Lesson 28 – Quotient Ideals

Recall the definition of the Zariski closure of a set...

Definition The **Zariski closure** of a subset of affine space is the smallest affine algebraic variety containing the set. If $S \subseteq k^n$, the Zariski closure of S is denoted by $\bar{S}$ and is equal to $\mathbf{V}(\mathbf{I}(S))$.
That is,

$$\bar{S} \stackrel{\text{def}}{=} \mathbf{V}\big(\mathbf{I}(S)\big).$$

Exercise 1 What is the Zariski closure of the set $S = V - W$ in $\mathbb{R}^2$, where $V = \mathbf{V}(xy - x^2)$ and $W = \mathbf{V}(x)$?

The ideal-theoretic analogue of $\overline{V - W}$ is an ideal known as the *quotient ideal*.

Definition If I and J are ideals in $k[x_1, \dots, x_n]$, then the **quotient ideal** or **colon ideal** is the ideal $I : J$ defined by

$$\begin{aligned} I : J &= \{f \in k[x_1, \dots, x_n] \mid fJ \subseteq I\} \\ &= \{f \in k[x_1, \dots, x_n] \mid fg \in I \ \forall g \in J\} \end{aligned}$$

Remarks.

1. We leave it as an exercise to show that $I : J$ is an ideal satisfying $I \subseteq I : J$.
2. In fact, $I : J$ is the largest ideal K satisfying $JK \subseteq I$. (For this reason, the notation I/J would better describe the quotient ideal, but that notation is taken!)

Sometimes the quotient ideal is denoted by $(I : J)$ or $[I : J]$.

Exercise 2 Compute the quotient ideal $\langle xy - x^2 \rangle : \langle x \rangle$ in $k[x, y]$.

The quotient ideal $I : J$ is very useful when I is a radical ideal. In particular, it is the vanishing ideal of the set difference $\mathbf{V}(I) - \mathbf{V}(J)$.

Proposition 1 If I is a radical ideal in $k[x_1, \dots, x_n]$, then $I{:}J = \mathbf{I}(\mathbf{V}(I) - \mathbf{V}(J))$

Applying $\mathbf{V}$ to both sides of $I{:}J = \mathbf{I}(\mathbf{V}(I) - \mathbf{V}(J))$ yields the following alternative way of stating Proposition 1 (this is Theorem 7 in your textbook).

Theorem 2 (Algebraic Analogue of $\overline{\mathbf{V}(I) - \mathbf{V}(J)}$) If k is an algebraically closed field and I is a radical ideal in $k[x_1, \dots, x_n]$, then

$$\mathbf{V}(I{:}J) = \overline{\mathbf{V}(I) - \mathbf{V}(J)})$$

Exercise 3 Prove that V and W are varieties in k^n, then $\mathbf{I}(V){:}\mathbf{I}(W) = \mathbf{I}(V - W)$.

Computing Quotient Ideals

The quotient $I : J$ can be computed via the intersection of quotients of I with generators of J, as formally stated in the following "useful formula".

Useful Formula Let I and J be ideals in $k[x_1, \dots, x_n]$, and assume $J = \langle f_1, f_2, \dots, f_r \rangle$. Then

$$I : J = I : \langle f_1, f_2, \dots, f_r \rangle = \bigcap_{i=1}^{r} (I{:}f_i) \qquad (*)$$

To justify the useful formula $(*)$, we need to show $I : (J + K) = I{:}J \cap I{:}K$ (Proposition 3 below), and to use this formula, we will need to see how to compute $(I{:}f_i)$ (Theorem 4 below).

Proposition 3 Let I and J_i be ideals in $k[x_1, \dots, x_n]$ for $1 \leq i \leq r$. Then

$$I : \left(\sum_{i=1}^{r} J_i\right) = \bigcap_{i=1}^{r} (I{:}J_i)$$

Theorem 4 Let I be an ideal and g a polynomial in $k[x_1, \dots, x_n]$. If $\{h_1, h_2, \dots, h_p\}$ is a basis of the ideal $I \cap \langle g\rangle$, then $\{\frac{h_1}{g}, \frac{h_2}{g}, \dots, \frac{h_p}{g}\}$ is a basis of $I : \langle g\rangle$.

The proof is straightforward and covered by Exercises 4 and 5 below.

Exercise 4 Show that any element in $\langle\frac{h_1}{g}, \frac{h_2}{g}, \dots, \frac{h_p}{g}\rangle$ belongs to $I : \langle g\rangle$.

Exercise 5 Show that every $f \in I{:}\langle g\rangle$ is a polynomial combination of $\{\frac{h_1}{g}, \frac{h_2}{g}, \dots, \frac{h_p}{g}\}$.

Exercise 6 Given two ideals, $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ in $k[x_1, \ldots, x_n]$, and use what you have learned to describe an algorithm for computing a basis of a quotient ideal $I : J$.

Example Consider the variety $V = \mathbf{V}(xz - y^2, x^3 - yz)$. The graph indicates that the variety contains two components; the line is easily seen to be the z-axis as both $xz - y^2$ and $x^3 - yz$ vanish at all points of the form (0, 0, z).

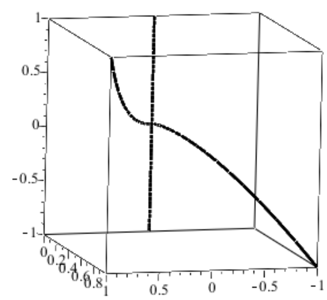

Figure 1: The variety $\mathbf{V}(xz - y^2, x^3 - yz)$

Exercise 7 The other component of the variety is $\overline{V - \mathbf{V}(x, y)}$. To find this variety, we compute the quotient ideal $\langle xz - y^2, x^3 - yz \rangle : \langle x, y \rangle$.

a) Define $I = \langle xz - y^2, x^3 - yz \rangle$. What does Proposition 3 say about $I : \langle x, y \rangle$?

b) Use the Maple output below to determine the variety $I : x$.

> $sys1 := [t\cdot(x\cdot z - y^2), t\cdot(x^3 - y\cdot z), (1-t)\cdot x]$;

$$sys1 := [t\,(x\,z - y^2), t\,(x^3 - y\,z), (1-t)\,x]$$

> $Groebner[Basis](sys1, plex(t, z, y, x))$;

$$[y^3\,x - x^5, z\,x^2 - y^2\,x, -x^4 + z\,y\,x, -y\,x^3 + z^2\,x, -x + x\,t, t\,y^2 - x\,z, t\,y\,z - x^3]$$

c) Use the Maple output below to determine the variety $I : y$.

> $sys2 := [t\cdot(x\cdot z - y^2), t\cdot(x^3 - y\cdot z), (1-t)\cdot y]$;

$$sys1 := [t\,(x\,z - y^2), t\,(x^3 - y\,z), (1-t)\,x]$$

> $Groebner[Basis](sys2, plex(t, z, y, x))$;

$$[y^4 - y\,x^4, z\,y\,x - y^3, -y\,x^3 + y^2\,z, -y^2\,x^2 + z^2\,y, t\,x^3 - y\,z, -y + t\,y, t\,x\,z - y^2]$$

d) Now find $I : \langle x, y\rangle = I{:}\,x \ \cap \ I{:}\,y$ and use this information to identify the second component of the original variety.

e) Guess what happens when you compute $I : \langle -yx^2 + z^2\rangle$.

Here we update our “algebra-geometry dictionary” :

The Algebra - Geometry Dictionary		
radical ideals I $\mathbf{I}(V)$	→ ←	**varieties** $\mathbf{V}(I)$ V
addition of ideals $I + J$ $\sqrt{\mathbf{I}(V) + \mathbf{I}(W)}$	→ ←	**intersection of varieties** $\mathbf{V}(I) \cap \mathbf{V}(J)$ $V \cap W$
product of ideals $I \cdot J$ $\sqrt{\mathbf{I}(V) \cdot \mathbf{I}(W)}$	→ ←	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$ $V \cup W$
intersection of ideals $I \cap J$ $\mathbf{I}(V) \cap \mathbf{I}(W)$	→ ←	**union of varieties** $\mathbf{V}(I) \cup \mathbf{V}(J)$ $V \cup W$
quotient of ideals $I : J$ $\mathbf{I}(V) : \mathbf{I}(W)$	Theorem 2 → Exercise 3 (really, Proposition 1) ←	**difference of varieties** $\overline{\mathbf{V}(I) - \mathbf{V}(J)}$ $\overline{V - W}$