUS:



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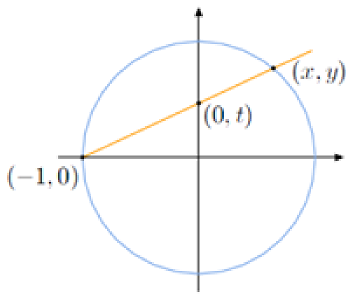
Example 3 Recall our parameterization of the unit circle obtained by the "sweeping line" method:



Rational Parameterization of Unit Circle:

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

By clearing denominators, the above parameterization yields the two polynomial equations $x(1+t^2) = 1 - t^2$ and $y(1+t^2) = 2t$. Defining $f_1 = x(1+t^2) - (1-t^2)$ and $f_2 = y(1+t^2) - 2t$, we find a Groebner basis for $I = \langle f_1, f_2 \rangle$:

> $\mathit{Basis}([x \cdot (1+t^2) - 1 + t^2,\ y \cdot (1+t^2) - 2 \cdot t], \mathit{plex}(t, x, y));$

$$[x^2 + y^2 - 1, y\,t - 1 + x, x\,t - y + t]$$

Aha...the first elimination ideal seems to give the implicit representation for the unit circle. Seems straightforward, right? Just to make your professor happy, let's be neurotic and sort out all of the notation and lingo. (It's easiest to do this with an example that makes perfect sense to you.)

a) The parameterization $F(t) = \{x(t), y(t)\} = \left\{\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right\}$ is a function $F: \mathbb{R} \longrightarrow \mathbb{R}^2$. Explain the meaning of this diagram (which the text uses; it's a nice way to "package" the information).

$$\begin{tikzcd} & \mathbb{R}^3 \arrow[dr] & \\ \mathbb{R} \arrow[ur] \arrow[rr, "F"] & & \mathbb{R}^2 \end{tikzcd}$$

b) Referring to the above Groebner basis of $I = \langle f_1, f_2 \rangle$, we see that $I_1 = \langle x^2 + y^2 - 1 \rangle$. And we know from last class, $\pi_1(V) \subseteq \mathbf{V}(I_1)$. What exactly is V here? How does the image of the parameterization, $F(\mathbb{R}^2)$, relate to $\pi_1(V)$ and to $\mathbf{V}(I_1)$?

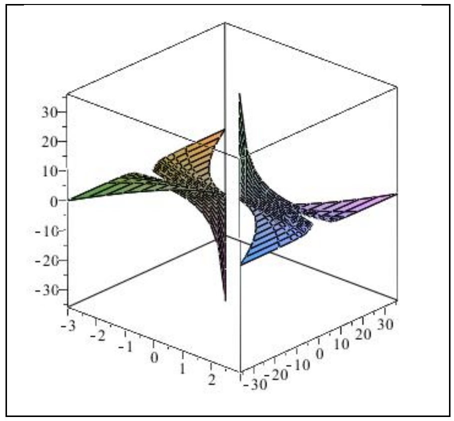
b) Use the Extension Theorem to show that the parameterization does not fill up the entire variety $\mathbf{V}(x^2 + y^2 - 1)$.

Example 4 Let's consider another rational parameterization; this time we'll consider a surface:

Rational Parameterization:

$$F(u,v) = \{x(u,v), y(u,v), z(u,v)\}$$

$$x = \frac{u^2}{v}$$
$$y = \frac{v^2}{u}$$
$$x = u$$



It's easy to see that the points $\left(\frac{u^2}{v},\ \frac{v^2}{u},\ u\right)$ lie on the surface $x^2y = z^3$. (Check this!)

a) Defining $I = \langle vx - u^2,\ uy - v^2, z - u\rangle \subseteq k[u,v,x,y,z]$, we compute a Groebner basis under the lex ordering with $u > v > x > y > z$. Here's Maple's output:

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> with(Groebner) :
  Basis([x·v − u^2, u·y−v^2, z − u], plex(u, v, x, y, z));
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$$[-z^4 + yzx^2,\ vz^2 - yzx,\ -z^2 + xv,\ -yz + v^2,\ -z + u]$$

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What is the second elimination ideal I_2? Is $\mathbf{V}(I_2)$ the smallest variety containing the parameterization?

b) Does it make sense to speak about the following diagram (which is similar to the one on the previous page but modified to reflect the fact that we have a surface in 3-space parameterized by *two* variables)?

$$\begin{tikzcd} & \mathbb{R}^5 \arrow[dr] & \\ \mathbb{R}^2 \arrow[ur] \arrow[rr, "F"] & & \mathbb{R}^3 \end{tikzcd}$$

To ensure the denominators appearing in the parameterization (u and v) never vanish, we have to be sneaky. The trick is to add another variable to the polynomial ring (we'll add the variable "D") and then enlarge the ideal I by the polynomial $1 - uvD$. That is, define the ideal J to be:

$$J = \langle 1 - uvD, vx - u^2, uy - v^2, z - u \rangle \subseteq k[D, u, v, x, y, z].$$

Observe that a point in $\mathbf{V}(J)$ cannot possibly be a point at which u or v vanishes. Now

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> with(Groebner) :
  Basis([1 − u·v·D, x·v − u^2, u·y−v^2, z − u], plex(D, u, v, x, y, z));
        [-z^3 + y x^2, z v − x y, -z^2 + x v, -y z + v^2, -z + u, z^3 D − x, y z^2 D − v, -1 + x y D]
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c) Which elimination ideal should we be examining in order to get our hands on the smallest variety containing the parameterization? Find it!

d) Which portion of $x^2y = z^3$ is parameterized by $F(u, v) = \left(\frac{u^2}{v}, \frac{v^2}{u}, u\right)$?

More generally, we have the Rational Implicitization Theorem (which you will prove for homework!)

Theorem (Rational Implicitization) If k is an infinite field, let $F: k^m - W \longrightarrow k^n$ be the function determined by the rational parameterization

$$x_1 = \frac{f_1(t_1, t_2, \dots, t_m)}{g_1(t_1, t_2, \dots, t_m)},$$
$$x_2 = \frac{f_2(t_1, t_2, \dots, t_m)}{g_2(t_1, t_2, \dots, t_m)},$$
$$\vdots$$
$$x_n = \frac{f_n(t_1, t_2, \dots, t_m)}{g_n(t_1, t_2, \dots, t_m)},$$

where $f_1, g_1, f_2, g_2, \dots, f_n, g_n$ are polynomials in $k[t_1, t_2, \dots, t_m]$. Let J be the ideal $J = \langle g_1x_1 - f_1, g_2x_2 - f_2, \dots, g_nx_n - f_n, 1 - gy \rangle \subseteq k[y, t_1, t_2, \dots, t_m, x_1, \dots, x_n]$, where $g = g_1g_2 \cdots g_n$, and let $J_{m+1} = J \cap k[x_1, \dots, x_n]$ be the $(m + 1)^{\text{st}}$ elimination ideal. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^n - W)$.

Exercise Given a rational parameterization:

$$x_1 = \frac{f_1(t_1, t_2, \dots, t_m)}{g_1(t_1, t_2, \dots, t_m)},$$
$$x_2 = \frac{f_2(t_1, t_2, \dots, t_m)}{g_2(t_1, t_2, \dots, t_m)},$$
$$\vdots$$
$$x_n = \frac{f_n(t_1, t_2, \dots, t_m)}{g_n(t_1, t_2, \dots, t_m)},$$

there is one case where the naïve ideal $I = \langle g_1x_1 - f_1, g_2x_2 - f_2, , \dots, g_nx_n - f_n \rangle \subseteq k[t, x_1, \dots, x_n]$ obtained by "clearing denominators" gives the right answer. Suppose that $x_i = \frac{f_i(t)}{g_i(t)}$ where there is only one parameter t. We can assume that for each i, $f_i(t)$ and $g_i(t)$ are relatively prime in $k[t]$ (so in particular, they have no common roots). If $k = \mathbb{R}$ or $k = \mathbb{C}$, prove that $\mathbf{V}(I_1)$ is the smallest variety containing $F(k^m - W)$, where $g = g_1g_2 \cdots g_n \in k[t]$ and $W = V(g) \subseteq k$. (This is actually true over any infinite field.)