Lesson 21 –Resultants

Elimination theory may be considered the origin of algebraic geometry; its history can be traced back to Newton (for special equations) and to Euler and Bezout. The classical method for computing varieties has been through the use of *resultants*, the focus of today's discussion. The alternative of using Groebner bases is more recent, becoming fashionable with the onset of computers. Calculations with Groebner bases may reach their limitations quite quickly, however, so resultants remain useful in elimination theory. The proof of the Extension Theorem provided in the textbook (see pages 164-165) also makes good use of resultants.

I. The Resultant Let R be a UFD and let $f, g \in R[x]$ be the polynomials

$$f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l, \qquad a_i \in k \text{ and } a_0 \neq 0$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \qquad b_i \in k \text{ and } b_0 \neq 0.$$

The definition and motivation for the resultant of $f, g \in R[x]$ lies in the following very simple exercise.

Exercise 1 Show that two polynomials f and g in $R[x]$ have a nonconstant common divisor in $R[x]$ if and only if there exist nonzero polynomials u, v such that $vf - ug = 0$, where $0 < \deg u < l$ and $0 < \deg v < m$.

The equation $vf - ug = 0$ can be turned into a numerical criterion for f and g to have a common factor. Actually, we use the equation $vf + ug = 0$ instead, because the criterion turns out to be a little cleaner (as there are fewer negative signs). Observe that $vf - ug = 0$ iff $vf + (-u)g = 0$, so $vf - ug = 0$ has a solution (u, v) iff $vf + ug = 0$ has a solution $(-u, v)$.

Exercise 2 (to be done outside of class) Given polynomials $f, g \in R[x]$ of positive degree, say

$$f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l, \qquad a_i \in R \text{ and } a_0 \neq 0$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \qquad b_i \in R \text{ and } b_0 \neq 0,$$

define

$$v(x) = c_0x^{m-1} + c_1x^{m-2} + \cdots + c_{m-1}$$
$$u(x) = d_0x^{l-1} + d_1x^{l-2} + \cdots + d_{l-1}$$

where the $l + m$ coefficients $c_0, c_1, \ldots, c_{m-1}, d_0, d_1, \ldots, d_{l-1}$ are treated as unknowns. Substitute the formulas for f, g, u and v into the equation $vf + ug = 0$ and compare coefficients of powers of x to produce the following system of linear equations with unknowns c_i, d_i and coefficients a_i, b_j, in R.

$$\begin{array}{lllll}
a_0c_0 & & + b_0d_0 & & = 0 \\
a_1c_0 + a_0c_1 & & + b_1d_0 + b_0d_1 & & = 0 \\
a_2c_0 + a_1c_1 + a_0c_2 & & + b_2d_0 + b_1d_1 + b_0d_2 & & = 0 \\
\vdots & & \vdots & & \vdots \\
 & a_lc_{m-1} & + & b_md_{l-1} & = 0
\end{array}$$

The system involves $l + m$ equations in $l + m$ unknowns. The key point here is that the system has a nonzero solution iff the determinant of the corresponding coefficient matrix is zero.

Definition Let $f, g \in R[x]$ be two polynomials of positive degree

$$f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l, \qquad a_i \in R \text{ and } a_0 \neq 0$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \qquad b_i \in R \text{ and } b_0 \neq 0.$$

Then the **Sylvester matrix** of f and g with respect to x, denoted Syl(f, g, x) is the coefficient matrix of the system of equations produced in the previous exercise. That is, Syl(f, g, x) is the following $(l + m) \times (l + m)$ matrix:

$$\mathrm{Syl}(f, g, x) = \begin{bmatrix}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & & & b_1 & b_0 & & \\
a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\
\vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\
 & \vdots & & a_1 & & & & b_1 \\
a_l & & & a_2 & b_m & & & b_2 \\
 & a_l & & \vdots & & b_m & & \vdots \\
 & & \ddots & & & & \ddots & \\
 & & & a_l & & & & b_m
\end{bmatrix}$$

where the empty spaces are filled in by zeros.

Definition The **resultant** of f and g with respect to x, denoted by Res(f, g, x), is the determinant of the Sylvester matrix. Thus,

$$\mathrm{Res}(f, g, x) = \det(\mathrm{Syl}(f, g, x)).$$

Remark: The Sylvester matrix is an $(l+m)\times(l+m)$- matrix, where $l = \deg(f)$ and $m = \deg(g)$, because there are m columns consisting of a_i's and l columns consisting of b_j's.

The gist of the previous page is summarized in the following theorem.

Theorem Let R be a UFD; then the polynomials

$$f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l, \qquad a_i \in R \text{ and } a_0 \neq 0$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \qquad b_i \in R \text{ and } b_0 \neq 0$$

in $R[x]$ have a non-constant common divisor in $R[x]$ if and only if the following equivalent conditions are satisfied:

(1) There exist nonzero polynomials $u, v \in R[x]$ with $\deg u < \deg f$, $\deg v < \deg g$, and $vf - ug = 0$.
(2) $\text{Res}(f, g, x) = 0$

We illustrate the process with a simple example.

Exercise 3 Take $f(x) = a_0x^2 + a_1x + a_2$ and $g(x) = b_0x + b_1$. Derive the Sylvester matrix Syl(f, g, x) from the equation $vf + ug = 0$ where $v(x) = c_0$ and $u(x) = d_0x + d_1$.

II. Finding Intersection Points of Curves Let us now apply resultants to curves. Consider the intersection of the two ellipses $\mathcal{C}_f = \mathbf{V}(x^2 + 2y^2 - 3)$ and $\mathcal{C}_g = \mathbf{V}(x^2 + xy + y^2 - 3)$ in $\mathbb{C}[x, y]$.

Exercise 4 Find $\mathcal{C}_f \cap \mathcal{C}_g = \mathbf{V}(x^2 + 2y^2 - 3, x^2 + xy + y^2 - 3)$.

Observe that, $\text{Res}(f(x,y), g(x,y), y)$ is a function of x and $\text{Res}(f(x,y), g(x,y), x)$ is a function of y. In fact, the following is true:

Theorem Let $f, g \in k[x_1, x_2, \dots, x_n]$ have positive degree in x_1. Then:

i) $\text{Res}(f, g, x_1)$ is in the first elimination ideal $\langle f, g \rangle \cap k[x_2, \dots, x_n]$.
ii) $\text{Res}(f, g, x_1) = 0$ if and only if f and g have a common factor in $k[x_1, x_2, \dots, x_n]$ which has positive degree in x_1.

Sketch of proof.

Exercise 5 Fill in the blanks to explain how resultants can be used to compute the points of intersection of two affine curves.

a) If (x_0, y_0) is a point on the intersection of $\mathcal{C}_f: f(x,y) = 0$ and $\mathcal{C}_g: g(x,y) = 0$, then Res$(f, g, y)$ ____

b) If (x_0, y_0) is a point on the intersection of $\mathcal{C}_f: f(x,y) = 0$ and $\mathcal{C}_g: g(x,y) = 0$, then $F(y) = f(x_0, y)$ and $G(y) = g(x_0, y)$ have the common factor ____________________________

III. Implicitization Resultants can be used for implicitization: this is the technique of finding an implicit equation of a parameterized curve. Consider

$$x = \frac{p(t)}{q(t)}, y = \frac{r(t)}{s(t)}, \qquad (*)$$

where GCD(p, q) =1 and GCD(r, s) =1.

Exercise 6 Show that $\mathbf{V}\big(xq(t) - p(t), ys(t) - r(t)\big) = \{(t, x, y): x = \frac{p(t)}{q(t)}, y = \frac{r(t)}{s(t)}\}$.

Exercise 7 Let $f = xq(t) - p(t)$ and $g = ys(t) - r(t)$ and define $R(x,y) = \text{Res}(f, g, t)$. Show that a point (x_0, y_0) is on the paramaterized curve iff $R(x_0, y_0) = 0$.

Exercise 8 Apply what you learned in Exercise 7 to derive the equation of the curve parameterized by

$$x = \frac{t^2 - 1}{t^2 + 1}, \quad y = \frac{2t}{t^2 + 1}$$

One final detail regarding resultants. Perhaps you noticed that our definition of the resultant of two polynomials, $\text{Res}(f, g, t)$, required that f and g be nonconstant (i.e., of positive degree). In fact, we can define the resultant in the case where one or both of the polynomials is constant. One just has to have the patience to work out the various cases. (See Exercise 14 on page 161 in your textbook). Here is what you get:

$f(x)$ nonconstant, $g(x)$ constant:
If $f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l$ and $g(x) = b_0$, then the Sylvester matrix is an $l \times l$ matrix with b_0 along the diagonal and zeros everywhere else. Hence
$\text{Res}(f, g, x) = \det(\text{Syl}(f, g, x)) = b_0^l$.

$f(x)$ constant, $g(x)$ nonconstant:
If $f(x) = a_0$ and $g(x) = b_0x^m + a_1x^{m-1} + \cdots + a_m$, then the situation is similar to the previous case and $\text{Res}(f, g, x) = \det(\text{Syl}(f, g, x)) = a_0^m$

$f(x)$ and $g(x)$ both constant:
If $f(x) = a_0$ and $g(x) = b_0$. Then we define $\text{Res}(f, g, x) = \begin{cases} 0 & \text{if either } a_0 = 0 \text{ or } b_0 = 0 \\ 1 & \text{if } a_0 \neq 0 \text{ and } b_0 \neq 0 \end{cases}$