Lesson 27 – Operations on Ideals

With the celebrated Nullstellensätze[1] under our belts we know exactly which ideals correspond to varieties, and this allows us to construct the so-called "dictionary" between geometry and algebra. Today we define a number of natural algebraic operations (sums, products and intersections) on ideals and study their geometric analogues. We'll start with sums and products, which are probably already familiar to you.

I. Sums and Products of Ideals (Briefly)

Definition If I and J are ideals $k[x_1, x_2, \dots, x_n]$, then

- the **sum** $I + J$ is defined by:
$$I + J = \{f + g \mid f \in I \text{ and } g \in J\}$$
- the **product** $I \cdot J$ is defined by:
$$I \cdot J = \{f_1 g_1 + \cdots + f_r g_r \mid f_1, \dots, f_r \in I \text{ and } g_1, \dots, g_r \in J, r \in \mathbb{N}\}$$

Remark. It is a straightforward computation to prove $I + J$ and $I \cdot J$ are both ideals. In fact, $I + J$ is the smallest ideal containing I and J.

Exercise 1
a) Given $I = \langle f_1, \dots, f_r\rangle$ and $J = \langle g_1, \dots, g_s\rangle$ are ideals in $k[x_1, \dots, x_n]$, what are the generators of $I + J$ and $I \cdot J$?

b) What are the geometric analogues of $I + J$ and $I \cdot J$? That is, what is $\mathbf{V}(I + J)$ and $\mathbf{V}(I \cdot J)$?

II. Intersections of Ideals

We all know what the intersection of two ideals is, and it's an easy exercise to check that $I \cap J$ is an ideal.

Definition The intersection $I \cap J$ of two ideals I and J in $k[x_1, x_2, \dots, x_n]$ is the set of polynomials which belong to both I and J.

For the rest of the lesson, we will focus on answering two questions:

1. What is the geometric analogue of $I \cap J$?
2. How can we compute $I \cap J$? That is, given $I = \langle f_1, \dots, f_r\rangle$ and $J = \langle g_1, \dots, g_s\rangle$, how can we find generators of $I \cap J$? Unlike sums and products of ideals, intersections take some work.

[1] Thanks to Andy Smith and David Mehrle for the German lesson!

Theorem 1 (Geometric Analogue of the Intersection) If I and J are ideals in $k[x_1, x_2, \dots, x_n]$, then $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$. (Hence $\mathbf{V}(I \cap J) = \mathbf{V}(I \cdot J) = \mathbf{V}(I) \cup \mathbf{V}(J)$.)

Theorem 1 says that the variety associated with an intersection of two ideals is actually the same as the variety associated with the product of the ideals, i.e., $\mathbf{V}(I \cap J) = \mathbf{V}(I \cdot J)$. Before proving this theorem, let's consider the following question...

Exercise 2 Given ideals I and J in $k[x_1, x_2, \dots, x_n]$, must $I \cap J = I \cdot J$?

Exercise 3 Prove Theorem 1.

The next result tells us how to compute intersections of ideals.

Theorem Let I and J be ideals in $k[x_1, x_2, \dots, x_n]$ and form the ideal $tI + (1-t)J \subseteq k[x_1, x_2, \dots, x_n, t]$.[2] Then $I \cap J = tI + (1-t)J \cap k[x_1, x_2, \dots, x_n]$.

Proof. We prove inclusion in each direction...

The previous theorem is slick, but somewhat unsatisfactory. The rough geometric picture of what the theorem describes is that there is a copy of $\mathbf{V}(I)$ at $t = 1$ and a copy of $\mathbf{V}(J)$ at $t = 0$. Taking the intersection sort of super-imposes the two, resulting in the variety $\mathbf{V}(I) \cup \mathbf{V}(J)$ which is predicted by the equality proved earlier: $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$.

The proof itself does not use any Groebner basis tools, but we will need them in order to carry out calculations.

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[2] Given an ideal $I \subseteq k[x_1, x_2, \dots, x_n]$, we use the notation $f(t)I$ to denote the ideal generated by the set of polynomials $\{f(t)h(x_1, \dots, x_n) | \, h \in I\}$.

Exercise 4 Describe an algorithm for computing intersection of two ideals I and J in $k[x_1, x_2, \dots, x_n]$.

Examples

Intersections come up in computing various things relating to algebraic sets, so it will be useful to consider some examples. In the world of PIDs, the difference between the product and the intersection of two ideals is the same as the difference between the product of two numbers and their least common multiple. For example:

Exercise 5 Find the product and intersection of $I = \langle x^2 - 1 \rangle$ and $J = \langle x^2 + 2x + 1 \rangle$ in $k[x]$.

Incidentally, a Groebner basis for $\langle t(x^2 - 1), (1 - t)(x^2 + 2x + 1) \rangle$ under the lex order with $t > x$ is:

> $Basis\left(\left[t\cdot\left(x^2 - 1\right), (1 - t)\cdot\left(x^2 + 2\cdot x + 1\right)\right], plex(t, x)\right);$

$$\left[x^3 - 1 + x^2 - x, 2\,t\,x - 1 - x^2 + 2\,t - 2\,x\right]$$

and

> $expand\left((x - 1)\cdot(x + 1)^2\right);$

$$x^3 - 1 + x^2 - x$$

Exercise 6 A polynomial ring in two or more variables is not a PID, and this introduces complication (hence the need for Groebner bases).

a) Given the ideals $I = \langle x, z \rangle$ and $J = \langle x, y \rangle$ in $\mathbb{R}[x, y, z]$, use the notion of LCMs to guess what the intersection $I \cap J$ might me.

Computing $\langle x, z\rangle \cap \langle x, y\rangle$ definitively will take a bit more work. We could use the algorithm outlined in Exercise 4, but let's try a different approach.

b) Prove that if I and J are radical ideals, then $I \cap J = \mathbf{I}(\mathbf{V}(I \cdot J))$.

c) Use part b) to show $\langle x, z\rangle \cap \langle x, y\rangle = \langle x, yz\rangle$.

Once again, if we rely on our algorithm for computing intersections we get:

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> Basis([t·x, t·z, (1 − t)·x, (1 − t)·y], plex(t, x, y, z));
                              [zy, x, tz, −y + yt]
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and we see that the first elimination ideal is $\langle x, yz\rangle$.

To summarize what we have discussed, sums, products and intersections of ideals translate into the geometric world of varieties as indicated in the table below. (Here I and J are assumed to be radical ideals.)

The Algebra - Geometry Dictionary		
radical ideals I $\mathbf{I}(V)$	$\longrightarrow$ $\longleftarrow$	**varieties** $\mathbf{V}(I)$ V
addition of ideals $I + J$	$\longrightarrow$	**intersection of varieties** $\mathbf{V}(I) \cap \mathbf{V}(J)$
product of ideals $I \cdot J$	$\longrightarrow$	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$
intersection of ideals $I \cap J$	$\longrightarrow$	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$

What about the reverse direction? We will be examining this in more detail, but here are the results:

The Algebra - Geometry Dictionary		
radical ideals I $\mathbf{I}(V)$	$\longrightarrow$ $\longleftarrow$	**varieties** $\mathbf{V}(I)$ V
addition of ideals $I + J$ $\sqrt{\mathbf{I}(V) + \mathbf{I}(W)}$	$\longrightarrow$ $\longleftarrow$	**intersection of varieties** $\mathbf{V}(I) \cap \mathbf{V}(J)$ $V \cap W$
product of ideals $I \cdot J$ $\sqrt{\mathbf{I}(V) \cdot \mathbf{I}(W)}$	$\longrightarrow$ $\longleftarrow$	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$ $V \cup W$
intersection of ideals $I \cap J$ $\mathbf{I}(V) \cap \mathbf{I}(W)$	$\longrightarrow$ $\longleftarrow$	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$ $V \cup W$